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On the Origin of Primary Cosmic Radiation

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(Received December 4, 1952)

At first it is illustrated that two important empirical facts concerning primary cosmic radiation, i.e., (i) the similarity of energy spectra expressed with respect to kinetic energy per nucleon of various nuclei independent of their charge numbers and (ii) fairly abundant existences of heavier nuclei, will not be able to be explained by any acceleration mechanisms. And it is shown that (i) and (ii) can be understood only if one assumes a new type of element formation, according to which, from the nucleon gas of extremely high temperature ($\approx 10^{10}$°K) and pressure, various atomic nuclei are created with energy spectra $N(\epsilon)=C\exp\{-(1+\epsilon)/kT\}$ (ϵ : kinetic energy per nucleon) and with ratio $C_p : C_\alpha : C_{CNO} : C_{Z>10}=10 : 1 : 1/20 : 1/60$. Physical inevitability with which this assumption is introduced and its physical significance are discussed. The theory itself of element formation of such a type, however, is not touched but only it is pointed out that the introduction of such an assumption is strongly required from present knowledges concerning primary cosmic ray particles.

§ 1. Introductory discussions

The theory on the origin of primary cosmic radiation should explain the following two important results of observation : (i) The energy spectra of primary cosmic radiation, which are expressed with respect to the kinetic energy per nucleon, have the same form for all nuclei. (ii) As to the chemical composition, the cosmic abundance is the same with that in primary cosmic radiation for the nuclei with rather small Z ($Z < 6$), but for heavier nuclei the abundance in primary cosmic radiation exceeds that in the universe and indeed the nuclei of $Z \gtrsim 10$ are about ten times more abundant in primary cosmic radiation than in the universe. Moreover, in order to understand the isotropy of primary radiation, it seems to be necessary to consider that the primary particles incident upon the earth have travelled through the universe for a long time after their origination. At the present time, these observational facts have been obtained fairly in detail in the kinetic energy region between 0.3 and 10 Bev per nucleon[1].

Owing to the empirical fact (ii), especially because of the fact that there exist few, or else none of, nuclei of Li^3, Be^4 and B^5 in primary cosmic radiation[2], we must conclude as follows : primary particles reach to the earth without too much collisions with the intersteller hydrogen gas, and therefore their chemical composition changes scarcely from that at the time of origination. From this point of view, it seems to be inadequate to assume some acceleration process in order to explain the problem of primary cosmic radiation as was done, for instance, by Fermi[3], because by any acceleration mechanism heavier particles are not so easily accelerated as lighter ones and it would, therefore, be difficult

for the chemical composition in the source to be maintained, and thus it can not be understood at all that the heavier nuclei are much more abundant in cosmic radiation than in the universe. With respect to (i), on the other hand, the integral energy spectra within the energy range (kinetic energy per nucleon in Bev) $0.3 \lesssim \epsilon \lesssim 10$ are well established by the following empirical formulae[1]:

$$N(\epsilon) = 4000/(1+\epsilon)^{1.67} \qquad \text{for protons,} \qquad (1)$$

and

$$N(\epsilon) = K(1+\epsilon)^{1.35} \qquad \text{for other nuclei.} \qquad (2)$$

Now we must emphasize that $(1+\epsilon)$ is the total energy per nucleon because the rest energy of a nucleon is 1 Bev. Therefore, the integral spectra (1) and (2) are the simple functions only of the total energy per nucleon. From this point also, Fermi's theory, according to which the forms of the energy spectra depend on the species of nuclei, must be considered to be unsatisfactory[4]. If, furthermore, one refers again to the fact that primary cosmic radiation seems to reach the earth after few collisions and therefore to experience acceleration process not so much as its chemical composition changes appreciably, it can be concluded that the energy spectra given by (1) and (2) are those of primary cosmic radiation at the time of their origination. Thus the energy spectra and chemical composition of primary cosmic radiation observed on the earth must be the same as those at the time of their origination. Then one should assume that, at some time after the origination of our universe, and from somewhere in our universe, various bare nuclei had been created with energy spectra given by (1) and (2). Although only for energy range (kinetic energy per nucleon in Bev) $0.3 \lesssim \epsilon \lesssim 10$ above mentioned results of observation (i) and (ii) are well established, there is some evidence that (i) and (ii) seem to hold at higher energies. At least it is certain that there are many primary particles of much higher energies than 10 Bev per nucleon. Now we refer to Cocconi's interesting discussion[5] about the origin (solar origin or galactic one) of cosmic primary radiation. According to Cocconi's discussion, cosmic ray particles of moderate energy ($1 \sim 10$ Bev) are those which were created locally near the sun and have been trapped in the closed orbits by the solar magnetic field, and reach the earth with relatively short lifetimes ($\approx 10^4$ years), thus passing through the matter layer less than 1 gram. The lifetime of this order of magnitude is sufficient to account for the isotropy of cosmic ray and selective absorption of electronic components[6]. Thus these particles of moderate energies can be considered to be incident upon the earth with energy spectra and chemical composition at their times of origination. On the other hand, particles of higher energies have the orbits with curvatures that are comparable with the size of solar system and, therefore, these particles cannot be confined only within the solar system but are distributed in the whole galaxy. That is, higher energy particles are those that were created at some star and leaked out of its trapping field and have travelled through the whole galaxy, being accelerated repeatedly for many years. Owing to the fact that its life time may have order of magnitude about 10^8 years, the chemical composition and energy spectra for these components of cosmic primary radiation must be changed entirely from those at the time of their origination.

§ 2. Our hypothesis concerning element formation and related discussions

As a result of the preceding considerations, we have enough evidence for the following important fact: That is, cosmic ray particles of energy less than 10 Bev, which can hardly be accounted for by any acceleration process, were created directly inside the solar system or else near it with the same chemical composition and energy spectra as observed on the earth concerning primary cosmic radiation. And, above in all, it should be noted that heavier nuclei must have been created much more abundantly than is predicted by the presently accepted theory of element formation.

According to G. Gamow[7], our universe has the age of $10^9 \sim 10^{10}$ years and in the early times it was in a gaseous state that consisted of nucleons with extremely high temperature and pressure, and its temperature, averaged over whole universe, was given by the following expression:

$$T = 1.5 \times 10^{10} / \sqrt{t}\ ^\circ K, \tag{3}$$

where t is the age of universe after its origination in second. It is believed also that the present universe with well known chemical composition was established through an element formation process that happened at the time when $T \approx 10^{10}\ ^\circ K$ or $kT \approx 1$ Mev (k= Boltzmann's constant). It must be, however, denied owing to the above mentioned reasons, especially on accout of marked predominance of heavier nuclei, to consider that after above process of element formation various nuclei were accelerated to cosmic ray energies under the operation of some acceleration mechanism. On the other hand, even if one assumes that the various nuclei with a certain kinetic energy were created directly from nucleon gas of extremely high temperature and pressure through the above element formation process, the emission of nuclei of kinetic energy ≈ 1 Bev could hardly be understood because $kT \approx 1$ Mev corresponding to temperature $T \approx 10^{10^\circ} K$ and the nuclear binding energies are of the same order.

In this paper, therefore, we wish to point out that, according to the discussions we have given above, our present knowledges with primary cosmic radiation give a strong evidence that there must be some new mechanism of element formation which is of quite different type from those considered and accepted up to the present. This new mechanism of element formation should be such that allows marked predominance of heavier nuclei and energy distribution of emitted nuclei agreeable with (1) and (2) in kinetic energy range per nucleon $0.3 \lesssim \epsilon \lesssim 10$ Bev.

From these considerations we seek an expression that approximates (1) and (2) most closely in kinetic energy range 0.3∼10 Bev. For instance it is the following one:

$$N(\epsilon) = C \exp\{-(1+\epsilon)/kT\}, \tag{4}$$

where $kT \cong 2.12$ Bev, or $T \cong 2.46 \times 10^{10^\circ} K$. (5)

Above expressions are plotted in Figure by solid curves while dashed ones are empirical spectra (1) and (2). The agreements of these curves are fairly good in this energy range. It should be noted that the exponent of the energy distribution (4) is not

$-A(1+\epsilon)/kT$ (A: mass number) but $-(1+\epsilon)/kT$, the physical significance of which is discussed later and the former exponent cannot reproduce the empirical curves entirely. Also in order to get the best agreement with empirical curves it is necessary to use the following ratio of abundance:

$$C_P : C_\alpha : C_{CNO} : C_{Z>10} = 10 : 1 : 1/20 : 1/60, \qquad (6)$$

which must be compared with the known chemical abundance of the whole universe given at present as follows[8]:

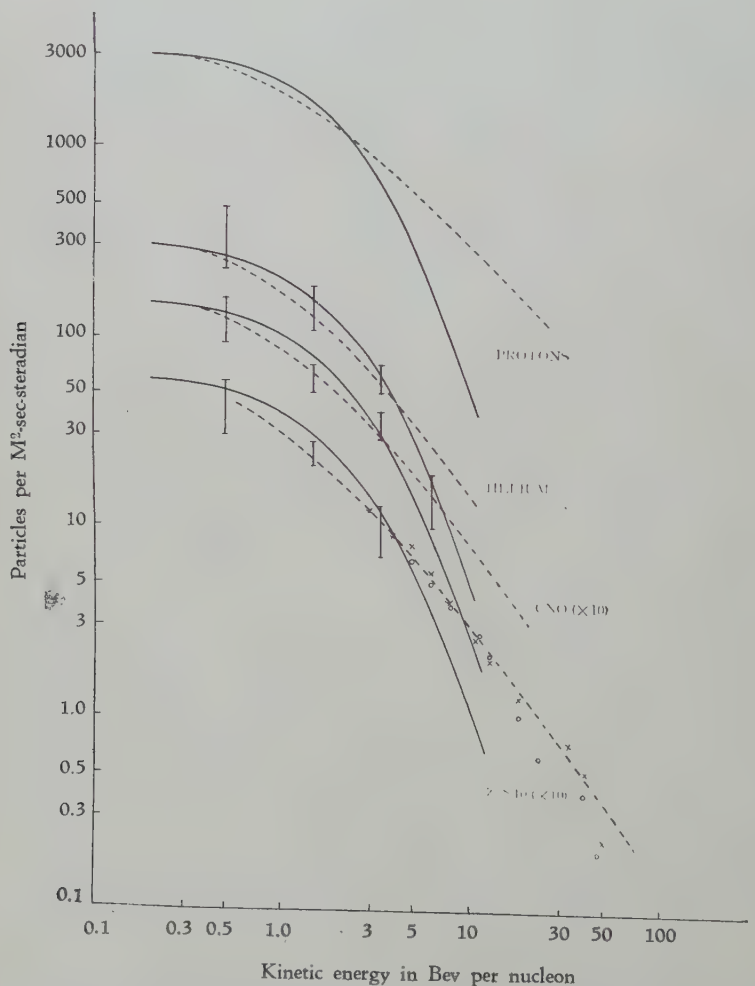

Figure: Integral energy spectrum of primary cosmic radiation. The dashed curves drawn for the heavy nuclei are represented by $N(\epsilon)=K/(1+\epsilon)^{1.75}$ and for the protons by $N(\epsilon)=4000/(1+\epsilon)^{1.07}$, where ϵ is kinetic energy in Bev per nucleon. The solid ones are obtained from the formulae assumed in this paper: $N(\epsilon)=C\exp\{-(1+\epsilon)/kT\}$. The dots, crosses and vertical bars represent the results of observation.

$$C_P : C_\alpha : C_{CNO} : C_{Z>10} = 10 : 1 : 1/75 : 1/500, \qquad (7)$$

which show quantitatively how the heavier elements are much more abundant in primary cosmic ray particles than in the present universe. Now, as the final step of our consideration, we wish to set up the following assumption : That is, somewhere in our universe, presumably in such places as inside the star, there existed, or exists, place or places of extremely high temperature ($T \cong 10^{13^\circ} K$) and there, as a matter of course, atomic nuclei were not yet formed and all the nucleons existed separately, forming a gaseous state of extremely high temperature and pressure, in which the Maxwell's energy distribution of nucleons of type (4), $(1+\epsilon)$ being the total energy per nucleon, was maintained. In addition to these, we assume that from above gas of nucleons of high temperature and pressure various nuclei were created according to some new type of mechanism with the chemical composition given by (6) and since nuclear binding energy ≈ 1 Mev is negligible compared with thermal kinetic energy ≈ 1 Bev, atomic nuclei still have the same kinetic energy per nucleon as before they are formed. Hence we may say that the emitted nuclei have the same energy distribution with regard to kinetic energy per nucleon as (4). Without introducing such an assumption, it seems very difficult to understand the empirical fact that cosmic ray components of any Z have the same energy spectra with respect to kinetic energy per nucleon. It should be remarked that even if one assumes Maxwell's distribution for the gas of extremely high temperature and pressure, if various atomic nuclei were formed already, the Maxwell's distribution in this case would be $\sim \exp\{-A(1+\epsilon)/kT\}$ (A : mass number of each nucleus), which quite disagree with the empirical fact mentioned just above. Therefore, the cosmic ray data requires evidences not merely for high temperature but also for a possible existence of a new type of element formation which takes place at extremely high temperature. Regarding this new type of element formation, of course, the possibility of it must be studied in detail but, since such a problem is out of lines of present authors, we will discontinue further discussions with this point. The authors wish merely to note that our present knowledges concerning primary cosmic radiation could be scarcely understood without introducing such an assumption and that our assumption is not necessarily unreasonable physically.

It is clear from our discussions that above considerations do not exclude any acceleration mechanism at all. Indeed it is certainly necessary to introduce some acceleration process in order to explain the existences of particles of energy $\gtrsim 10$ Bev. If, for instance, one adopts Fermi's assumption[3] in addition to our one, the heavier nuclei will also be accelerated because according to our assumption there exist already various nuclei having kinetic energy of order of magnitude of 1 Bev which far exceeds Fermi's minimum injection energy. Hence it will not be the case that, as in Fermi's theory, only protons can be accelerated on account of which one can not understand the abundant existence of heavier nuclei in primary radiations. We do not continue, however, to discuss the accleration mechanism in detail theoretically, but only point out the possibility that the better agreement may be gained between solid curves and empirical dashed ones in Figure by considering some acceleration processes.

Finally, with respect to lattitude effect, there appears a knee at $\lambda \simeq 50°$ in the numbers of counts of primary particles versus lattitude curve, which has been considered as an evidence against the existence of particles of energy less than 2 Bev ($\approx$ magnetic cut-off momentum at $\lambda \simeq 50°$).[9] According to our assumption, however, $kT \simeq 2$ Bev and therefore the appearance of knee in lattitude effect seems to be understood naturally.

The authors wish to express their appreciation for the valuable and stimulating discussions to Dr. Y. Ôno and Dr. S. Hayakawa (Osaka City University) throughout the course of this work. They also wish to thank Professor S. Tomonaga for his kind and helpful advices.

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On the Spin Wave Theory of Bloch Wall

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§ 1. Introduction

In a ferromagnetic crystal, generally there exists domain structure as a natural consequence of the various contributions to the energy. Between adjacent domains magnetized in different direction, a transition layer is set up. It is denoted as Bloch wall. The theoretical researches on Bloch wall have been performed by Bloch[1] and Landau-Lifshitz[2]. The theories are confirmed with a fair success by the experiments of magnetic powder pattern methods. Their theories, however, are constructed from the point of view of Ising model, in which a spin operator is regarded as a vector, so it seems worth while to reconsider this problem on the basis of quantum mechanical Heisenberg model.

In order to discuss the Bloch wall based on Heisenberg model it is most convenient to use the spin wave theory. Recently, the spin wave theories of the antiferromagnetics are evolved by Anderson[3] based on Hellar and Kramers'[4] approximation and by Kubo[5], Nakamura[6] and Ziman[7] based on Holstein and Primakoff's[8] method. In these theories, it is assumed that the state of the antiferromagnet is not greatly different from the classical ground state in which the spins of one sublattice point in one direction, the spins of the other sublattice in the opposite direction, and in this state, the zero point energy characteristic to the quantum mechanics is calculated as a correction term. This process of approach suggests the possibility of the spin wave theory of Bloch wall. In addition, though the spin wave theories of ferro- and antiferromagnetics contain some difficulties that many thermodynamical quantities are divergent in the case of vanishing anisotropy, fortunately, in the Bloch wall, an anisotropy energy plays an important role, so these divergent difficulties are removed of itself.

§ 2. Hamiltonian of the Bloch wall

We consider about 180° wall which separates the adjacent domains oppositely magnetized. The treatments to the other kinds of walls are also easily done. Various types of 180° wall may be considered, but from the consideration of the magnetostatic energy, the magnetization component normal to the Bloch wall must be continuous lest free magnetic pole should appear throughout the Bloch wall, so that only one type of 180° wall such as shown in Fig. 1 can exist. From the point of view of classical theory, spin directions change in a gradual way in the Bloch wall. The reason for the gradual nature of the

change is the fact that for a given total change of spin direction, the exchange energy is lower when the change is distributed over many spins than when the change occur abruptly.

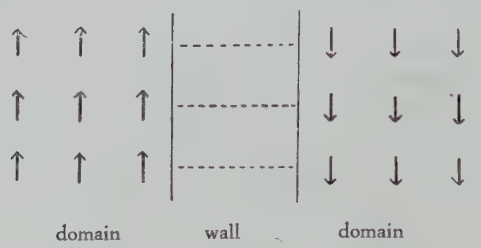

Fig. 1.

We take up a linear array of spins perpendicular to the Bloch wall and show schematically in fig. 2. As mentioned above, the spin rotations are constrained in x_0 z_0 plane owing to the continuity of the normal component of the magnetization. From the quantum mechanical

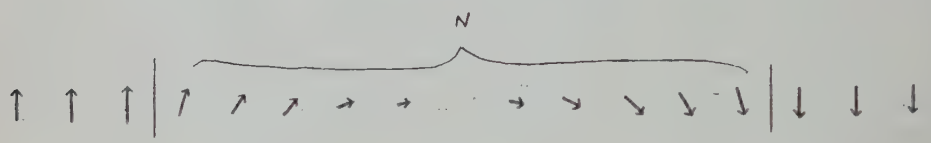

Fig. 2.

point of view, the rotation of spins may be regarded as the rotation of the axis of quantization of spins. The behavior of the rotation should be solved from the variational method of the total energy, but here we assume that the axes of quantization of neighboring two spins make the same small angle $\pi/N=\theta$ with each other, where N is the thickness of the Bloch wall measured in atomic separations. Suppose that, for the sake of simplicity, we still take only one-dimensional Bloch wall such as shown in fig. 2, although the real Bloch wall is three-dimensional one. Extension of our treatment to the three-dimensional Bloch wall is possible except that the last integral cannot be evaluated by the elementary functions.

Let the spin operator of m-th spin be denoted by $\boldsymbol{S}_m$, and the exchange integral by J. The Hamiltonian by the exchange energy of the Bloch wall and of the domains of both sides is

$$H_{exch}=-J\sum_m(\boldsymbol{S}_m\cdot\boldsymbol{S}_{m+1}). \tag{1}$$

In the quantum statistical mechanics, the Hamiltonian (1) cannot be separated into two parts, the Bloch wall and the domains of both sides, because this is the many body problem. The Bloch wall, however, is the stable one having the constant thickness for any observation, so that the state that the Bloch wall has a constant thickness should exist with exceedingly large probability. For this reason, we are allowed to take up only this state and to separate the Hamiltonian of the Bloch wall from the total Hamiltonian. Thus we regard anew (1) as the exchange Hamiltonian of the Bloch wall and the summation is limited to the spins in the Bloch wall.

We do not use the coordinate system (x_0, y_0, z_0) fixed to the space, but adopt the

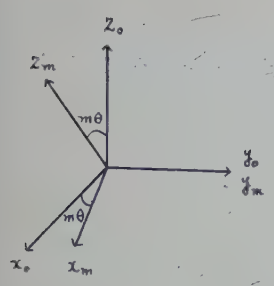

Fig. 3.

rotational coordinate systems (x_m, y_m, z_m) $(m=1, 2, \cdots\cdots N)$, which rotate about y_0-axis through an angle θ during the translation by one lattice constant (see, fig. 3). The z_m-axis of these coordinate system is regarded as the direction of the quantization of m-th spin. The spin operators of m-th spin are transformed in the following:

$$\boldsymbol{S}_m^x = \begin{pmatrix} -\sin m\theta, & \cos m\theta \\ \cos m\theta, & \sin m\theta \end{pmatrix} = \cos m\theta\ \boldsymbol{S}_m^{x_0} - \sin m\theta\ \boldsymbol{S}_m^{z_0},$$

$$\boldsymbol{S}_m^y = \begin{pmatrix} 0, & -i \\ i, & 0 \end{pmatrix} = \boldsymbol{S}_m^{y_0}, \tag{2}$$

$$\boldsymbol{S}_m^z = \begin{pmatrix} \cos m\theta, & \sin m\theta \\ \sin m\theta, & -\cos m\theta \end{pmatrix} = \sin m\theta\ \boldsymbol{S}_m^{x_0} + \cos m\theta\ \boldsymbol{S}_m^{z_0}.$$

Next, we consider the anisotropy energy, which acts to limit the thickness of the Bloch wall. For the sake of simplicity, we take up the uniaxial crystal like a cobalt and regard the z_0-axis as direction of easy magnetization. Let the anisotropy constant per one spin be denoted by K, then the anisotropy energy of m-th spin may be written approximately by $K\boldsymbol{S}_m^z \sin m\theta$. (The phenomenological anisotropy energy of an uniaxial crystal is well represented by $K \sin^2\theta + K' \sin^4\theta$. Our anisotropy energy, though may be rough approximation, has the same tendency as above.) The total Hamiltonian of the Bloch wall is the sum of the exchange and anisotropy energy Hamiltonian:

$$\begin{aligned} H = H_{exch} + H_{aniso} &= -J \sum_m (\boldsymbol{S}_m \cdot \boldsymbol{S}_{m+1}) + K \sum_m \boldsymbol{S}_m^z \sin m\theta \\ &= -J \sum_m \{\cos\theta \boldsymbol{S}_m^x \boldsymbol{S}_{m+1}^x + \sin\theta \boldsymbol{S}_m^x \boldsymbol{S}_{m+1}^z - \sin\theta \boldsymbol{S}_m^z \boldsymbol{S}_{m+1}^x + \boldsymbol{S}_m^y \boldsymbol{S}_{m+1}^y \\ &\quad + \cos\theta \boldsymbol{S}_m^z \boldsymbol{S}_{m+1}^z\} + K \sum_m \boldsymbol{S}_m^z \sin m\theta. \end{aligned} \tag{3}$$

§ 3. Diagonalization of the Bloch wall's Hamiltonian

We shall solve in this section the eigenvalues of the Bloch wall's Hamiltonian (3). We assume that the Bloch wall is approximately in the classical ground state, that means the spins are almost pointing in the direction of each quantization. Now, according to Anderson's approximation, we put

$$\boldsymbol{S}_m^z \simeq S_c - \frac{\boldsymbol{S}_m^{x2} + \boldsymbol{S}_m^{y2}}{2S_c}, \tag{4}$$

$$S_c^2 = \boldsymbol{S}^{x2} + \boldsymbol{S}^{y2} + \boldsymbol{S}^{z2} = S(S+1), \tag{5}$$

and S is the spin quantum number. Substituting (4) to (3), and neglecting the terms more than fourth power of $\boldsymbol{S}^x$ and $\boldsymbol{S}^y$, (this is equivalent to neglecting the large spin

deviation in Holstein and Primakoff's method), we obtain,

$$H=-JN\cos\theta S_c^2+J\cos\theta\sum_m(S_m^{x2}+S_m^{y2})-J\sum_m(\cos\theta\, S_m^x S_{m+1}^x+S_m^y S_{m+1}^y)$$

$$+KS_c\cot\frac{\theta}{2}-\frac{K}{2S_c}\sum(S_m^{x2}+S_m^{y2})\sin m\theta. \tag{6}$$

Now introduce a set of Fourier transform variable defined by

$$S_m^x=\sqrt{\frac{S}{N}}\sum_\lambda e^{i\lambda m}Q_\lambda,\quad S_m^y=\sqrt{\frac{S}{N}}\sum_\lambda e^{i\lambda m}P_\lambda, \tag{7}$$

where the wave number λ's are

$$\lambda=2\pi n/N,\quad n=-\frac{N}{2}+1,\cdots\cdots,0,\cdots\cdots,\frac{N}{2}, \tag{8}$$

then (6) is transformed into,

$$H=-JN\cos\theta\, S_c^2+KS_c\cot\frac{\theta}{2}$$

$$+JS\sum_\lambda\{\cos\theta\,(1-\cos\lambda)Q_\lambda^2+(\cos\theta-\cos\lambda)\,P_\lambda^2\}. \tag{9}$$

The new set of operators Q_λ and P_λ proves to satisfy the commutation relations:

$$\begin{aligned}&[Q_\lambda,\ P_{\lambda'}]=i\delta_{\lambda\lambda'},\\ &[Q_\lambda,\ Q_{\lambda'}]=0,\\ &[P_\lambda,\ P_{\lambda'}]=0.\end{aligned} \tag{10}$$

The first term of (9) is the exchange energy, the second term is the anisotropy energy and the third term is the zero-point energy. Using (10), the Hamiltonian (9) can be diagonalized and the eigenvalues are

$$H=-NJS_c^2\cos\theta+KS_c\cos\frac{\theta}{2}+2JS\sum_\lambda(n_\lambda+\frac{1}{2})\{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}},$$

$$n_\lambda=0,\ 1,\ 2,\cdots\cdots. \tag{11}$$

In the ground state, the energy is

$$H_{gr}=-NJS_c^2\cos\theta+KS_c\cot\frac{\theta}{2}+JS\sum_\lambda\{\cos\theta\,(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}}, \tag{12}$$

and the frequency of the spin wave is

$$\omega=\frac{2JS}{\hbar}\{\cos\theta\,(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}}. \tag{13}$$

The zero-point energy is obtained replacing the sum $\sum_\lambda$ by integration over λ and putting θ for the lower limit of the integration, namely,

$$JS\sum_{\lambda}\{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}}=\frac{JSN}{\pi}\int_{\theta}^{\pi}\{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}}d\lambda$$

$$=JSN(\cos\theta)^{\frac{1}{2}}(\cos\frac{\theta}{2})^{2}. \tag{14}$$

It is worth while to note that the frequency which satisfies $|\lambda|<\theta$ becomes imaginary. In other words, the spin wave whose wave length is longer than the thickness of the Bloch wall, in our case, the spin wave of $n=0$, cannot exist. It is the effect of the anisotropy energy which makes the Bloch wall finite thickness against the exchange energy, so a restraining force or a kind of the boundary condition, which makes the domains of both sides oppositely magnetize, eliminates these spin waves of $|\lambda|<\theta$, and remove the divergent property resulting from the spin wave of $n=0$.

We examine the mean values of the spin components whether they converge or not. The mean value of the spin component in the direction of the quantization in the ground state is

$$<\sum_{m}\boldsymbol{S}_{m}^{z}>=S_{c}-\frac{S}{2NS_{c}}<\sum_{\lambda}(Q_{\lambda}^{2}+P_{\lambda}^{2})>, \tag{15}$$

where

$$<Q_{\lambda}^{2}>=\frac{1}{2}\left\{\frac{\cos\theta-\cos\lambda}{\cos\theta(1-\cos\lambda)}\right\}^{\frac{1}{2}},$$

$$<P_{\lambda}^{2}>=\frac{1}{2}\left\{\frac{\cos\theta(1-\cos\lambda)}{\cos\theta-\cos\lambda}\right\}^{\frac{1}{2}}, \tag{16}$$

and, again, replacing the sum by the integration, we obtain

$$<\sum_{m}\boldsymbol{S}_{m}^{z}>=S_{c}-\frac{S}{4S_{c}}\frac{1-\sin\theta/2+\cos\theta}{(\cos\theta)^{\frac{1}{2}}}$$

$$\fallingdotseq N(S+\frac{1}{4S}). \tag{17}$$

This result is in agreement with that of the ferromagnet. In the same way, we can see that $<\sum\boldsymbol{S}_{m}^{x}>$ and $<\sum\boldsymbol{S}_{m}^{y}>$ converge. So the ground state obtained above is the stable state.

§ 4. The thickness of the Bloch wall

We can obtain the free energy of the Bloch wall from (9), considering that (9) and (10) are the same forms as those of the Hamiltonian and the commutation relations of the harmonic oscillator, namely;

$$F=-NJS_{c}^{2}\cos\theta+KS_{c}\cot\frac{\theta}{2}$$

$$+\sum_{\lambda}kT\log\left[2\sinh\{JS\sqrt{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)}/kT\}\right]$$

$$= -NJS_c^2 \cos\theta + KS_c \cot\frac{\theta}{2} + \frac{NJS}{\pi}\int_\theta^\pi \{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}} d\lambda$$

$$+ \frac{NkT}{\pi}\int_\theta^\pi \log[1-e^{-\frac{2JS}{kT}\{\cos\theta(1-\cos\lambda)(\cos\theta-\cos\lambda)\}^{\frac{1}{2}}}] d\lambda. \tag{18}$$

Since in the last term of (18), only the part of λ nearly equal to θ gives the dominant contribution to the integral, we may roughly evaluate the integral. This approximation is guaranteed because only long waves of spin wave are excited at lower temperatures. Thus we obtain

$$F = -NJS_c^2 \cos\theta + KS_c \cot\frac{\theta}{2} + JSN(\cos\theta)^{\frac{1}{2}}(\cos\frac{\theta}{2})^2$$

$$+ N\left\{\frac{(kT)^3}{4\pi JS(\cos\theta)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\zeta\left(\frac{3}{2}\right). \tag{19}$$

On the other hand, if these layers of the same thickness are not the transition layers but the domain, the free energy of them is calculated in the same way as above,

$$F_d = -NJS^2 + N\left\{\frac{(kT)^3}{4\pi JS}\right\}^{\frac{1}{2}}\zeta\left(\frac{3}{2}\right). \tag{20}$$

We can regard $F-F_d$ as the increase of the free energy owing to the construction of the wall. The thickness of the Bloch wall is determined so as to be the minimum increase of $F-F_d$, that is, by the root of $\partial(F-F_d)/\partial N=0$, as follows,

$$N = \left\{\frac{1}{2}\pi^2 JS^2 - \frac{1}{4}\pi KS_c + \frac{\pi^2}{8}\left(\frac{(kT)^3}{4\pi JS}\right)^{\frac{1}{2}}\zeta\left(\frac{3}{2}\right)\right\}^{\frac{1}{2}} \Big/ \frac{2}{\pi}KS_c. \tag{21}$$

For the order of magnitude, in cobalt,

$$\frac{1}{2}\pi^2 JS^2 = 7\times 10^{-13},$$

$$\frac{1}{4}\pi KS_c = 1\times 10^{-16}, \tag{22}$$

$$\frac{\pi^2}{8}\left(\frac{(kT)^3}{4\pi JS}\right)^{\frac{1}{2}}\zeta\left(\frac{3}{2}\right) = 1\times 10^{-18}\, T^{\frac{3}{2}}.$$

We can neglect the second and third terms in the numerator of (21) and reduce to

$$N = \left\{\frac{JS\pi^3}{4KS_c}\right\}^{\frac{1}{2}}. \tag{23}$$

We have the result that the thickness of the Bloch wall is in proportion to the square root of the ratio of the exchange integral to the anisotropy energy. This result is in

agreement with the classical treatment of the Bloch wall. The cases in which the result of the spin wave treatment is in agreement with that of the classical treatment, are already known in the perpendicular magnetic susceptibility and the resonance absorption frequency in antiferromagnetics, and our problem of the Bloch wall is the third example of them.

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The General Theory of the Interaction Representation, I

—The Local Field—

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The general relation between the covariant commutation relation and the Green function of the field equation is given. This relation can be easily applied to find the commutation relation of the free field. With help of this relation we develop the general quantization theory for the system involving the general fields. Moreover, it is proved that the normal dependent term has no effects on the S-matrix and on the expectation values of the physical quantities. We shall point out the several erroneous conclusions in the other developments of the quantization theory.

§ 1. Introduction

In the present quantum field theory, the quantization of the force free field, which may be interpreted as a system of the harmonic oscillators with infinitely many degrees of freedom, can be uniquely carried out by starting from the given Lagrangian. However, the general rule for the derivation of the covariant commutation relations has been still unknown* and so they were derived in a suitable way so as to make them to be compatible with the field equations for each particular case[1].

Moreover, it is necessary to find the general quantization theory for the system involving the general interaction, because in such case we can not always adopt the procedure of the canonical field quantization. In particular such general theory will be needed for the case of the non-local interaction, which seems to be required in connection with the recent analysis of the divergence difficulties in the renormalization theory[2]. The Feynman's theory[3] of the position, in which the Green functions of the field equations play the central role and which was extended to the cases of the scalar and the vector fields by Yang and Feldman[4], seems to be adequate to search for such general quantization theory. Nevertheless it has not been generally proved that this formulation can be extended to the general theory which is equivalent to the Tomonaga-Schwinger's quantum theory in the usual cases, because we have not known on the intimate connection between the commutation relations and the Green functions.

In Feynman's treatment the results are essentially obtained with no help of the terms

* One important rule for the derivation of the commutation relation was shown by G. Wentzel in his famous text-book (Einführung in die Quantentheorie der Wellenfelder 1943). However his rule is rather cumbersome, because we must use explicitly the independent canonical conjugate variables and treat the space- and time-coordinates in an unsymmetrical manner.

depending on the normal n_μ of the spacelike surface. As shown by Koba[5], the fact that the normal-dependent terms have no effects on the S-matrix can be represented as the *integrability condition* of the Schrödinger equation. Koba's treatment, which is based on the assumption[5)6)] that the interaction Hamiltonian can be determined by the integrability condition and on the use of the Δ_F^*-function, leads often to some erroneous results (cf. § 5), because the normal independent term in $H-(-L')$ is not obtained by the integrability condition, where H and L' mean the interaction Hamiltonian and the interaction Lagrangian, respectively, and the Green function cannot be given by the Δ_F^*-function. On the other hand we can show that to introduce the unitary transformation, which combines the field equations in the Heisenberg representation with those in the interaction representation, makes it possible to determine uniquely the interaction Hamiltonian which is also compatible with the integrability condition (cf. § 4).

The main object of this and following papers is to develop the general quantization method which corresponds to the generalized Feynman's formulation for the cases of general types of fields and interactions. We shall establish in § 2 the general relation between the covariant commutation rule for the free field and the Green function of the field equation. The use of this relation leads to the very simple rule for determining the commutation relations of the free field operators, which is concretely exemplified in § 3. Using the relation between the covariant commutation rule and the Green function we shall discuss the general quantization method in § 4, by which the interaction Hamiltonian is uniquely determined by means of the above-mentioned unitary transformation. Moreover, it is proved that the normal-dependent term has no effects not only on the S-matrix but also on the expectation values of the physical quantities. The energy-momentum four-vector which has not been defined in Feynman's formulation can be easily constructed as the time- and space-translation operators for the field quantities in the Heisenberg representation. Thus we can generally recognize the equivalence between the Feynman theory and the Tomonaga-Schwinger theory. In this paper we treat the simple case in which the degree of the derivation operators in the interaction Lagrangian is at most 1. The possibility for the extension of our theory to the cases of the interactions containing higher derivatives, especially, non-local interactions and non-local actions, will be discussed in detail in the subsequent paper (Part II.)

§ 2. On the relation between the covariant commutation relation of the free field and the Green function

The free field quantities $Q_\alpha(x)$ satisfy the field equations

$$\Lambda_{\alpha\beta}(\partial)Q_\beta(x)=0, \tag{2·1}$$

which can be reduced to the equations of the harmonic oscillators

$$(\Box-\varkappa^2)Q_\alpha(x)=0 \tag{2·2}$$

and the several subsidiary conditions. In (2·1) $\Lambda_{\alpha\beta}(\partial)$ is the derivation operator.

Now we shall prove the relation

$$\Lambda_{\alpha\beta}(\partial)R_{\beta\tau}(\partial)=(\Box-\varkappa^2)\delta_{\alpha\tau}, \tag{I_2}$$

where $R_{\alpha\beta}(\partial)$ is the derivation operator appearing in the commutation relation, i.e.,

$$[Q_\alpha(x), Q_\beta(x')]_\pm = iR_{\alpha\beta}(\partial)\Delta(x-x'). \tag{2·3}^{*)**)}$$

[I$_2$] *shows the intimate connection between the commutation relation and the Green function of the field equation.*

Substituting the solution of (2·2) into the Lagrangian, from which the equation (2·1) is given, we have

$$\int L dt = \frac{1}{2}\int \Lambda_{\alpha\beta}(k) Q_\beta(-k) Q_\alpha(k) \{\delta(k^2+\varkappa^2)\}^2 \frac{d^4k}{(2\pi)^2}, \tag{2·4}$$

where $Q_\alpha(k)$ is the Fourier amplitude of $Q_\alpha(x)$, i.e.,

$$Q_\alpha(x) = \int Q_\alpha(k)\delta(k^2+\varkappa^2) e^{ikx} \frac{d^4k}{(2\pi)^3}. \tag{2·5}$$

The delta function $\delta(k^2+\varkappa^2)$ ensures that the individual term of the expansion obeys the equation (2·2).

Introducing the positive and negative frequency parts $Q_\alpha^\pm$ of Q_α as follows:

$$\begin{aligned} Q_\alpha(x) &\equiv Q_\alpha^+(x) + Q_\alpha^-(x) \\ &= \int_{k_0>0} \delta(k^2+\varkappa^2)\{Q_\alpha^+(k) e^{ikx} + Q_\alpha^-(k) e^{-ikx}\} \frac{d^4k}{(2\pi)^3}, \end{aligned} \tag{2·6}$$

$$\left.\begin{aligned} Q_\alpha^+(k) &= Q_\alpha(k), \\ Q_\alpha^-(k) &= Q_\alpha(-k), \end{aligned}\right\} (k_0>0), \tag{2·7}$$

one obtains from (2·2)

$$[Q_\alpha^+(x), Q_\beta^-(x')]_\pm = iR_{\alpha\beta}(\partial)\Delta^+(x-x'), \tag{2·8}$$

where $\Delta^+(x)$ is the positive frequency part of $\Delta(x)$.[7]

(2·8) is equivalent to the following commutation relation in momentum space:

$$[Q_\alpha^+(k), Q_\beta^-(k')]_\pm = 2(2\pi)^3 K R_{\alpha\beta}(k)\delta(\boldsymbol{k}'-\boldsymbol{k}), \tag{2·9}$$

where $K = \sqrt{\boldsymbol{k}^2+\varkappa^2}$ and the bold letter means the three dimensional vector.

Since we can easily show (by regarding the imaginary i due to the pseudo-property of the Lorentz space as the real factor) that $R = (R_{\alpha\beta})$ is the Hermite matrix, there exists the unitary matrix which transforms R into the diagonal matrix $\bar{R}$, i.e.,

$$\bar{R} = U^{-1}RU = (\bar{R}(r;k)\delta_{rs}). \tag{2·10}^{***)}$$

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* Our notations are the usual ones: $[a,b]_+ = ab+ba$ and $[a,b] = [a,b]_- = ab-ba$. As is well known, $[\ ]_+$ and $[\ ]_-$ in (2·3) correspond to the cases of the integer- and the half-integer-spin, respectively.

** The derivation operator in the commutation relation is able to have the arbitrary additive term which is proportional to $(\Box-\varkappa^2)$ (e.g., the second term in (2·17)). Since this term can be chosen as one wishes, it is determined in [I$_2$] so that the intimate connection between the commutation relation and the Green function may be clear. Therefore [I$_2$] is the sufficient (but not necessary) condition for $R(\partial)_{\alpha\beta}$ in order that the quantized field may be equivalent to the system of the harmonic oscillators.

*** Thus we have verified that the transformation U picks up the independent components of the field Q_α. As the example of (2·12) and (2·13) we shall discuss the case of the vector field in Appendix I.

According to (2·3) we have

$$\bar{R}(r;k)\delta(k^2+\varkappa^2)\begin{cases}\neq 0 & \text{for } r=1,2,\cdots,q,\\ =0 & \text{for } r\geq q+1.\end{cases} \tag{2·11}$$

The rank q of the matrix $R\delta(k^2+\varkappa^2)$ is $(2s+1)$ or $2(2s+1)$ for the real or complex fields with the integer spin s respectively and $2(2s+1)$ for the field with the half integer spin s because of the existence of the two states of the particle and the antiparticle (i.e., the both signs of the energy).

If we introduce $q_r(k)$ as follows:

$$q_r(k)=\{\bar{R}(r;k)\}^{-1/2}Q_\alpha(k)U_{\alpha r}, \qquad r=1,2,\cdots,q, \tag{2·12}$$

$$q_r(k)\delta(k^2+\varkappa^2)=0, \qquad r\geq q+1, \tag{2·13}$$

the commutator (2·9) can be written in the diagonal form:

$$[q_r^+(k),q_s^-(k)]_\pm=2(2\pi)^3K\delta_{rs}\delta(\boldsymbol{k}-\boldsymbol{k}'), \quad \text{for } r,s=1,2,\cdots,q. \tag{2·14}$$

The requirement that the quantized free field is equivalent to the system of the harmonic oscillators is satisfied, when (2·4) is transformed by (2·12) into the following form:

$$\int L\,dt=-\frac{1}{2}\int(k^2+\varkappa^2)q_r(-k)q_s(k)\delta_{rs}\{\delta(k^2+\varkappa^2)\}^2\frac{d^4k}{(2\pi)^2}. \tag{2·15}$$*)

By substituting (2·12) into (2·15) and comparing (2·15) with (2·4), we have

$$\Lambda_{\alpha\beta}(\partial)\bar{R}(t;\partial)U_{\alpha t}^{-1}U_{\beta s}=(\Box-\varkappa^2)\delta_{ts}. \tag{2·16}$$

Thus, the proof for the relation $[I_2]$ was completed.

Therefore, representing $R_{\alpha\beta}(\partial)$ with the sum of the all possible derivation operators with the degree $\leq 2s\equiv b^s$ **)[2·8], the same transformation property as $Q_\alpha\times Q_\beta$ and the undeterminate constants and determining the latter constants according to the relation $[I_2]$, we can easily obtain the derivation operator $R_{\alpha\beta}(\partial)$ and the covariant commutation relation

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* The condition that (2·4) can be rewritten into (2·15) is equivalent to the requirement that the energy-momentum vector T_μ has the following form:

$$T_\mu=-i\sum_{r,k}K_\mu n_r(K)+\text{c-number}, \qquad n_r(K)=\text{number operator}.$$

** We shall prove this relation for the case of the Bose-field, but the circumstances are similar to the Fermi-field.

If we denote $R_{\alpha\beta}(\partial)$ of the field of spin s and mass $\varkappa\neq 0$ by $R_{\mu_1\cdots\mu_{2s}}$, $R_{\mu_1\cdots\mu_{2s}}$ will be obtained from $R_{\mu_1\cdots\mu_{2s-2}}$ by use of $\delta_{\mu\nu}$ and $\partial_\mu\partial_\nu$. However the field with $R_{\mu_1\cdots\mu_{2s-2}}\delta_{\mu_{2s-1}\mu_{2s}}$ as $R_{\alpha\beta}$ has the independent component of $f_s=4f_{s-1}$, where f_s is the number of the independent component of spin s-field, but this fact is incompatible with $f_s=2s+1$ and $f_{s-1}=2(s-1)+1$. Therefore $R_{\mu_1\cdots\mu_{2s-2}}$ must be obtained by producing the linear combination of $\delta_{\mu\nu}$ and $\partial_\mu\partial_\nu$. Since $b^s=0$ for $s=0$, one obtains the identity $b^s=2s$ by the mathematical induction. For the field with mass $\varkappa=0$, we have

$$b^s=\begin{cases}0 & \text{for integer } s, \qquad \varkappa=0\\ 1 & \text{for half-integer } s, \qquad \varkappa=0\end{cases}$$

from the consideration that ∂_μ has the dimension of [length^{-1}].

(2·3) for the case of the spin s. This method for the determination of $R_{\alpha\beta}(\partial)$ is exemplified in § 3.

It should be pointed out that, as will be shown by the example in § 3, $R_{\alpha\beta}(\partial)$ generally contains the term which is proportional to $(\Box-\varkappa^2)$:

$$R_{\alpha\beta}(\partial)\equiv R_{\alpha\beta}'(\partial)+(\Box-\varkappa^2)R_{\alpha\beta}''(\partial). \tag{2·17}$$

The second term of the right hand side of (2·17) has no contribution to the commutation relation, i.e.,

$$[Q_\alpha(x), Q_\beta(x')]_\pm = iR_{\alpha\beta}'(\partial)\Delta(x-x'). \tag{2·18}$$

Of course, $R_{\alpha\beta}'(\partial)$ does not satisfy the relation $[\mathrm{I}_2]$. On the other hand, in the treatment of the S-matrix we must calculate the following quantity :

$$\langle P(Q_\alpha(x), Q_\beta(x'))\rangle_0 = \frac{1}{2}R_{\alpha\beta}(\partial)\Delta_F(x-x')+f_{\alpha\beta}(n_\mu), \tag{2·19}*$$

$$=\frac{1}{2}R_{\alpha\beta}'(\partial)\Delta_F(x-x')+f_{\alpha\beta}'(n_\mu), \tag{2·20}$$

where

$$f_{\alpha\beta}(n_\mu)=-\frac{i}{2}[R_{\alpha\beta}(\partial), \varepsilon(x-x')]\Delta(x-x') \tag{2·19'}**$$

and

$$f_{\alpha\beta}'(n_\mu)=-\frac{i}{2}[R_{\alpha\beta}'(\partial), \varepsilon(x-x')]\Delta(x-x') \tag{2·20'}$$

are the terms depending on the normal n_μ of the space-like surface. When we determine the interaction Hamiltonian by the requirement of the integrability condition only, we have two possibilities to eliminate the contributions of $f_{\alpha\beta}(n_\mu)$ or $f_{\alpha\beta}'(n_\mu)$ respectively. However it is clarified in § 4 that the interaction Hamiltonian obtained for the latter elimination is erroneous, because such an interaction Hamiltonian does not lead to the correct field equation in Heisenberg representation which is derived from the total Lagrangian containing the interaction term (cf. § 5). Determining the correct interaction Hamiltonian by means of the unitary transformation which connects the field equations in the Heisenberg representation with those in the interaction representation, we shall verify in § 4 that the first term of the right hand side of (2·19) *which is the Green function of* (2·1) has

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* Symbol P means the Wick's T in this paper, that is :

$$P(Q_\alpha(x), Q_\beta(x'))=\frac{\eta(x,x')}{2}\{[Q_\alpha(x), Q_\beta(x')]_+ + \varepsilon(x-x')[Q_\alpha(x), Q_\beta(x')]_-\},$$

where

$$\eta(x,x')=\begin{cases}1 & \text{for the Bose-field,}\\ \varepsilon(x-x') & \text{for the Fermi-field.}\end{cases}$$

**
$$\varepsilon(x-x')=\begin{cases}1 & x_0>x_0',\\ -1 & x_0<x_0'.\end{cases}$$

the main physical effects*). It is easy to see that $f_{\alpha\beta}(n_\mu)$ can be uniquely determined by $R_{\alpha\beta}(\partial)$ and has the following form:

$$f_{\alpha\beta}(n_\mu)=g_{\alpha\beta}(n_\mu\,;\partial_\mu)\delta(x-x'), \tag{2·21}$$

where $g_{\alpha\beta}(n_\mu\,;\partial_\mu)$ is a functional of the normal n_μ and the derivation operator.

§ 3. Some examples of the quantization

Now we shall determine the commutation relations for the Duffin-Kemmer field with the spin $(1,0)$ and the Rarita-Schwinger field[9] with the spin $3/2$ by use of the relation $[I_2]$.

i) The quantization of the Duffin-Kemmer field.

According to $(2\cdot1)$ we have:

$$\Lambda\equiv(\Lambda_{\alpha\beta})=-(\beta_\mu\partial_\mu+\varkappa). \tag{3·1}$$**)

Taking into account the fact that the highest degree of derivation operators in $R_{\alpha\beta}(\partial)$ is $2s=2$, we can write it in the following form:

$$R\equiv(R_{\alpha\beta})=a+b\beta_\mu{}^2+c\beta_\mu\partial_\mu+d\beta_\mu\beta_\nu\partial_\mu\partial_\nu+e\Box+f\beta_\mu{}^2\Box. \tag{3·2}$$

Substituting $(3\cdot1)$ and $(3\cdot2)$ into $[I_2]$, the factors (a,b,c,d,e,f) are determined as follows:

$$a=-\varkappa,\quad b=0,\quad c=1,\quad d=-\varkappa^{-1},\quad e=\varkappa^{-1},\quad f=0. \tag{3·3}$$

Thus one finds the correct expression of R:

$$\left.\begin{aligned} &R=-\left\{\frac{1}{\varkappa}(\Box-\varkappa^2)+\beta_\mu\partial_\mu-\frac{1}{2\varkappa}(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu)\partial_\mu\partial_\nu\right\},\\ &[\psi(x),\bar\psi(x')]=iR\Delta(x-x'). \end{aligned}\right\} \tag{3·4}$$

The commutation relation in the text-book[10] consists of the second and the third terms in $(3\cdot4)$.

One obtains from $(3\cdot4)$

$$\langle P(\psi(x),\bar\psi(x'))\rangle_0=\frac{1}{2}R(\partial)\Delta_F(x-x')+f(n_\mu). \tag{3·5}$$

According to $(2\cdot19')$ we have:

$$\begin{aligned} f(n_\mu)&=-\frac{i}{2}[R,\varepsilon(x-x')]\Delta(x-x')\\ &=-\frac{i}{\varkappa}(1+(\beta_\mu n_\mu)^2)\delta(x-x'). \end{aligned} \tag{3·6}$$

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* Koba's rule[5] based on the Δ_F*-function $((\Box-\varkappa^2)\Delta_F{}^*=0)$ leads to the erroneous results for the case $R'_{\alpha\beta}{}''(\partial)\neq0$, because the main physical effects come from $R_{\alpha\beta}\Delta_F{}^*=R_{\alpha\beta}{}'\Delta_F{}^*\neq R'_{\alpha\beta}\Delta_F$ in his theory.

** We must notice that the field equation obtained from the Lagrangian $L=-\psi(\beta\partial+\varkappa)\psi$ is not $(\beta\partial+\varkappa)\psi=0$, but $-(\beta\partial+\varkappa)\psi=0$.

The first term of the right hand side of (3·5) is the Green function of the free field equation[3].

ii) The quantization of the Rarita-Schwinger field with the spin 3/2.

The Lagrangian is as follows:

$$L=-\bar{\psi}_\mu(\gamma\partial+\varkappa)\psi_\mu+\frac{1}{3}\bar{\psi}_\mu(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\psi_\nu-\frac{1}{3}\bar{\psi}_\mu\gamma_\mu(\gamma\partial-\varkappa)\gamma_\nu\psi_\nu. \tag{3·7}$$

The variation of ψ_μ yields

$$\Lambda_{\mu\nu}\psi_\nu=-\left\{(\gamma\partial+\varkappa)\delta_{\mu\nu}-\frac{1}{3}(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)+\frac{1}{3}\gamma_\mu(\gamma\partial-\varkappa)\gamma_\nu\right\}\psi_\nu=0. \tag{3·8}$$

Taking into account the fact that the highest degree of derivation operators in $R_{\mu\nu}(\partial)$ is $2s=3$, we can put $R_{\mu\nu}$ as follows:

$$\begin{aligned}R_{\mu\nu}=&a\varkappa\delta_{\mu\nu}+b\varkappa\gamma_\mu\gamma_\nu+c(\gamma\partial)\delta_{\mu\nu}+d(\gamma\partial)\gamma_\mu\gamma_\nu+e\gamma_\mu\partial_\nu+f\gamma_\nu\partial_\mu\\&+g\frac{1}{\varkappa}\Box\delta_{\mu\nu}+h\frac{1}{\varkappa}(\gamma\partial)^2\gamma_\mu\gamma_\nu+i\frac{1}{\varkappa}(\gamma\partial)\gamma_\nu\partial_\mu+j\frac{1}{\varkappa}\Box\gamma_\mu\gamma_\nu\\&+k\frac{1}{\varkappa}\partial_\mu\partial_\nu+l\frac{1}{\varkappa^2}\Box(\gamma\partial)\delta_{\mu\nu}+m\frac{1}{\varkappa^2}\Box(\gamma\partial)\gamma_\mu\gamma_\nu\\&+n\frac{1}{\varkappa^2}(\gamma\partial)\partial_\mu\partial_\nu+o\frac{1}{\varkappa^2}\Box\gamma_\mu\partial_\nu+p\frac{1}{\varkappa^2}\Box\gamma_\nu\partial_\mu\end{aligned}$$

and

$$[\psi_\mu(x),\bar{\psi}_\nu(x')]_+=iR_{\mu\nu}(\partial)\Delta(x-x'). \tag{3·9}$$

Determining the factors $(a, b, \cdots, o, p)$ according to $[I_2]$, we have

$$\begin{aligned}R_{\mu\nu}=&-(\gamma\partial-\varkappa)\left\{\delta_{\mu\nu}-\frac{1}{3}\gamma_\mu\gamma_\nu+\frac{1}{3\varkappa}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)-\frac{2}{3\varkappa^2}\partial_\mu\partial_\nu\right\}\\&+\frac{2}{3\varkappa^2}(\Box-\varkappa^2)\{\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)+(\gamma\partial-\varkappa)\gamma_\mu\gamma_\nu\},\end{aligned} \tag{3·10}$$

$$R_{\mu\nu}'=-(\gamma\partial-\varkappa)\left\{\delta_{\mu\nu}-\frac{1}{3}\gamma_\mu\gamma_\nu+\frac{1}{3\varkappa}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)-\frac{2}{3\varkappa^2}\partial_\mu\partial_\nu\right\}.$$

§ 4. The quantum theory of fields with the general interaction

In conformity with the identity $[I_2]$, we shall derive the method by which the system with the general interaction is treated in the relativistically covariant way. This is the formalism by which we seek the unitary transformation combining the free equation of motion with that obtained from the Lagrangian in Heisenberg representation. Let L' be the interaction Lagrangian,* so the equation of motion of the field quantity $\underline{Q}_\alpha(x)$ in Heisenberg representation is**

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* In this paper L' contains the derivation operators, the degree of which is 1 at most. More general case will be discussed in the following paper, in which the same notations as those here will be used.

** In this paper, we denote the field quantity and its functional in Heisenberg representation by the under-line and α labels the different types of the field as well as the components of each field.

$$\Lambda_{\alpha\beta}(\partial)\underline{Q}_\beta(x)=\underline{J}_\alpha(x), \tag{4·1}$$

where

$$\underline{J}_\alpha(x)=\frac{\partial \underline{L}'}{\partial \underline{Q}_\alpha}-\partial_\mu\frac{\partial \underline{L}'}{\partial \underline{Q}_{\alpha\mu}}+\cdots \tag{4·2}$$

$$\equiv\sum_{n=0}^{\infty} D_{\mu_1\cdots\mu_n}\underline{j}_{\alpha\mu_1\cdots\mu_n}(x) \tag{4·2'}$$*)

with

$$D_{\mu_1\cdots\mu_n}\equiv(-1)^n\partial_{\mu_1}\cdots\partial_{\mu_n}\equiv D_a,$$

$$\underline{j}_{\alpha,\mu_1\cdots\mu_n}(x)\equiv\frac{\partial \underline{L}'}{\partial \underline{Q}_{\alpha,\mu_1\cdots\mu_n}}\equiv\underline{j}_{\alpha a},$$

$$\underline{Q}_{\alpha\mu_1\cdots\mu_n}(x)\equiv\partial_{\mu_1}\cdots\partial_{\mu_n}\underline{Q}_\alpha(x).$$

Hereafter we shall symbolically denote (4·2') as

$$\underline{J}_\alpha(x)\equiv D_a\underline{j}_{\alpha a}(x), \tag{4·2''}$$

furthermore,

$$D_a f(x-x')\underline{j}_{\alpha a}(x')\equiv\sum_{n=0}^{\infty} D_{\mu_1\cdots\mu_n} f(x-x')\underline{j}_{\alpha\mu_1\cdots\mu_n}(x'). \tag{4·2'''}$$

Using $R_{\alpha\beta}(\partial)$ in [I$_2$], we can rewrite (4·1) into the integral form

$$\underline{Q}_\alpha(x)=Q_\alpha(x)+\int R_{\alpha\beta}(\partial)D_a\Delta_G(x-x')\underline{j}_{\beta a}(x')dx', \tag{4·3}$$

where $\Delta_G(x)$ which is the linear combination of $\Delta^{(1)}$, Δ and $\bar{\Delta}$ satisfies the equation

$$\Lambda_{\alpha\beta}(\partial)R_{\beta\gamma}(\partial)\Delta_G(x-x')=(\Box-\varkappa^2)\Delta_G(x-x')\delta_{\alpha\gamma}=\delta(x-x')\delta_{\alpha\gamma} \tag{4·4}$$

and $Q_\alpha(x)$ is the solution of the free field equation

$$\Lambda_{\alpha\beta}(\partial)Q_\beta(x)=0. \tag{4·5}$$

In order to find the unitary transformation combining (4·1) and (4·5), we introduce the subsidiary field operator, which, as is easily seen, satisfies the free field equation,

$$Q_\alpha(x,\sigma)=Q_\alpha(x)+\int^{\sigma}R_{\alpha\beta}(\partial)D_a\Delta(x-x')\underline{j}_{\beta a}(x')dx', \tag{4·6}$$

where x does not always lie on σ. We can adopt an arbitrary surface as the lower limit in the second term of the right hand side of (4·6). Since we can equally well treat even if we adopted any surface as this limit**), let us appoint $-\infty$ to the lower limit. So, from (4·6),

$$Q_\alpha(x,-\infty)\equiv Q_\alpha(x).$$

This adoption leads to

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* $a\equiv(\mu_1\cdots\mu_n)$.

** This arbitrariness corresponds to the existence of the various interaction representations.

$$\underline{Q}_\alpha(x)=Q_\alpha(x)+\int R_{\alpha\beta}(\partial)D_a\Delta^{ret}(x-x')\,\underline{j}_{\beta a}(x')dx', \tag{4·7}$$

with

$$\Delta^{ret}(x-x')=\frac{1+\varepsilon(x-x')}{2}\Delta(x-x').$$

From (4·6) and (4·7), we obtain

$$\underline{Q}_\alpha(x)=Q_\alpha(x/\sigma)+\frac{1}{2}\int R_{\alpha\beta}(\partial)D_a,\ \varepsilon(x-x')]\Delta(x-x')\,\underline{j}_{\beta a}(x')dx', \qquad [\mathrm{II}_4]^{*)}$$

therefore, as is easily tried,

$$\equiv Q_\alpha(x/\sigma)+i\underline{g}_\alpha(x\,;n), \qquad [\mathrm{II}_4']^{**)\,***)}$$

so $Q_\alpha(x)$ is expressed successively in terms of $Q_\alpha(x/\sigma)$ by use of $[\mathrm{II}_4']$ and $[\mathrm{II}_4'']$.

Now, let us require the following commutation relation for $Q_\alpha(x\,;\sigma)$:

$$[Q_\alpha(x,\sigma),\,Q_\beta(x',\sigma)]_{\pm}=iR_{\alpha\beta}(\partial)\Delta(x-x'), \qquad [\mathrm{III}_4]$$

because $Q_\alpha(x;\sigma)$ satisfies the free field equation. The requirement $[\mathrm{III}_4]$ is satisfied only when the unitary transformation $U(\sigma)$ in (4·8) exists. The condition for the existence of $U(\sigma)$ is represented by $[\mathrm{IV}_4]$ which is the requirement for the interaction Hamiltonian. (When there is no $U(\sigma)$, we can repeat the following discussions by using the slightly modified free field $Q_\alpha'(x,\sigma)$, for which there is the unitary transformation $U'(\sigma)$ combining $Q_\alpha'(x,\sigma)$ with $Q_\alpha(x)$.)$^{****)}$ $[\mathrm{III}_4]$ is required for every surface σ whence it follows that there exists a unitary transformation $U(\sigma\,;\sigma')$ such that

$$\begin{aligned}Q_\alpha(x,\sigma)&=U^{-1}(\sigma\,;\sigma')Q_\alpha(x,\sigma')U(\sigma\,;\sigma')\\&=U^{-1}(\sigma)Q_\alpha(x)U(\sigma),\end{aligned} \tag{4·8}$$

with $U(\sigma)\equiv U(\sigma\,;-\infty)$. When $F(x)$ is a functional of $Q_\alpha(x)$, the notation $F(x,\sigma)$ means in this section the same form functional but with $Q_\alpha(x,\sigma)$ replacing $Q_\alpha(x)$, so

$$U(\sigma)F(x,\sigma)=F(x)U(\sigma) \tag{4·9}$$

* x/σ means x laid on σ.

** n_μ means the normal of σ. For the independent canonical variable, the second term in $[\mathrm{II}_4']$ does not appear.

*** Generally speaking, it must be noticed that the field $Q_\alpha(x/\infty)$, which is combined with $Q_\alpha(x)$ through the transformation, is equal to the field $Q_{out}\equiv\lim_{x_0\to\infty}\underline{Q}(x)$, only when we can assume that the second term in $[\mathrm{II}_4']$ is switched off in the limit $\sigma\to\infty$ through the suitable adiabatic process. The detailed and concrete criticism must be done for the latter assumption.

**** For the usual local interaction (with the derivation operators of the low degree) there is always $H(x';n)$ satisfying $[\mathrm{IV}_4]$ and so $U(\sigma)$ exists. It must be noticed that it is not always possible to require the commutation relation (2·3) for the arbitrary free field $Q_\alpha''(x,\sigma)$, because there is not always the unitary transformation combining $Q_\alpha''(x,\sigma)$ with $Q_\alpha(x)$. We have the example of such circumstance in the case of $Q_\alpha''(x,\sigma)=Q_\alpha(x)+\int^\sigma R_{\alpha\beta}(\partial)\Delta(x-x')D_{a'}\underline{j}_{\beta a}(x')dx'$. Moreover, there are many types of the non-local $\underline{I}'$, for which there is no $H(x';n)$ even for $[\mathrm{IV}_4]$, and then we must use the slightly modified $Q_\alpha'(x,\sigma)$ instead of (4·6). The detailed discussion on these accounts will be given in the next paper.

is obtained, so if we determine $U(\sigma)$ by

$$i\frac{\delta U(\sigma)}{\delta\sigma(x)}=[H(x\,;n)U(\sigma)]_{x/\sigma}=U(\sigma)H(x/\sigma\,;n), \qquad (4\cdot10)^{*)}$$

$H(x\,;n)$ is Hermite on account of the unitarity of $U(\sigma)$. With the help of $(4\cdot9)$ and $(4\cdot10)$, one obtains

$$i\frac{\delta F(x,\sigma)}{\delta\sigma(x')}=[F(x,\sigma),H(x'/\sigma\,;n)]. \qquad (4\cdot11)$$

When $F(x,\sigma)$ contains explicitly σ, we have, instead of $(4\cdot11)$,

$$i\frac{\delta F(x,\sigma)}{\delta\sigma(x')}=i\frac{\partial F(x,\sigma)}{\partial\sigma(x')}+[F(x,\sigma),H(x'/\sigma\,;n)]. \qquad (4\cdot12)^{**)}$$

For the special case of $Q_\alpha(x,\sigma)$ as $F(x,\sigma)$, the field equation of motion

$$i\frac{\delta Q_\alpha(x,\sigma)}{\delta\sigma(x')}=[Q_\alpha(x,\sigma),H(x'/\sigma\,;n)] \qquad (4\cdot13)$$

is derived and from $(4\cdot6)$ one obtains

$$[Q_\alpha(x,\sigma),H(x'/\sigma\,;n)]=iR_{\alpha\beta}(\partial)D_\alpha\Delta(x-x')\underline{j}_{\beta\alpha}(x'). \qquad [\mathrm{IV}_4]^{***)}$$

This equation determines $H(x'/\sigma\,;n)$ in co-operation with $[\mathrm{III}_4]$ and $[\mathrm{II}_4']$ by which the right hand side of $[\mathrm{IV}_4]$ is successively expressed with $Q_\alpha(x'/\sigma)$. $H(x'/\sigma\,;n)$ may have an undeterminate additive c-number term, but we neglect such a term in our formalism. The meaning of this neglect will become clear in connection with the integrability condition of the equation $(4\cdot11)$.

We combine the respective state-vector $\Psi[\sigma]$ and Φ of the interaction- and the Heisenberg-representations by

$$\Psi(\sigma)=U(\sigma)\Phi \qquad (4\cdot14)^{****)}$$

and have the Schrödinger equation for $\Psi(\sigma)$

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)}=H(x\,;n)\Psi(\sigma), \qquad [\mathrm{V}_4]$$

from $(4\cdot10)$ and $(4\cdot14)$.

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* H depends on n_μ as shown later.

** That is, $\partial/\partial\sigma$ means the differentiation in which $U(\sigma)$ is considered as constant.

*** $[\mathrm{IV}_4]$ and $(4\cdot6)$ will be generalized in part II for the cases of $\underline{L}'$ with higher derivatives,

**** Therefore, the state $\Psi(\sigma)$ gives the information with regard to the field $Q_\alpha(x,\sigma)$. Accordingly, for the usual local interaction for which $Q_\alpha(x/\sigma)$ uniquely corresponds to $Q_\alpha(x)$ for the every surface σ, $\Psi(\sigma)$ gives the information with regard to the field $Q_\alpha(x)$. However, for the non-local interaction for which $Q_\alpha(x/\sigma)$ does not correspond to $Q_\alpha(x)$ for the every surface but to the summary of the properties of the field $Q_\alpha(x)$ in some small region $\Delta\sigma$, $\Psi[\sigma]$ gives the information for the field $Q_\alpha(x,\sigma)$ (and so averaged information for the field $Q_\alpha(x)$). The physical interpretation of this circumstance will be discussed in the next paper.

Especially, it must be noticed that the state $\Psi(\infty)$, which is combined with $\Psi(-\infty)=\Phi$ through the S-matrix $S(\infty)$, gives the information for the outgoing wave only when the second term in $[\mathrm{II}_4']$ is switched off in the limit $\sigma\to\infty$ through the suitable adiabatic process. (cf. the footnote of $[\mathrm{II}_4']$).

Thus, *we can take* $[I_2]$, $[II_4]$, $\cdots$ *and* $[V_4]$ *as the foundation of the formalism of the interaction representation.*

From (4·14), it is seen that $U(\infty)$ gives the S-matrix.

In the next place, let us examine the integrability condition of the equations (4·11) and $[V_4]$.

As will be shown in Appendix II, we obtain from (4·11) and (4·12)

$$[H(x'/\sigma\,;n), H(x''/\sigma\,;n)]-i\frac{\partial H(x''/\sigma\,;n)}{\partial\sigma(x')}+i\frac{\partial H(x'/\sigma\,;n)}{\partial\sigma(x'')}=c\text{-}number. \quad (4\cdot15)$$

When c-number is zero, (4·15) is just the integrability condition. Previously we mentioned that the arbitrary c-number term of $H(x'/\sigma\,;n)$ was neglected when it is determined by $[IV_4]$, so the left hand side of (4·15) cannot include any c-number terms, therefore the right hand side ought to be zero. As is easily seen from (4·6) and (4·7), for an arbitrary derivative-operator $M(\partial)$, we have

$$M(\partial)\underset{\sim}{Q}_\alpha(x)=M(\partial)Q_\alpha(x,\sigma)|_{x/\sigma}$$
$$+\frac{1}{2}\int[M(\partial)D_\alpha R_{\alpha\beta}(\partial),\varepsilon(x-x')]\,\Delta(x-x')\underset{\sim}{j}_{\beta\alpha}(x')dx'. \qquad [II_4'']$$

We can express $\underset{\sim}{g}_\beta(x\,;n)$ in terms of $M(\partial)Q_\alpha(x,\sigma)|_{x/\sigma}$ by the repeated use of $[II_4'']$ and denote this by $g_\beta(x,\sigma\,;n)$. In the next, if one denotes the same form functional of $Q_\alpha(x)$ as $g_\beta(x,\sigma\,;n)$ of $Q_\alpha(x,\sigma)$ by $g_\beta(x\,;n)$, we have

$$\underset{\sim}{Q}_\alpha(x)=U^{-1}(\sigma)\{Q_\alpha(x)+ig_\alpha(x\,;n)\}U(\sigma). \qquad [II_4''']$$

One finds easily, from $[II_4''']$,

$$[Q_\alpha(x)+ig_\alpha(x\,;n), H(x'\,;n)]$$
$$-i\frac{\partial}{\partial\sigma(x')}\{ig_\alpha(x\,;n)\}=0 \qquad \text{for} \quad x\neq x', \qquad (4\cdot16)$$

because we started with $\underset{\sim}{Q}_\alpha(x)$ as a point function independent of σ, therefore, for all x we have

$$[Q_\alpha(x\,;n), H(x'\,;n)]-[\bar{Q}_\alpha(x'\,;n), H(x\,;n)]$$
$$=i\frac{\partial}{\partial\sigma(x')}\bar{Q}_\alpha(x\,;n)-i\frac{\partial}{\partial\sigma(x)}Q_\alpha(x'\,;n), \qquad (4\cdot17)$$

with the abbreviation

$$\bar{Q}_\alpha(x\,;n)=Q_\alpha(x)+ig_\alpha(x\,;n),$$

which is the mathematical expression of the property of point function of $\underset{\sim}{Q}_\alpha(x)$ and is the important relation together with the integrability condition (4·15).

Accordingly we shall introduce the energy-momentum four-vector of the system with

the general interaction*).

We introduce the following quantities by use of the energy-momentum tensor of the free field $T_{\mu\nu}^{0}$ and $H(x;n)$ previously obtained:

$$\left.\begin{aligned} \underline{T}_{\mu\nu}(x) &= U^{-1}(\sigma)\{T_{\mu\nu}^{0} - \delta_{\mu\nu}H(x;n)\}U(\sigma), \\ \underline{T}_{\mu} &= -i\int_{\sigma}\underline{T}_{\mu\nu}(x')d\sigma_{\nu}'. \end{aligned}\right\} \qquad [\mathrm{VI}_4]$$

We can easily prove that $\underline{T}_{\mu}$ satisfies the conservation low and has the role of the displacement operator for the Heisenberg operator, i.e., for the arbitrary quantity $\underline{F}(x)$ in Heisenberg representation,

$$-\frac{\partial \underline{F}(x)}{\partial x_{\mu}} = [\underline{F}(x), \underline{T}_{\mu}]. \qquad (4\cdot 18)$$

The strict proof is given in Appendix II and III. Therefore, *we can say that the quantity* $[\mathrm{VI}_4]$ *is the energy-momentum tensor of the interacting system.* Now, one finds through the method of Appendix V that the S-matrix given by the interaction Hamiltonian $H(x;n)$ thus obtained is constructed independent of the intermediate surface of the integration. Furthermore, we can derive that, when we calculate the expectation value of any physical quantity by the Dyson formalism, this value is also indepennent of the intermediate surface and of that including the point in question. These facts are also proved in Appendix. On that occasion, we must calculate the correction of the quantity

$$\bar{\underline{Q}}_{\alpha}(x;n) \equiv \underline{Q}_{\alpha}(x) + ig_{\alpha}(x;n),$$

the second term of the right hand side of this equation has the role of the elimination of the normal dependent terms based on the non-covariant chronological operator P, i.e., from $[\mathrm{II}_4'']$, the expectation value including only one internal line in mixed representation of

$$\underline{Q}_{\alpha}(x) = U^{-1}(\sigma)\bar{\underline{Q}}_{\alpha}(x)U(\sigma) \qquad (4\cdot 19)$$

is

$$\langle \underline{Q}_{\alpha}(x)\rangle = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_m \langle P(Q(x;n), H(x_1;n), \cdots, H(x_m;n))\rangle \qquad (4\cdot 19')$$

$$\begin{aligned} &= Q_{\alpha}(x) + ig_{\alpha}(x;n) \\ &\quad - \frac{i}{2}\int R_{\alpha\beta}(\partial)D_{\alpha}\Delta_F(x-x')j_{\beta\alpha}(x')dx' \\ &\quad - \frac{1}{2}\int [R_{\alpha\beta}(\partial)D_{\alpha}, \varepsilon(x-x')]\Delta(x-x')j_{\beta\alpha}(x')dx' \\ &\quad + \cdots \end{aligned}$$

* This has been discussed roughly by several authors, but their discussions are incomplete as will be shown later, because *the energy-momentum four-vector cannot be determined uniquely by the conservation condition, only.*

S. Kanesawa, Prog. Theor. Phys. **5** (1950), 157.
K. Nishijima, Prog. Theor. Phys. **5** (1950), 187.

$$=Q_\alpha(x)+\frac{(-i)}{2}\int R_{\alpha\beta}(\partial)D_\alpha\Delta_F(x-x')j_{\beta\alpha}(x')dx'+\cdots$$

and the normal dependent terms inherent in $[R_{\alpha\beta}(\partial)D_\alpha, \varepsilon(x-x')]$ vanish consequently. This cancellation holds true in the higher order term according to the proof in Appendix IV. Since (4·19) holds for the arbitrary functional $\bar{F}(x, n)$ of $\bar{Q}_\alpha(x, n)$, when we calculate the expectation value of an arbitrary physical quantity, we may neglect the term $ig_\alpha(x; n)$ and use the following replacement in (4·19′) :

$$Q_\alpha(x; n) \to Q_\alpha(x),$$
$$H(x; n) \to -L'(x),$$

from $[\mathrm{IV}_4]$ and (4·2).

Thus we have the following rule of calculation : *The correction of the physical quantity $F(x)$ consisted of Q_α or the matrix element of S-matrix in Dyson formalism is given by the procedure that $H(x; n)$ is replaced by $-L'$ and P-bracket by the following normal independent term**)*

$$\langle P(M(\partial)Q_\alpha(x), N(\partial')Q_\beta(x')\rangle_0 \to \frac{1}{2}M(\partial)N(\partial')R_{\alpha\beta}(\partial)\Delta_F(x-x'). \quad (4\cdot20)$$

In (4·20), we must notice not to use $R_{\alpha\beta}'(\partial)$ but $R_{\alpha\beta}(\partial)$.***)

In conclusion, let us pay our attention to the following fact : in our formalism, $H(x; n)$ and T_μ are uniquely determined and the various requirements in our general discussion for the interaction Hamiltonian are also satisfied in the special case of the theory of the canonical form, so the general discussion in this section holds true in such case.

§ 5. Some concrete remarks on the interaction Hamiltonian***)

The several authors assert that the interaction Hamiltonian must be determined by the integrability condition of the Schrödinger equation. However, there are many interactions satisfying the integrability condition in practice, but most of these lead to the wrong equation of motion in Heisenberg representation. The circumstaces are explained in the case of the Duffin-Kemmer meson interacting with the electromagnetic field.

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* This rule of the replacement was exemplified in our previous paper on the self-stress. See Y. Takahashi and H. Umezawa, Prog. Theor. Phys. 8 (1952), 193.

** Koba's discussion[5] regarding these replacements is unsatisfactory, because, it does not prove the disappearance of the normal independent terms in $[H(x:n)-(-L'(x))]$. These terms appear when $R_{\alpha\beta}(\partial)$ contains the derivation $\Box$ or L' contains heiger derivatives of degree $n(>1)$, and cannot determined by the integrability condition.

*** The Koba's method of Δ_F* is equivalent to adopt the former, i.e.,

$$\langle P(Q_\alpha(x), Q_\beta(x'))\rangle_0 \to \frac{1}{2}R_{\alpha\beta}(\partial)\Delta_F{}^*(x-x') = \frac{1}{2}R_{\alpha\beta}'(\partial)\Delta_F(x-x')$$

and leads to the erroneous result for the field with term $R_{\alpha\beta}''(\partial)$ (see (2·17)).

The interaction Lagrangian of Duffin-Kemmer meson interacting with the electromagnetic field is

$$L' = ie\bar{\psi}\beta_\mu\psi A_\mu , \tag{5·1}$$

so the interaction Hamiltonian satisfying the integrability condition (4·15) (*c-number* $=0$) is

$$H(x\,;n) = -ie\bar{\psi}\beta_\mu\psi\underline{A}_\mu + \frac{e^2}{m}\bar{\psi}\beta_\mu(\beta n)^2\beta_\nu\psi\underline{A}_\mu\underline{A}_\nu + e^2w(x) + \text{higher order tems } o(e^3), \tag{5·2}$$

where $e^2w(x)$ is an arbitrary function proportional to e^2 and independent of n_μ. Whatever function $w(x)$ may be, according to the integrability condition we can determine the higher order normal-dependent terms successively by starting with $w(x)$. In this manner, the interaction Hamiltonian is not determined uniquely by the integrability condition. Unfortunately the simplest possibility $w(x)=0$ leads to the wrong result, i.e., by use of the unitary transformation $U(\sigma)$ composed of such interaction Hamiltonian, the following erroneous equation of motion in Heisenberg representation is given from the equation of motion in interaction representation :

$$(\beta_\mu\partial_\mu + m)\underline{\psi} = \frac{ie}{m}\beta_\nu\partial_\nu(\beta_\mu\underline{A}_\mu\underline{\psi}). \tag{5·3}$$

According to our method previously developed by which $H(x\,;n)$ is uniquely determined including $w(x)$, from [II$_4$'] and (3·4'), we have

$$\left.\begin{aligned}\underline{\psi}(x) &= \psi(x/\sigma) + \frac{ie}{m}(1+(\beta n)^2)\beta_\nu\underline{\psi}(x)\underline{A}_\nu(x),\\ \underline{A}_\mu(x) &= A_\mu(x/\sigma),\end{aligned}\right\} \tag{5·4}$$

and from (5·4)

$$\begin{aligned}\underline{\psi}(x) &= \psi(x/\sigma) + \frac{ie}{m}(1+(\beta n)^2)\beta_\nu A_\nu(x/\sigma)\psi(x/\sigma)\\ &= U^{-1}(\sigma)\left\{1+\frac{ie}{m}(1+(\beta n)^2)\beta_\nu A_\nu(x)\right\}\psi(x)U(\sigma),\end{aligned} \tag{5·5}$$

where the identity

$$(1+(\beta n)^2)\beta_\nu(1+(\beta n)^2) = 0 \tag{5·6}$$

was used. If we substitute (5·5) into [IV$_4$], we obtain

$$\begin{aligned}[\psi(x,\sigma), H(x'/\sigma\,;n)] &= -eR\Delta(x-x')\beta_\mu A_\mu(x'/\sigma)\\ &\times\left\{1+\frac{ie}{m}(1+(\beta n)^2)\beta_\nu A_\nu(x'/\sigma)\right\}\psi(x'/\sigma).\end{aligned} \tag{5·7}$$

Therefore, one obtains

$$H(x\,;n) = -ie\bar{\psi}\beta_\mu\psi A_\mu + \frac{e^2}{m}\bar{\psi}\beta_\mu(1+(\beta n)^2)\beta_\nu\psi\underline{A}_\mu\underline{A}_\nu, \tag{5·8}$$

which is the right interaction Hamiltonian. The second term may be neglected by use of the replacement (4·20).

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Appendix I

The example of the transformation (2·10)

We shall discuss the vector field U_μ with the mass $\varkappa$, for which:

$$R=(R_{\mu\nu})=(\delta_{\mu\nu}+a_\mu a_\nu), \tag{A·1}$$

$$a_\mu=\frac{1}{\varkappa}k_\mu\,,\quad a_\mu a_\mu=-1. \tag{A·2}$$

One easily verifies that the secular equation $|R_{\mu\nu}-\lambda\delta_{\mu\nu}|=0$ has the triple roots $\lambda=1$ and the single root $\lambda=a_\mu a_\mu+1=0$ and so the rank of R is 3. Therefore only three of the wave functions can be considered as independent.

Using the coordinate system in which $k_1=k_2=0$, we find the transformation U in (2·12) which transforms U_μ into U_r $(r=1, 2, 3, 4)$ as follows:

$$\begin{aligned}&\bar{U}_1(k)=U_1(k),\quad \bar{U}_2(k)=U_2(k),\\ &\bar{U}_3(k)=-a_4U_3(k)+a_3U_4(k),\\ &\bar{U}_4(k)=a_3U_3(k)+a_4U_1(k)=\frac{1}{\varkappa}k_\mu U_\mu\,.\end{aligned} \tag{A·3}$$

$\bar{U}_1$, $\bar{U}_2$ and $\bar{U}_3$ are the independent components which correspond to the transverse and longitudinal parts, and $\bar{U}_4$ vanithes according to the subsidiary condition $\partial_\mu U_\mu=0$.

Appendix II

On the integrability condition (4·15)

Since $Q_\alpha(x,\sigma)$ depends only on given x and σ,* the functional $F(x,\sigma)$ of $Q_\alpha(x,\sigma)$ also satisfies the following relation

$$\left[\frac{\delta}{\delta\sigma(x')}\frac{\delta}{\delta\sigma(x'')}-\frac{\delta}{\delta\sigma(x'')}\frac{\delta}{\delta\sigma(x')}\right]F(x,\sigma)=0.$$

Therefore, it follows from (4·11) and (4·12)

$$\begin{aligned}0&=i\frac{\delta}{\delta\sigma(x')}[F(x,\sigma),H(x''/\sigma\,;n)]-i\frac{\delta}{\delta\sigma(x'')}[F(x,\sigma),H(x'/\sigma\,;n)]\\ &=[[F(x,\sigma),H(x'/\sigma\,;n)],H(x''/\sigma\,;n)]+\left[F(x,\sigma),i\frac{\partial H(x''/\sigma\,;n)}{\partial\sigma(x')}\right]\\ &\quad-[[F(x,\sigma),H(x''/\sigma\,;n)],H(x'/\sigma\,;n)]-\left[F(x,\sigma),i\frac{\partial H(x'/\sigma\,;n)}{\partial\sigma(x'')}\right]\end{aligned}$$

* See the fourth foot-note in page 22.

$$=\left[F(x,\sigma),[H(x'/\sigma\,;n),H(x''/\sigma\,;n)]-i\frac{\partial H(x''/\sigma\,;n)}{\partial\sigma(x')}+i\frac{\partial H(x'/\sigma\,;n)}{\partial\sigma(x'')}\right].$$

Now that we may have any functional $F(x,\sigma)$, (4·15) is obtained. As is mentioned in section 4, since the arbitrary c-number term of $H(x\,;n)$ is neglected, one finds

$$[H(x'\,;n),H(x''\,;n)]-i\frac{\partial H(x''\,;n)}{\partial\sigma(x')}+i\frac{\partial H(x'\,;n)}{\partial\sigma(x'')}=0$$

and therefore the Schrödinger equation $[V_4]$ is also integrable.

Appendix III

On the role of T_μ as the displacement operator and the energy conservation

When the function $f(x\,;n)$ explicitly contains $\sigma(x)$ and

$$F_\lambda(\sigma)=\int_\sigma f(x'\,;n)\,d\sigma_\lambda', \tag{A·4}$$

let us notice the following two formulas:

$$\frac{\delta F_\lambda(\sigma)}{\delta\sigma(x)}=\frac{\partial f(x\,;n)}{\partial x_\lambda}+\int_\sigma\frac{\partial f(x'\,;n)}{\partial\sigma(x)}\,d\sigma_\lambda' \tag{A·5}$$

and

$$[f(x\,;n),\int_\sigma T_{\mu\nu}{}^0(x')\,d\sigma_\nu']=-i\frac{\partial f(x\,;n)}{\partial x_\mu}-i\int_\sigma\frac{\partial f(x\,;n)}{\partial\sigma(x')}\,d\sigma_\mu'. \tag{A·6}$$

Therefore it follows from (A·5), (A·6) and $[VI_4]$

$$\begin{aligned}\frac{\delta T_\mu}{\delta\sigma(x)}&=U^{-1}(\sigma)\,[H(x\,;n),\int\{T_{\mu\nu}{}^0(x')-\delta_{\mu\nu}H(x'\,;n)\}\,d\sigma_\nu']\,U(\sigma)\\&\quad-iU^{-1}(\sigma)\left\{\frac{\partial T_{\mu\nu}{}^0(x)}{\partial x_\nu}-\frac{\partial H(x\,;n)}{\partial x_\mu}-\int_\sigma\frac{\partial H(x'\,;n)}{\partial\sigma(x)}\,d\sigma_\mu'\right\}U(\sigma)\\&=-iU^{-1}(\sigma)\left\{\frac{\partial H(x\,;n)}{\partial x_\mu}+\int_\sigma\frac{\partial H(x\,;n)}{\partial\sigma(x')}\,d\sigma_\mu'\right\}U(\sigma)\\&\quad-U^{-1}(\sigma)\int_\sigma[H(x\,;n),H(x'\,;n)]\,d\sigma_\mu'U(\sigma)\\&\quad+iU^{-1}(\sigma)\left\{\frac{\partial H(x\,;n)}{\partial x_\mu}+\int_\sigma\frac{\partial H(x'\,;n)}{\partial\sigma(x)}\,d\sigma_\mu'\right\}U(\sigma)\\&=-U^{-1}(\sigma)\int_\sigma\left\{[H(x\,;n),H(x'\,;n)]+i\frac{\partial H(x\,;n)}{\partial\sigma(x')}-i\frac{\partial H(x'\,;n)}{\partial\sigma(x)}\right\}d\sigma_\mu'U(\sigma),\end{aligned} \tag{A·7}$$

which vanishes definitly on account of the integrability condition, i.e., the energy-momentum $[VI_4]$ is conserved.

In the next, let us ascertain that $[VI_4]$ plays a role of the displacement operator to

the Heisenberg operator. For the functionals $F(x;n)$ and $F(x,\sigma;n)$,

$$F(x,\sigma;n)=U^{-1}(\sigma)F(x;n)U(\sigma) \tag{A·8}$$

and

$$\begin{aligned}\underline{Q}_\alpha(x)&=\underline{Q}_\alpha(x/\sigma)+i\underline{g}_\alpha(x;n)\\&=U^{-1}(\sigma)\bar{Q}_\alpha(x;n)U(\sigma).\end{aligned} \tag{A·9}$$

By taking the variation of both sides of

$$\int_\sigma \underline{Q}_\alpha(x)d\sigma_\mu=U^{-1}(\sigma)\int_\sigma \bar{Q}_\alpha(x';n)d\sigma'_\mu U(\sigma), \tag{A·10}$$

one obtains

$$\begin{aligned}i\frac{\partial \underline{Q}_\alpha(x)}{\partial x_\mu}&=U^{-1}(\sigma)\Big[\int_\sigma \bar{Q}_\alpha(x';n)d\sigma_\mu',H(x;n)\Big]U(\sigma)\\&\quad+iU^{-1}(\sigma)\int_\sigma\frac{\partial \bar{Q}_\alpha(x';n)}{\partial\sigma(x)}d\sigma_\mu' U(\sigma)\\&\quad+iU^{-1}(\sigma)\frac{\partial \bar{Q}_\alpha(x;n)}{\partial x_\mu}U(\sigma).\end{aligned} \tag{A·11}$$

On the other hand, we have

$$\begin{aligned}&(-i)[\underline{Q}_\alpha(x),\underline{T}_\mu]\\&=-U^{-1}(\sigma)\{[\bar{Q}_\alpha(x;n),\int_\sigma T_{\mu\nu}{}^0(x')d\sigma_\nu']+[\bar{Q}_\alpha(x;n),\int H(x';n)d\sigma_\mu']\}U(\sigma)\\&=iU^{-1}(x)\Big\{\frac{\partial \bar{Q}_\alpha(x;n)}{\partial x_\mu}+\int_\sigma\frac{\partial \bar{Q}_\alpha(x;n)}{\partial\sigma(x')}d\sigma'_\mu\Big\}U(\sigma)\\&\quad+U^{-1}(\sigma)[\bar{Q}_\alpha(x;n),\int H(x';n)d\sigma_\mu']U(\sigma).\end{aligned} \tag{A·12}$$

Therefore, we can identify (A·11) with (A·12) by (4·5), i.e.,

$$\frac{\partial \underline{Q}_\alpha(x)}{\partial x_\mu}=-[\underline{Q}_\alpha(x),\underline{T}_\mu]. \tag{A·13}$$

This is the mathematical expression of the displacement operator. Thus, we can call $[\mathrm{VI}_4]$ the energy-momentum four-vector of interacting system.

Appendix IV

On the normal-dependent term in the calculation of the expectation value

$$\langle \underline{Q}_\alpha(x)\rangle=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_m\langle P(\bar{Q}(x;n),H(x_1;n),\cdots,H(x_m;n))\rangle.$$

If we pick up the m-th order term and take the variation with respect to $\sigma(x')$ $(x\neq x')$

$$\frac{\delta}{\delta\sigma(x')}\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_m P(Q_\alpha(x;n), H(x_1;n), \cdots, H(x_m;n))$$

$$=\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_m P\Big(\frac{\delta\bar{Q}_\alpha(x;n)}{\delta\sigma(x')}, H(x_1;n), \cdots, H(x_m;n)\Big)$$

$$+\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_{m-1} P(m[\bar{Q}_\alpha(x;n), H(x';n)], H(x_1;n), \cdots, H(x_{m-1};n)).$$

The first term of right hand side is the change of $\bar{Q}_\alpha(x,n)$ itself and the second term comes from the rearrangement of the operators according to the definition of the P-bracket, since a point x', which lies before deformation on the future side of x, is now found on the past side. Therefore,

$$\frac{\delta\langle Q_\alpha(x)\rangle}{\delta\sigma(x')}=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int dx_1\cdots\int dx_m P\Big(\frac{\delta\bar{Q}_\alpha(x;n)}{\delta\sigma(x')}$$

$$-i[\bar{Q}_\alpha(x;n), H(x';n)], H(x_1;n), \cdots, H(x_m;n)\Big),$$

which vanishes from (4·16). The variation with respect to a intermediate surface $\sigma_j(x_j)$ of integral vanishes also from the integrability condition (4·15).

Appendix V

On the cancellation of the terms in $[H-(-L')]$ *in the S-matrix*

We consider the $S(\infty)$ in g^2 (g: coupling constant)-approximation, but it is easy to take into account the higher order terms. Then we have

$$S(\infty)\approx-\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx' P(L'(x), L'(x'))-i\int_{-\infty}^{\infty}dx w_{\mu\nu}(x)n_\mu n_\nu, \qquad (A\cdot14)$$

where $\approx$ means the equality in the g^2-approximation and we assume that the interaction Hamiltonian is given by

$$H(x;n)=-L'(x)+w_{\mu\nu}(x)n_\mu n_\nu. \qquad (A\cdot15)$$

We can rewrite (A·14) into the form

$$S(\infty)=-\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx' j_{\cdot\alpha}(x)j_{\beta b}(x')P(D_\alpha Q_\alpha(x), D_b'Q_\beta(x'))$$

$$-i\int_{-\infty}^{\infty}dx w_{\mu\nu}(x)n_\mu n_\nu. \qquad (A\cdot16)$$

On the other hand we obtain from $[\mathrm{II}_4]$:

$$[Q_\alpha(x,\sigma), w_{\mu\nu}(x'/\sigma)]n_\mu n_\nu\approx\frac{i}{2}R_{\alpha\beta}(\partial)D_\alpha'\Delta(x-x')\frac{\partial j_{\beta a}}{\partial Q_{\rho b}}$$

$$\times\int dx''[D_b'D_c''R_{\rho\tau}(\partial'), \varepsilon(x'-x'')]\Delta(x'-x'')j_{\tau c}(x''), \qquad (A\cdot17)$$

where

$$\frac{\partial j_{\beta a}}{\partial Q_{\rho b}} = \frac{\partial j_{\beta\mu_1\cdots\mu_n}}{\partial Q_{\rho\mu_1'\cdots\mu_m'}} = \frac{\partial^2 L'}{\partial Q_{\rho b}\partial Q_{\beta a}} = \frac{\partial j_{\rho b}}{\partial Q_{\beta a}}. \tag{A·18}$$

Therefore, we have

$$\int \tau_{\mu\nu}(x')\, n_\mu n_\nu dx' = \frac{1}{4}\iint j_{\alpha a}(x')\, j_{\beta b}(x'') \times [R_{\alpha\beta}(\partial) D_a' D_b'', \varepsilon(x'-x'')]\, \Delta(x'-x'')\, dx' dx''. \tag{A·19}$$

Since we have the relation

$$\langle P(D_a Q_\alpha(x), D_b' Q_\beta(x'))\rangle_0 = iR_{\alpha\beta}(\partial) D_a D_b' \Delta_F(x-x') - \frac{i}{2}[R_{\alpha\beta}(\partial) D_a D_b', \varepsilon(x-x')]\, \Delta(x-x'), \tag{A·20}$$

by substituting (A·20) and (A·19) into (A·16), one obtains

$$S(\infty) \approx -\frac{1}{4}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, j_{\alpha a}(x)\, \{R_{\alpha\beta}(\partial) D_a D_b' \Delta_F(x-x')\}\, j_{\beta b}(x'). \tag{A·21}$$

Thus we find that we must replace $P(M(\partial) Q_\alpha(x), N(\partial') Q_\beta(x'))$ by $1/2 \cdot M(\partial) \times N(\partial') R_{\alpha\beta}(\partial) \Delta_F(x-x')$ in order to neglect the term $\tau_{\mu\nu}(x) n_\mu n_\nu$.

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Note added in proof: The extension for the interactions containing higher derivatives in Part II is based on the generalization of relations (4·6) and [IV] of the present paper. In part II the general structure of the theory of the inreraction representation and some differences between our discussions and usual ones for the interaction representation will be also synthetically discussed.

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An Algebraic Treatment of the Many Electron Problem

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The algebraic structure of the hamiltonian of the many electron system is studied. Based upon the second quantization scheme, a purely algebraic-theoretical method is established in order to determine the total energies of the many electron system, where each energy eigenstate of one electron in the Hartree-Fock field of these electrons is occupied by a given number of electrons. This occupation number is either 0, 1 or 2, because of the value, 1/2, of the electron spin. Only the part of the hamiltonian related to the singly-occupied states are found to be algebraically important, in relation to which the so-called exchange algebra emerges. We state the general principle how to manipulate this algebra, and will work out in some cases of a few numbers of singly occupied states. In the case that the energy eigenstates of the Hartree-Fock field have degeneracies, the theory so far mentioned does not suffice, and more detailed and somewhat complicated study is required.

§ 1. Introduction

Among the methods of treating the many electron problem we have the well-known group-theoretical one, where a certain kind of representation of the symmetric group is studied. This is based upon the so-called Dirac's vector model[1], a notion originating from the fact that the interchange operation of the i-th and j-th electrons can be represented by the operator,

$$V_{ij}=\frac{1}{2}\{1+4(\boldsymbol{S}_i, \boldsymbol{S}_j)\} \qquad (1\cdot1)$$

in the spin spaces of these electrons; $\boldsymbol{S}_i$, $\boldsymbol{S}_j$ being the spin vector operators. This model is revealed to be very fruitful in a large number of problems. Prof. T. Yamanouchi[2] has successfully invented a method of obtaining the irreducible representation matrices for every V_{ij}, and worked out the theory up to seven electrons. On the other hand H. Ostertag[3], who started with the second quantization algebra of the electron field, suggested an algebraic treatment of this problem. Though his method promises to be practically useful as well as theoretically, he appeared, to our regrets, to stop before obtaining any interesting results. We have developed his idea unto the last stage. In § 2 the preliminary subjects are explained, the selection operators are prepared and the hamiltonian for a certain electron configuration where λ_i electrons are present in the i-th energy eigenstate of one electron problem in the Hartree-Fock field of the electrons. When these eigenstates are not degenerate at all, the study of only this hamiltonian is sufficient. How to treat in those

cases where this condition is not fulfilled are explained in § 7. In the hamiltonian set up in § 2 we find that only the parts contributed from the singly occupied states are algebraically important. This part is constituted by the elements of what we call the exchange algebra. In § 3 we investigate the structure of the so-called magnetic algebra as the preliminary for the study of the exchange algebra. The detailed structure of the exchange algebra is made clear in § 4, and, based on this knowledge, a general principle of finding out all the inequivalent irreducible representation matrices of the exchange quantities is established. In § 6 these procedures are traced concretely in the cases where the number of the singly occupied states are 2, 3, 4, 5, 6 and 7. How to find to which 'multiplet' (a terminology in the spectroscopy) each of our representations corresponds is explained in § 5, and also in note at the end of this paper.

§ 2. Preliminary

1. The commutative subalgebra, and the selection operators. We first consider the commutative subalgebra $\boldsymbol{N}=\{1, N_1, N_2,\ldots, N_i,\ldots, N_K\}$ of the second quantization algebra of electrons, where $i(=1, 2,\ldots, K)$ is the suffix for numerating the energy eigenstates of a single electron in the Hartree-Fock field of the others. The quantities N_i have the well-known relations,

$$N_i^2=N_i, \qquad N_iN_j=N_jN_i, \tag{2·1}$$

which have the meaning of the number of electrons existing in the i-th eigenstate. This subalgebra $\mathbf{N}$ has the order 2^K, and the 2^K pairwise orthogonal primitive idempotents are

$$E_\sigma(i_1, i_2,\ldots, i_\sigma)=\prod_{i=i_1}^{i_\sigma} N_i \prod_{j=i_{\sigma+1}}^{i_K}(1-N_j), \tag{2·2}$$

where $i_1, i_2,\ldots, i_\sigma, i_{\sigma+1},\ldots, i_K$ is a certain permutation of the natural order $1, 2, 3,\ldots, K$. The idempotents of the form of (1·2) corresponds to the configuration that the eigenstates $i_1, i_2,\ldots, i_\sigma$ are occupied by electrons and in total σ electrons are present. There exist $\binom{K}{\sigma}$ quantities of this sort in all. On account of the relation $\sum_\sigma \binom{K}{\sigma}=2^K$, the total number of idempotents is 2^K. These idempotents sum up to the unity in all, namely

$$\sum_{\sigma=0}^{K} \sum_{i_1<i_2<\cdots<i_\sigma} E_\sigma(i_1, i_2,\ldots, i_\sigma)=1. \tag{2·3}$$

The selection operator, destined to select out all possible states wherever σ electrons are present, is

$$N(\sigma)=\sum_{i_1<i_2<\cdots<i_\sigma} E_\sigma(i_1, i_2,\ldots, i_\sigma), \tag{2·4}$$

which is a part of the left-hand side of (1·3).

2. The hamiltonian of the many electron system. The hamiltonian operator of the system is

$$H=H_0-H_1, \tag{2·5a}$$

$$H_0=\sum_i \sum_\rho W_i \cdot N_{i\rho}\,, \tag{2·5b}$$

$$H_1=\frac{1}{2}\sum_{i,j,k,l}\sum_{\rho=\rho'} H_{ijkl}\, a^+_{i\rho}\, a^+_{j\rho'}\, a_{l\rho'}\, a_{k\rho}\,, \tag{2·5c}$$

in the scheme of the second quantization[4], where the second suffices of the Greek letters represents the two spin states, to be taken as 1 or 2. We can obtain the selection operator in order to select out such a state of the system that λ_i electrons are present on the i-th eigenstate, as follows

$$S(\lambda_1, \lambda_2, \ldots, \lambda_K)=\prod_{i=1}^{K} N_i(\lambda_i). \tag{2·6}$$

where $\lambda_i=0$, 1, or 2, and we have made use of the abbreviations

$$\left.\begin{aligned} N_i(0)&=(1-N_{i1})(1-N_{i2}),\\ N_i(1)&=N_{i1}(1-N_{i2})+(1-N_{i1})N_{i2},\\ N_i(2)&=N_{i1}N_{i2}. \end{aligned}\right\} \tag{2·7}$$

They are obtained at once by applying (1·2), (1·4) to the case of $K=2$. In the case of the Hartree-Fock field having no degeneracies, it is sufficient to discuss only the part of the hamiltonian

$$H_1'=S(\lambda_1, \lambda_2, \ldots, \lambda_K)H_1S(\lambda_1, \lambda_2, \ldots, \lambda_K). \tag{2·8}$$

By making use of the well-known relations

$$\left.\begin{aligned} &N_{i\rho}=a^+_{i\rho}\, a_{i\rho}\,,\\ &a^+_{i\rho}\, a_{j\sigma}+a_{j\sigma}\, a^+_{i\rho}=\delta_{ij}\,\delta_{\rho\sigma}\,,\\ &a_{i\rho}\, a_{j\sigma}+a_{j\sigma}\, a_{i\rho}=0,\\ &a^+_{i\rho}\, a^+_{j\sigma}+a^+_{j\sigma}\, a^+_{i\rho}=0, \end{aligned}\right\} \tag{2·9}$$

we arrive at the reduced form of H_1', as follows.

$$H_1'=A+B, \tag{2·10a}$$

$$A=S(\lambda_1, \lambda_2, \ldots, \lambda_K)\Big[\sum_j H(j)N_j(2) + \sum_{i<j} J(ij)\,\{N_i(1)+2N_i(2)\}\{N_j(1)+2N_j(2)\}\Big]S(\lambda_1, \lambda_2, \ldots, \lambda_K), \tag{2·10b}$$

$$B=-S(\lambda_1, \lambda_2, \ldots, \lambda_K)\Big[\sum_{i<j} J(ij)P_{ij}N_i(1)N_j(1)+\sum_{i<j} K(ij)\cdot \cdot\{N_i(1)N_j(2)+N_i(2)N_j(1)+2N_i(2)N_j(2)\}\Big]\,S(\lambda_1, \lambda_2, \ldots, \lambda_K), \tag{2·10c}$$

where we make use of the abbreviations $H(i)$, $J(ij)$ and $K(ij)$ for H_{iiii}, H_{ijij} and H_{ijji}; and also

$$P_{ij}=\frac{1}{2}(1+A_iA_j+B_iB_j+C_iC_j), \tag{2·11a}$$

in which

$$A_i=a_{i1}^{+}a_{i2}+a_{i2}^{+}a_{i1}, \tag{2·11b}$$

$$B_i=-i(a_{i1}^{+}a_{i2}-a_{i2}^{+}a_{i1}), \tag{2·11c}$$

$$C_i=N_{i1}-N_{i2}. \tag{2·11d}$$

Those parts of A and B contributed from the representative pair of the eigenstates, the i-th and j-th states, are shown in table 1 for each value of the occupation number λ_i, λ_j.

Table 1.

λ_i	0	0	1	1	2
λ_j	0	1	1	2	2
A	0		$J(ij)$	$2J(ij)+H(j)$	$H(i)+H(j)+4J(ij)$
B	0		$-K(ij)P_{ji}$	$-K(ij)$	$2K(ij)$

We can at once see that non-commutative elements appear only in relation to the singly-occupied states, and therefore only these states are important in algebraical point of view. Thus we shall be engaged exculsively in the detailed study of these states in the following paragraphs.

§ 3. The study of the magnetic algebra M in connection with the singly-occupied states

By the reason explained in the last paragraph we shall study the algebraic structure of the parts relating to the singly-occupied states in detail. We shall assume the number of these states to be n. Each set $(1, A_i, B_i, C_i)$ $(i=1, 2,\ldots, n)$, as easily proved, forms a quarternion, and any two elements belonging to different sets are commutative each other*. We transfer to the new combinations

$$\begin{aligned} a(ij) &= \frac{1}{2}(1+C_iC_j)=\sum_{\rho} N_{i\rho}N_{j\rho}, \\ a'(ij) &= \frac{1}{2}(1-C_iC_j)=\sum_{\rho} N_{i\rho}N_{j\bar{\rho}}, \\ b(ij) &= \frac{1}{2}(A_iA_j+B_iB_j)=\sum_{\rho} a_{i\rho}^{+}a_{i\bar{\rho}}a_{j\bar{\rho}}^{+}a_{j\rho}, \end{aligned} \tag{3·1}$$

* Therefore A_i, B_i, C_i are exactly equivalent to twice the operators of the spin vector components of the i-th electron, S_{ix}, S_{iy}, S_{iz}, and we may write $P_{ij}=\frac{1}{2}\{1+4(\boldsymbol{s}_i, \boldsymbol{s}_j)\}$.

where $\bar{\rho}$ takes the values 1, 2 respectively corresponding to the values 2, 1 of ρ. Then we can obtain the relation.

$$P_{ij}=a(ij)+b(ij),\ (i,j=1,2,\ldots,n), \tag{3·2}$$

and see that the a and a' factors belong to the commutative part of the algebra and the b factors break the commutativity, but all quantities having no common suffices commute with each other. We shall study the structure of the algebra $\mathbf{M}=\{a(12), a'(12), b(12), \ldots, a(ij), a'(ij), b(ij), \ldots, a(n-1,n), a'(n-1,n), b(n-1,n)\}$, what we call the magnetic algebra. The algebraic relations in this algebra are as follows,

$$\begin{aligned}
&a(ij)+a'(ij)=1,\\
&a(ij)^2=a(ij),\ a'(ij)^2=a'(ij),\ b(ij)^2=a'(ij),\\
&a(ij)b(ij)=b(ij)a(ij)=0,\ a'(ij)b(ij)=b(ij)a'(ij)=b(ij),\\
&a(ij)b(ik)=b(ik)a'(ij),\ a'(ij)b(ik)=b(ik)a(ij),\\
&a(ij)a(ik)+a'(ij)a'(ik)=a(jk),\\
&a(ij)a'(ik)+a'(ij)a(ik)=a'(jk),\\
&b(ij)b(ik)+b(ik)b(ij)=b(jk),\\
&b(ij)b(ik)b(il)=b(il)b(ik)b(ij),\\
&b(ij)b(ik)b(ij)=0.
\end{aligned} \tag{3·3a}$$

Here and hereafter we indicate the different numbers by the different letters in the brackets. Furthermore from these we can derive the following useful relations.

$$\begin{aligned}
&\Big[\prod_{\sigma=1}^{\mu-1} b(ij_\sigma)b(ik_\sigma)b(ij_\mu)\Big]^2=\prod_{\sigma=1}^{\mu-1} a'(ij_\sigma)a(ik_\sigma)a'(ij_\mu),\\
&\Big[\prod_{\sigma=1}^{\mu-1} b(ij_\sigma)b(ik_\sigma)b(ij_\mu)\Big]\cdot\Big[\prod_{\sigma=1}^{\mu-1} a'(ij_\sigma)a(ik_\sigma)a'(ij_\mu)\Big]\\
&=\Big[\prod_{\sigma=1}^{\mu-1} a'(ij_\sigma)a(ik_\sigma)a'(ij_\mu)\Big]\cdot\Big[\prod_{\sigma=1}^{\mu-1} b(ij_\sigma)b(ik_\sigma)b(ij_\mu)\Big]\\
&=\prod_{\sigma=1}^{\mu-1} b(ij_\sigma)b(ik_\sigma)b(ij_\mu).
\end{aligned} \tag{3·3b}$$

According to (3·3b), in a product of any number of b factors all having one common suffix it matters only which factors appear in the even or odd sites, and the more detailed order is irrelevant.

We can obtain all pairwise orthogonal primitive idempotents[5] which sum up to the unity quantity in all, viz.,

$$c_\mu^{(n)}(i_1,i_2,\ldots,i_\mu;\ k_1,k_2,\ldots,k_{n-\mu-1})=\prod_{\sigma=1}^{\mu} a'(1i_\sigma)\prod_{\tau=1}^{n-\mu-1} a(1k_\tau), \tag{3·4}$$

when n is odd or n is even and $\mu < \frac{n}{2}$. In the case where n is even and $\mu = \frac{n}{2}$, the idempotent of this form is not primitive, and resolves into the two primitive ones as follows

$$e^{(n)\pm}_{n/2}(i_1, i_2, \ldots, i_{n/2}; k_1, k_2, \ldots, k_{n/2-1})$$

$$= \frac{1}{2}\Big\{ \prod_{\sigma=1}^{n/2-1} a'(1i_\sigma)a(1k_\sigma)a'(1i_{n/2}) \pm \prod_{\sigma=1}^{n/2-1} b(1i_\sigma)b(1k_\sigma)b(1i_{n/2}) \Big\}. \tag{3·5}$$

We prove the primitivity of these idempotents in appendix I. Now we had better study the two cases seperately.

(1) n=odd.

We construct following quantities

$$\varepsilon^{(n)}_\mu = \sum_{1<i_1<i_2<\cdots<i_\mu\leq n} e^{(n)}_\mu(i_1, i_2, \ldots, i_\mu; k_1, k_2, \ldots, k_{n-\mu-1})$$

$$+ \sum_{1<i_1<i_2<\cdots<i_{n-\mu}\leq n} e^{(n)}_{n-\mu}(i_1, i_2, \ldots, i_{n-\mu}; k_1, k_2, \ldots, k_{\mu-1}), \tag{3·6}$$

$$\left(\mu = 0, 1, 2, \ldots, \frac{n-1}{2}\right),$$

where the summation extends over all the possible combinations of μ numbers $i_1 < i_2 < \ldots < i_\mu$ and $(n-\mu)$ numbers $i_1 < i_2 < \ldots < i_{n-\mu}$ taken from the $(n-1)$ numbers, 2, 3, ..., n. Every of these $\varepsilon^{(n)}_\mu$ is the unity quantity of the simple two-sided ideal[4] of the magnetic algebra which corresponds to the irreducible representation $R^{(n)}_\mu$ of this algebra, and pairwise orthogonal to one another. These representations are all inequivalent, and the total of these are all possible irreducible representations of **M**. Therefore we have $(n+1)/2$ irreducible representations, inequivalent to one another, viz. $R^{(n)}_0$, $R^{(n)}_1$, $R^{(n)}_2$, ..., $R^{(n)}_{\frac{n-1}{2}}$. These representations are shown schematically in fig. 1.

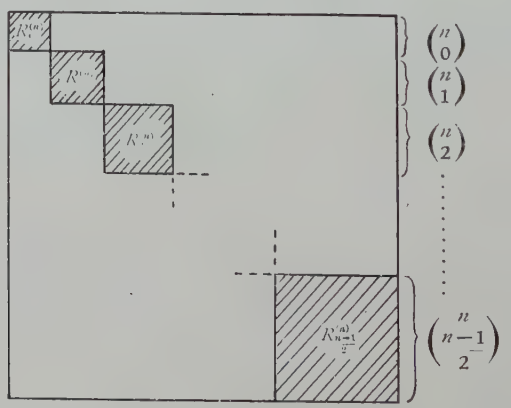


Fig. 1. The numbers on the right show the degrees of the corresponding representation matrices. This use is made also in Fig. 2.

In order to find the representation matrices derived from $\varepsilon_\mu^{(n)}$ explicitly, it is required to find out such a quantity, e_{ij}, that has the property,

$$e_i e_{ij} = e_{ij} e_j = e_i e_{ij} e_j = e_{ij} ,$$

when we regard any two constituent primitive idempotents of $\varepsilon_\mu^{(n)}$ having the form $e_\mu^{(n)}(i_1, i_2, \ldots, i_\mu ; k_1, k_2, \ldots, k_{n-\mu-1})$ or $e_{n-\mu}^{(n)}(i_1, i_2, \ldots, i_{n-\mu} ; k_1, k_2, \ldots, k_{\mu-1})$ as e_i and e_j. We describe the study about this problem in appendix II. We shall picture the quantities having the above properties schematically just like in Table 2, where the pairwise orthogonal idempotents occupy the diagonal places, and we call the table of this property the algebraic table. On account of the relation (II. 1), (II. 2) in appendix II, we obtain the representative parts of the algebraic table for the representation $R_\mu^{(n)}$, which are shown in table 3 (a), (b).

Table 3.

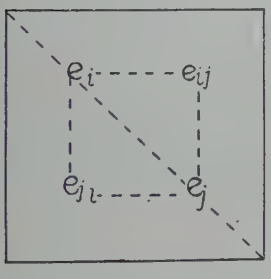
$\prod_{\sigma=1}^{\mu} a'(1i_\sigma) \prod_{\tau=1}^{n-\mu-1} a(1k_\tau)$	$\prod_{\sigma=1}^{\nu} b(1i_\sigma)\, b(1k_\sigma) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\rho=\nu+1}^{n-\mu-1} a(1k_\rho)$
$\prod_{\sigma=1}^{\nu} b(1k_\sigma) b(1i_\sigma) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\rho=\nu+1}^{n-\mu-1} a(1k_\rho)$	$\prod_{\sigma=1}^{\nu} a'(1k_\sigma) a(1i_\sigma) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\rho=\nu+1}^{n-\mu-1} a(1k_\rho)$

(a)

$\prod_{\sigma=1}^{\mu} a'(1i_\sigma) \prod_{\tau=1}^{n-\mu-1} a(1k_\tau)$	$\prod_{\sigma=1}^{\nu-1} b(1i_\sigma) b(1k_\sigma) b(1i_\nu) \prod_{\tau=\nu+1}^{\mu} a(1i_\tau) \prod_{\rho=\nu}^{n-\mu-1} a'(1k_\rho)$
$\prod_{\sigma=1}^{\nu-1} b(1i_\sigma) b(1k_\sigma) b(1i_\nu) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\tau=\nu}^{n-\mu-1} a(1k_\tau)$	$\prod_{\sigma=1}^{\nu-1} a'(1i_\sigma) a(1k_\sigma) a'(1i_\nu) \prod_{\tau=\nu+1}^{\mu} a(1i_\tau) \prod_{\rho=\nu}^{n-\mu-1} a'(1k_\rho)$

(b)

Table 2.



(2) n=even.

When $\mu < n/2$, we can discuss quite similarly to the case (1). Such quantities as

$$\varepsilon_\mu^{(n)} = \sum_{1<i_1<\cdots<i_\mu\leq n} \prod_{\sigma=1}^{\mu} a'(1i_\sigma) \prod_{\tau=1}^{n-\mu-1} a(1k_\tau) + \sum_{1<i_1<\cdots<i_{n-\mu}\leq n} \prod_{\sigma=1}^{n-\mu} a'(1i_\sigma) \prod_{\tau=1}^{\mu-1} a(1k_\tau)$$

with different values of μ are pairwise orthogonal idempotents which behave as the unity quantities in the corresponding two-sided ideals of **M**. Therefore they produce the different irreducible representations $R_\mu^{(n)}\left(\mu=0, 1, 2, \ldots, \frac{n}{2}-1\right)$ inequivalent one another; $e_\mu^{(n)} \leftarrow\rightarrow R_\mu^{(n)}$. In the case corresponding to the value $\frac{n}{2}$ of μ, we have the two types of pairwise orthogonal idempotents as follows:

$$\varepsilon_{\frac{n}{2}}^{(n)\pm} = \sum_{1<i_1<\cdots<i_{n/2}\leq n} e_{n/2}^{(n)\pm}(i_1, i_2, \ldots, i_{n/2} ; k_1, k_2, \ldots, k_{\frac{n}{2}-1}). \qquad (3\cdot7)$$

These are the idempotents which are the unity quantities of the two-sided ideals corresponding to the irreducible representations $K_{n/2}^{(n)\pm}$. Thus we derive the scheme of all irreducible representations inequivalent to one another as in fig. 2., and can write down the representative part of the algebraic table as shown in table 4 by making use of the relation (II. 3) in appendix II.

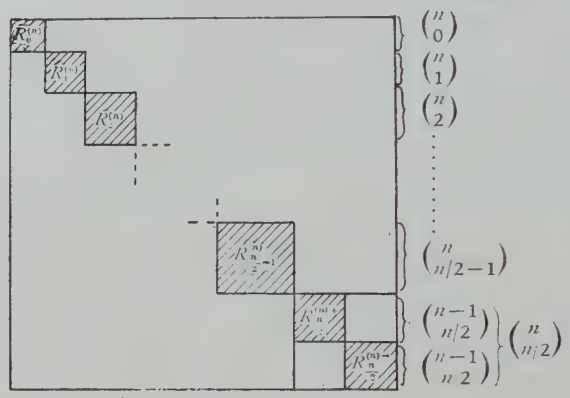


Fig. 2.

Table 4.

$\frac{1}{2}\left\{\prod_{\sigma=1}^{(n/2)-1} a'(1i_\sigma)a(1k_\sigma)a(1i_{n/2}) \pm \prod_{\sigma=1}^{(n/2)-1} b(1i_\sigma)b(1k_\sigma)b(1i_{n/2})\right\}$	$\frac{1}{2}\left\{\prod_{\sigma=1}^{\nu-1} b(1i_\sigma)b(1k_\sigma)b(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} a(1i_{\tau+1})a'(1k_\tau) \pm \prod_{\sigma=1}^{\nu-1} a'(1i_\sigma)a(1k_\sigma)a'(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} b(1i_{\tau+1})b(1k_\tau)\right\}$
$\frac{1}{2}\left\{\prod_{\sigma=1}^{\nu-1} b(1i_\sigma)b(1k_\sigma)b(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} a'(1i_{\tau+1})a(1k_\tau) \pm \prod_{\sigma=1}^{\nu-1} a'(1i_\sigma)a(1k_\sigma)a'(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} b(1k_\tau)b(1i_{\tau+1})\right\}$	$\frac{1}{2}\left\{\prod_{\sigma=1}^{\nu-1} a'(1i_\sigma)a(1k_\sigma)a'(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} a(1i_{\tau+1})a'(1k_\tau) \pm \prod_{\sigma=1}^{\nu-1} b(1i_\sigma)b(1k_\sigma)b(1i_\nu)\prod_{\tau=\nu}^{(n/2)-1} /(1i_{\tau+1})b(1k_\tau)\right\}$

In concluding this paragraph we call the reader's attention to the fact that the number μ indicating each irreducible representation has the relation to the total spin magnetic quantum number M of the system of all electrons in the singly-occupied states, as follows :

$$M=\frac{n}{2}-\mu. \tag{3·8}$$

§ 4. The structure of the exchange algebra, E

The magnetic algebra, **M**, itself is not important for us, but only the quantities of the combinations P_{ij} in (3·2) matter. In a word we have much interest in the exchange algebra which is engendered by all the quantity of the form of P_{ij} and is a subalgebra of **M**. The origin of the name 'exchange algebra' is made clear in appendix III. Every irreducible representation of **M** generally resolves to several irreducible representations of its subalgebra **E**, when it is regarded as the representation of **E**. In order to investigate the

detail of this resolution we introduce another subset **B** of **M** which is composed of the elements B_μ having the form

$$B_\mu = \sum_{1\leq i_1<i_2<\cdots<i_\mu\leq n} \sum_{1\leq k_1<k_2<\cdots<k_{\mu-1}\leq n\ (k_\tau\neq i_\sigma)} \prod_{\sigma=1}^{\mu-1} b(1i_\sigma)b(1k_\sigma)b(1i_\mu)$$

$$+ \sum_{1\leq i_1<i_2<\cdots<i_\mu\leq n} \sum_{1<k_1<k_2<\cdots<k_\mu\leq n\ (k_\tau\neq i_\sigma)} \prod_{\sigma=1}^{\mu} b(1i_\sigma)b(1k_\sigma) \qquad (4\cdot1)$$

$$= \frac{1}{\mu!} \sum_{1\leq i_1<i_2<\cdots<i_\mu\leq n} \sum_{k_1,k_2,\cdots,k_\mu=1}^{n\ (k_\tau\neq i_\sigma,\ k_\sigma\neq k_\tau)} \prod_{\sigma=1}^{\mu} b(i_\alpha k_\sigma),$$

$$\mu = \begin{cases} 0, 1, 2, \ldots, \dfrac{n-1}{2}, & \text{for odd } n, \\ 0, 1, 2, \ldots, \dfrac{n}{2}, & \text{for even } n. \end{cases}$$

The summations are to be extended over all possible combinations of 2μ or $2\mu-1$ numbers $(i_1, i_2, \ldots, i_\mu)$ and $(k_1, k_2, \ldots, k_{\mu-1})$ or $(k_1, k_2, \ldots, k_\mu)$ taken from 2, 3, ..., n in the first expression. In the second expression the number 1 is also allowed to be taken and the order of $k_1, k_2, \ldots, k_\mu$ is permutated freely. The equality of the first to the second expression can be proved by making use of the relation (3·3). If we define $B_0=1$, the subset engendered from all the elements of the form of B_μ constitute a subalgebra, **B**, of **M** and this is the commutative subalgebra as proved in appendix IV. Furthermore any element B_μ of **B** commutes with any element of **E**, viz., **B** and **E** are commutative each other. We find the proof in appendix III. When we consider **M**, **E** and **B** only in the ideal having the idempotent $\varepsilon_\mu^{(n)}$ or $\varepsilon_{n/2}^{(n)\pm}$ as its unity quantity, then **M** is normal simple[5], and **E** and **B** are both semisimple subalgebra[5] of **M** commuting each other. In this case the algebraist concludes that the centra of **E** and **B** coincide exactly[5]. As the centrum of the commutative subset **B** is **B** itself as shown in appendix IV, that of **E** is also **B**. Now we can construct the idempotent quantities by the proper linear combinations of B_μ, which play the rôle of the unity quantity of the two-sided ideals of **E** and therefore correpond to the irreducible representation of **E** which appears upon the resolution of $R_\mu^{(n)}$ or $R_{n/2}^{(n)\pm}$. The above consideration tells us that the number of irreducible representations of $R_\mu^{(n)}$ appearing when $R_\mu^{(n)}$ or $R_{n/2}^{(n)\pm}$ is regarded as the representation of **E** is equal to the number of independent elements of **B** considered in the matric form in $R_\mu^{(n)}$ or $R_{n/2}^{(n)\pm}$.

§ 5. The determination of the correspondence between our irreducible representations of the exchange algebra and the multiplicities of the spectroscopic terminology

We have seen in § 3 the relation (3·7) between the index μ of $R_\mu^{(n)}$ and the total spin magnetic quantum number M of all electrons in the singly-occupied states. Then

the representation $R_\mu^{(n)}$ contains all states having the value $M=\frac{n}{2}-\mu$ of the total spin magnetic quantum number, or the states having the values S of the total spin quantum number equal to or greater than $\frac{n}{2}-\mu$, or

$$S=\frac{n}{2}-\mu,\quad \frac{n}{2}-\mu+1,\quad \frac{n}{2}-\mu+2,\ldots,\quad \frac{n}{2}-1,\quad \frac{n}{2}. \tag{5·1}$$

Therefore the character $\chi_\mu'^{(n)}(x)$ of any quantity x of $\mathbf{E}$ in the representation space $R_\mu^{(n)}$ is the diagonal sum of this quantity extended over the states having the value $\frac{n}{2}-\mu$ of the total spin magnetic quantum number, or the values S equal to or greater than $\frac{n}{2}-\mu$ of the total spin quantum number, and we can obtain the character $\chi_S^{(n)}(x)$ in the irreducible representation of $\mathbf{E}$ corresponding to the value S of the total spin quantum number (multiplicity $2S+1$) by means of the relation*

$$\chi_S(x)=\chi'_{\frac{n}{2}-S}(x)-\chi'_{\frac{n}{2}-S-1}(x). \tag{5·2}$$

By taking the unity quantity as x, we can find the dimension $D_S^{(n)}$ of the irreducible representation of multiplicity $2S+1$, viz.

$$D_S^{(n)}=\chi'^{(n)}_{\frac{n}{2}-S}(1)-\chi'^{(n)}_{\frac{n}{2}-S-1}(1)=\binom{n}{\frac{n}{2}-S}. \tag{5·3}$$

The value of $\chi_\mu'^{(n)}(P_{ij})$ can be derived, by making use of the fact that this is equal to the number of the pairwise orthogonal idempotents, constituting $\varepsilon_\mu^{(n)}$ which have the factor $a(1j)$, when $i=1$, and the factor $a(1i)a(1j)$ or $a'(1i)a'(1j)$ when $i\neq 1$, $j\neq 1$. This can be evaluated easily, and is in both cases

$$\chi_\mu'^{(n)}(P_{ij})=\binom{n-2}{\mu}+\binom{n-2}{\mu-2}$$

Thus we derive

$$\begin{aligned}\chi_S^{(n)}(P_{ij})&=\chi'^{(n)}_{\frac{n}{2}-S}(P_{ij})-\chi'^{(n)}_{\frac{n}{2}-S-1}(P_{ij})\\&=\frac{(n-2)!(2S+1)\left\{2S(S+1)-\frac{n^2}{2}-2n\right\}}{\left(\frac{n}{2}-S\right)!\left(\frac{n}{2}+S+1\right)!}.\end{aligned} \tag{5·4}$$

We can now decide to which multiplet each particular representation to be acquired corresponds by comparing the dimension and the character of P_{ij} in this representation space and the values derived from (5·3) and (5·4). These values which will be required

* Out of the states indicated by an S value of the total spin quantum number in (5·1), only one state, viz. the one having the total spin-magnetic quantum number M, appears in $R_\mu^{(n)}$.

in § 6 are tabulated in table 5. Because the representation space $R_{\mu}^{(n)}$ having the maximum value of μ contains all multiplets, it is sufficient to study only this. It may be noted that $R_{\frac{n}{2}}^{(n)}$ is used in the meaning of the direct sum of $R_{\frac{n}{2}}^{(n)+}$ and $R_{\frac{n}{2}}^{(n)-}$ viz. $R_{\frac{n}{2}}^{(n)} = R_{\frac{n}{2}}^{(n)+} + R_{\frac{n}{2}}^{(n)-}$ throughout this paragraph.

Table 5.

n	2		3		4			5			6				7			
S	1	0	$\frac{3}{2}$	$\frac{1}{2}$	2	1	0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	3	2	1	0	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$D_S^{(n)}$	1	1	1	2	1	3	2	1	4	5	1	5	9	5	1	6	14	14
$\chi_S^{(n)}(P_{tt})$	1	−1	1	0	1	1	0	1	2	1	1	3	3	1	1	4	6	4

§ 6. Some concrete representation matrices

We shall work out the general method mentioned in the preceding paragraphs in the cases of a few numbers n of singly-occupied states. We make use of the symbols a_i, a_i', b_i instead of the former symbols $a(1i)$, $a'(1i)$, $b(1i)$ in order to save the space in this paragraph.

$n=2$. The algebraic table is shown in Table 6. It is sufficient to consider $R_1^{(2)\pm}$ only. In $R_1^{(2)+}$, $P_{12} \simeq 1$, and in $R_1^{(2)-}$, $P_{12} \simeq -1$. The former corresponds to $S=1$, and the latter to $S=0$, as can be seen from table 5.

$$R_0^{(2)} \to \boxed{a_2}$$

$$R_1^{(2)+} \to \boxed{\frac{1}{2}(a_2+b_2)}$$

$$R_1^{(2)-} \to \boxed{\frac{1}{2}(a_2-b_2)}$$

Table 6.

$n=3$ The algebraic table in this case is shown in Table 7. According to the rule of appendix V, we can see in $R_1^{(3)}$,*

$$P_{12} \simeq \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline \end{array}, \quad P_{13} \simeq \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & \\ \hline 1 & & \\ \hline \end{array}, \quad P_{23} \simeq \begin{array}{|c|c|c|}\hline & 1 & \\ \hline 1 & & \\ \hline & & 1 \\ \hline \end{array} \qquad (6\cdot1)$$

$$B_0 \simeq \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array}, \quad B_1 \simeq \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline 1 & & 1 \\ \hline 1 & 1 & \\ \hline \end{array}.$$

These matrices of B_0 and B_1 have the relation

$$B_1^2 = 2B_0 + B_1, \qquad (6\cdot2)$$

which leads to two pairwise orthogonal idempotent matrices

* Zero matrix elements are as usual omitted to be written down througout this paragraph.

$$\varepsilon_1=\frac{1}{3}(B_0+B_1)=\begin{vmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{vmatrix}, \tag{6·3}$$

$$\varepsilon_2=\frac{1}{3}(2B_0-B_1)=\begin{vmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{vmatrix}. \tag{6·4}$$

$R_0^{(3)} \to \left| \; a_2a_3 \; \right|$

$R_1^{(3)} \to \begin{array}{|c|c|c|} \hline a_2a_3' & b_3b_2 & a_2b_3 \\ \hline b_2b_3 & a_2'a_3 & a_3b_2 \\ \hline a_2'b_3 & a_3'b_2 & a_2'\,a_3' \\ \hline \end{array}$ Table 7.

Let the basic vectors of $R_1^{(3)}$ be u_1, u_2, u_3, and we write the proper vector of ε_1 and ε_2 in the form as

$$U=\xi_1u_1+\xi_2u_2+\xi_3u_3=(u_1, u_2, u_3)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}.$$

In order that this may be the proper vector of ε_1, the following condition must be satisfied.

$$\varepsilon_1\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{vmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{vmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}.$$

From this we obtain the relation $\xi_1=\xi_2=\xi_3$, whence the proper vector of ε_1 is

$$U=u_1+u_2+u_3. \tag{6·5}$$

According to their matric form (6·1) we see that each P_{ij} transforms the basic vectors in the following manner.

$$P_{12}:\ u_1\to u_1,\ u_2\to u_3,\ u_3\to u_2,$$

$$P_{13}:\ u_1\to u_3,\ u_2\to u_2,\ u_3\to u_1,$$

$$P_{23}:\ u_1\to u_2,\ u_2\to u_1,\ u_3\to u_3.$$

All these transform U to U itself, viz. every $P_{ij}\simeq 1$ (the identity representation $S=3/2$).

The proper vector of ε_2 must satisfy the condition

$$\varepsilon_2\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{vmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{vmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}.$$

Independent solutions of this equation are $\xi_1=1$, $\xi_2=0$, $\xi_3=-1$; and $\xi=0$, $\xi_2=1$, $\xi_3=-1$, and thus there exist two idependent proper vectors U_1, U_2 as follows:

$$U_1=u_1-u_3,\quad U_2=u_2-u_3. \tag{6·6}$$

These vectors are transformed by each P_{ij}, as follows,

$$P_{12}: U_1 \to U_1 - U_2,\ U_2 \to -U_2,$$
$$P_{13}: U_1 \to -U_1,\ U_2 \to -U_1 + U_2,$$
$$P_{23}: U_1 \to U_2,\ U_2 \to U_1,$$

whence the representation matrices can be written down.

$$P_{12} \simeq \begin{vmatrix} 1 & & \\ -1 & -1 & \\ & & \end{vmatrix}, \quad P_{13} \simeq \begin{vmatrix} -1 & & \\ & -1 & \\ & 1 & \end{vmatrix}, \quad P_{23} \simeq \begin{vmatrix} & & \\ 1 & & \\ & & 1 \end{vmatrix}. \tag{6·7}$$

This representation can be seen to correspond to $S=1/2$.

$n=4$. We show the algebraic table for $R_2^{(4)\pm}$ in Table 8, where the double sign $\pm$ corresponds to $R_2^{(4)+}$ and $R_2^{(4)-}$ respectively. First we study $R_2^{(4)+}$.

The matric forms of the quantities in this space are known from this table, as follows,

$$P_{12} \simeq P_{34} \simeq \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}, \quad P_{13} \simeq P_{24} \simeq \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}, \quad P_{23} \simeq P_{14} \simeq \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix},$$

$$B_0 \simeq \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}, \quad B_1 \simeq \begin{vmatrix} & 2 & 2 \\ 2 & & 2 \\ 2 & 2 & \end{vmatrix}, \quad B_2 \simeq \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} \simeq B_0.$$

From these we can obtain the idempotents,

$$\varepsilon_1 = \frac{1}{3}(B_0 + B_1) \simeq \begin{vmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{vmatrix}, \quad \varepsilon_2 = \frac{1}{3}(2B_0 - B_1) \simeq \begin{vmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{vmatrix}.$$

All circumstances are quite similar to the case $n=3$ as far as P_{12}, P_{13}, P_{23} concern. We attain the identity representation, $S=2$, and another representation, $S=0$, as follows,

$$P_{12} \simeq P_{34} \simeq \begin{vmatrix} 1 & & \\ -1 & -1 & \\ & & \end{vmatrix}, \quad P_{13} \simeq P_{24} \simeq \begin{vmatrix} -1 & & \\ & -1 & \\ & 1 & \end{vmatrix}, \quad P_{14} \simeq P_{23} \simeq \begin{vmatrix} & & \\ 1 & & 1 \\ & & \end{vmatrix}. \tag{6·8}$$

In $R_2^{(4)-}$, matrices attached to the exchange quantities are

$$P_{12} \simeq \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}, \quad P_{13} \simeq \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}, \quad P_{14} \simeq \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix}, \tag{6·9}$$

$$P_{23} \simeq \begin{vmatrix} & -1 & \\ -1 & & \\ & & 1 \end{vmatrix}, \quad P_{24} \simeq \begin{vmatrix} & & -1 \\ & 1 & \\ -1 & & \end{vmatrix}, \quad P_{34} \simeq \begin{vmatrix} 1 & & \\ & & -1 \\ & -1 & \end{vmatrix}.$$

Those for B_μ are

$$B_0 \simeq \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}, \quad B_1 \simeq 0, \quad B_2 \simeq \begin{vmatrix} -1 & & \\ & -1 & \\ & & -1 \end{vmatrix} \simeq -B_0.$$

We have only B_0, the identity matrix, as the idempotent derived from linear combinations of B_μ. Then $R_2^{(4)-}$ remains irreducible still when restricted to **E**, and the irreducible representation matrices of P_{ij} are just those in Exp. (6·7), which correspond to $S=1$.

Table 8.

$\frac{1}{2}(a_3'a_2a_4'\pm b_3b_2b_4)$	$\frac{1}{2}(a_3'a_2b_4\pm b_3b_2a_4')$	$\frac{1}{2}(a_4'a_2b_3\pm b_4b_2a_3')$
$\frac{1}{2}(a_2'a_3b_4\pm c_2b_3a_4')$	$\frac{1}{2}(a_2'a_3a_4'\pm b_2b_3b_4)$	$\frac{1}{2}(a_4'a_3b_2\pm b_4b_3b_2')$
$\frac{1}{2}(a_2'a_4b_3\pm b_2b_3a_3')$	$\frac{1}{2}(a_3'a_4b_2\pm b_3b_4a_2')$	$\frac{1}{2}(a_2'a_4a_3'\pm b_2b_4b_3)$

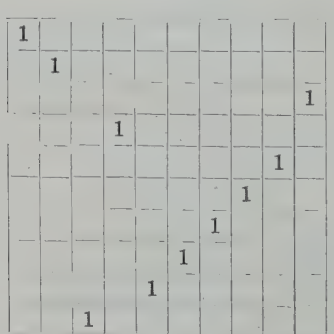
$n=5$ The algebraic table of $R_2^{(5)}$ is shown in table 9.

Table 9.

$a_2a_3a_4'a_5'$	$a_2a_5'b_4b_3$	$a_3a_5'b_4b_2$	$a_2a_4'b_5b_3$	$a_3a_4'b_5b_2$	$b_4b_2b_5b_3$	$a_3b_4b_2b_5$	$a_2b_4b_3b_5$	$a_2a_3a_4'b_5$	$a_2a_3a_5'b_4$
$a_2a_5'b_3b_4$	$a_2a_5'a_4a_5'$	$a_4a_5'b_3b_2$	$a_2a_3'b_5b_4$	$b_3b_2b_5b_4$	$a_3'a_4b_5b_2$	$a_4b_3b_2b_5$	$a_2b_3'a_4b_5$	$a_2b_3b_4b_5$	$a_2a_4a_5'b_3$
$a_3a_5'b_2b_4$	$a_4a_5'b_2b_3$	$a_2'a_3a_4a_5'$	$b_2b_3b_5b_4$	$a_2'a_3b_5b_4$	$a_2'a_4b_5b_3$	$a_2'a_3a_4b_5$	$a_4b_2b_3b_5$	$a_3b_2b_4b_5$	$a_3a_4a_5'b_2$
$a_2a_4'b_3b_5$	$a_2a_3'b_4b_5$	$b_3b_2b_4b_5$	$a_2a_3'a_4'a_5$	$a_4'a_5b_3b_2$	$a_3'a_5b_4b_2$	$a_5b_4b_2b_3$	$a_2a_3'a_5b_4$	$a_2a_4'a_5b_3$	$a_2b_4b_5b_3$
$a_3a_4'b_2b_5$	$b_2b_3b_4b_5$	$a_2'a_3b_4b_5$	$a_4'a_5b_2b_3$	$a_2'a_3a_4'a_5$	$a_2'a_5b_4b_3$	$a_2'a_3a_5b_4$	$a_5b_4b_3b_2$	$a_3a_4'a_5b_2$	$a_3b_4b_5b_2$
$b_2b_4b_3b_5$	$a_3'a_4b_2b_5$	$a_2'a_4b_3b_5$	$a_3'a_5b_2b_4$	$a_2'a_5b_3b_4$	$a_2'a_3'a_4a_5$	$a_2'a_3a_5b_3$	$a_3'a_4a_5b_2$	$a_5b_2b_4b_3$	$a_4b_2b_5b_3$
$a_3'b_4b_2b_5$	$a_4'b_3b_2b_5$	$a_2a_3'a_4'b_5$	$a_5'b_3b_2b_4$	$a_2a_3'a_5'b_4$	$a_3a_4'a_5'b_3$	$a_2a_3'a_4'a_5'$	$a_4'a_5'b_3b_2$	$a_3'a_5'b_4b_2$	$a_3'a_4'b_5b_2$
$a_2'b_4b_3b_5$	$a_2'a_3a_4'b_5$	$a_3'b_2b_3b_5$	$a_2'a_3a_5'b_4$	$a_5'b_2b_3b_4$	$a_3a_4'a_5'b_2$	$a_4'a_5'b_2b_4$	$a_2'a_3a_4'a_5'$	$a_2'a_5'b_4b_3$	$a_2'a_4'b_5b_3$
$a_2'a_3'a_4'b_5$	$a_2'b_3b_4b_5$	$a_3'b_2b_4b_5$	$a_2'a_4a_5'b_3$	$a_3'a_4a_5'b_2$	$a_5'b_2b_4b_3$	$a_3'a_5b_2b_4$	$a_2'a_5'b_3b_4$	$a_2'a_3'a_4a_5'$	$a_2'a_3'b_5b_4$
$a_2'a_5'a_5b_4$	$a_2'a_4'b_5b_3$	$a_3'a_4'a_5b_2$	$a_2'b_3b_5b_4$	$a_3'b_2b_5b_4$	$a_4'b_2b_5b_3$	$a_3'a_4'b_3b_5$	$a_2'a_4'b_3b_5$	$a_2'a_3'b_4b_5$	$a_2'a_3'a_4'$

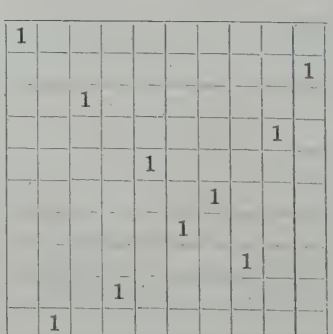
The matrices of the quantities are

$P_{12} \simeq$

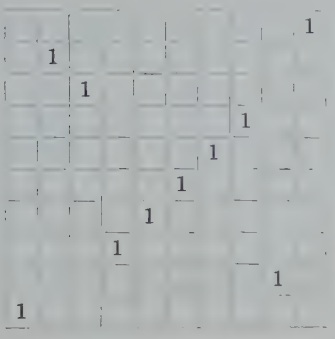


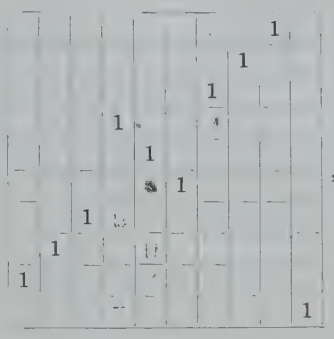
,

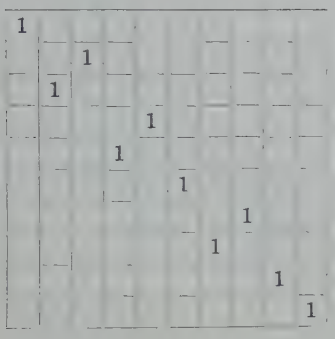
$P_{13} \simeq$

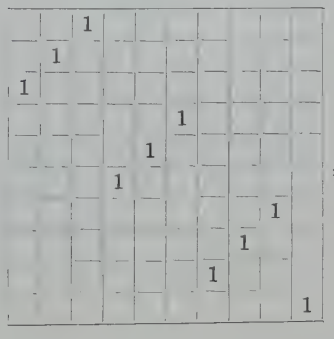


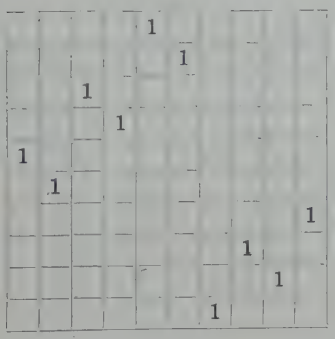
,

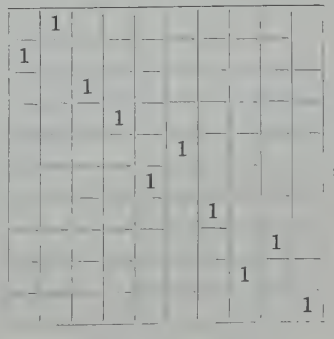
$P_{14} \simeq$  ,

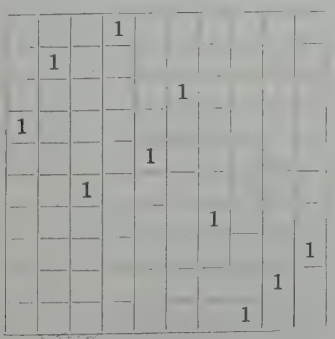
$P_{15} \simeq$  ,

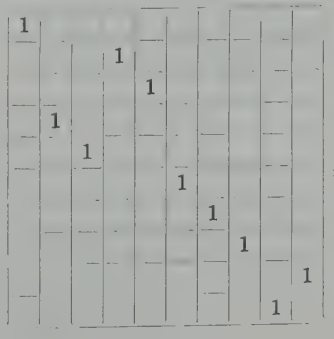
$P_{23} \simeq$  ,

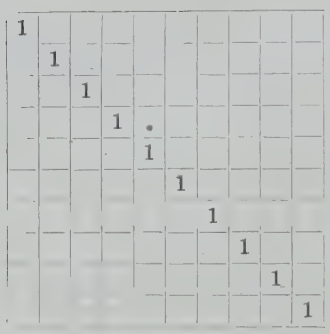
$P_{24} \simeq$  ,

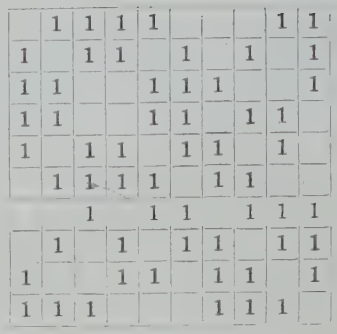
$P_{25} \simeq$  ,

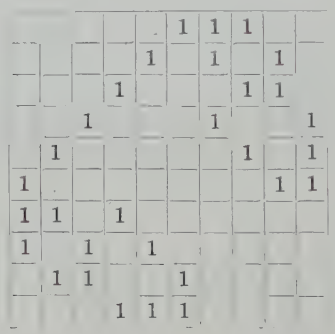
$P_{34} \simeq$  ,

$P_{35} \simeq$  ,

$P_{45} \simeq$  ,

$B_0 \simeq$  ,

$B_1 \simeq$  ,

$B_2 \simeq$  .

The multiplicative laws among these matrices of B_μ are following.

$$B_1^2=6B_0+3B_1+4B_1,\quad B_1B_2=B_2B_1=2B_1+2B_2,$$

$$B_2^2=3B_0+B_1.$$

By making use of these laws, we derive the following three pairwise orthogonal idempotents.

$$\varepsilon_1=\frac{1}{10}(B_0+B_1+B_2),\qquad \varepsilon_2=\frac{1}{15}(6B_0+B_1-4B_2),$$

$$\varepsilon_3=\frac{1}{6}(3B_0-B_1+B_2).$$

There is only one proper vector of ε_1, $U=\sum_{i=1}^{10} u_i$, in terms of the original basic vector u_i. The identity representation, $S=5/2$, is derived from this, viz. every $l'_{ij}\simeq 1$. The proper vectors of ε_2 are

$$U_1=u_3+u_5+u_6-2(u_1+u_2+u_4)+3u_7,$$

$$U_2=u_2+u_4+u_6-2(u_1+u_3+u_5)+3u_8,$$

$$U_3=u_1+u_4+u_5-2(u_2+u_3+u_6)+3u_9,$$

$$U_4=u_1+u_2+u_3-2(u_4+u_5+u_6)+3u_{10}.$$

This vector space gives the following representation, $S=3/2$,

$$P_{12} \simeq \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} & -\tfrac{2}{3} \\ \hline \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ \hline \tfrac{1}{3} & -\tfrac{2}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \\ \hline \end{array}, \quad P_{13} \simeq \begin{array}{|c|c|c|c|}\hline \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} & -\tfrac{2}{3} \\ \hline & 1 & & \\ \hline -\tfrac{2}{3} & \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ \hline -\tfrac{2}{3} & \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \\ \hline \end{array}, \quad P_{14} \simeq \begin{array}{|c|c|c|c|}\hline \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ \hline -\tfrac{2}{3} & \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ \hline & & 1 & \\ \hline -\tfrac{2}{3} & -\tfrac{2}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\ \hline \end{array}, \quad P_{15} \simeq \begin{array}{|c|c|c|c|}\hline \tfrac{1}{3} & -\tfrac{2}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \\ \hline -\tfrac{2}{3} & \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \\ \hline -\tfrac{2}{3} & -\tfrac{2}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\ \hline & & & 1 \\ \hline \end{array},$$

$$P_{23} \simeq \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline 1 & & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array}, \quad P_{24} \simeq \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline \end{array}, \quad P_{25} \simeq \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline 1 & & & \\ \hline \end{array}, \quad P_{34} \simeq \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & 1 & \\ \hline & 1 & & \\ \hline & & & 1 \\ \hline \end{array},$$

$$P_{35} \simeq \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & & 1 & \\ \hline & 1 & & \\ \hline 1 & & & \\ \hline \end{array}, \quad P_{45} \simeq \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array}. \qquad (6\cdot10)$$

Lastly we derive the proper vectors of $\mathfrak{s}_3$ in the following.

$$U_1=u_1-u_6+u_7+u_8-u_9-u_{10},$$
$$U_2=u_2-u_6+u_7-u_{10},$$
$$U_3=u_3-u_6+u_8-u_{10},$$
$$U_4=u_4-u_6+u_7-u_9,$$
$$U_5=u_5-u_6+u_8-u_9.$$

This vector space leads to the representation corresponding to the state, $S=1/2$, and gives the matrices,

$$P_{12} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & 1 & & & \\ \hline -1 & -1 & -1 & & \\ \hline & & & 1 & \\ \hline -1 & & & -1 & -1 \\ \hline \end{array}, \quad P_{13} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline -1 & -1 & -1 & & \\ \hline & & 1 & & \\ \hline -1 & & & -1 & -1 \\ \hline & & & & 1 \\ \hline \end{array}, \quad P_{14} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & -1 & -1 & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline 1 & & 1 & & 1 \\ \hline 1 & 1 & & 1 & \\ \hline \end{array},$$

$$P_{15} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & & & -1 & -1 \\ \hline 1 & & 1 & & \\ \hline 1 & 1 & & 1 & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array}, \quad P_{23} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline & & & & 1 \\ \hline & & & 1 & \\ \hline \end{array}, \quad P_{24} \simeq \begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline & & & & 1 \\ \hline \end{array}, \qquad (6\cdot11)$$

$$P_{25} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & -1 & -1 & -1 & 1 \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline 1 & & & & \\ \hline \end{array}, \quad P_{34} \simeq \begin{array}{|c|c|c|c|c|}\hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline \end{array}, \quad P_{35} \simeq \begin{array}{|c|c|c|c|c|}\hline & & & 1 & \\ \hline & 1 & & & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline 1 & & & & \\ \hline & & & & 1 \\ \hline \end{array}, \quad P_{45} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & & 1 & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & & 1 \\ \hline \end{array}.$$

$n=6$ We show the algebraic table of $R_3^{(6)\pm}$ in Table 10. We at first consider $R_3^{(6)+}$, where the matrices representing the quantities in **B** are the following.

$B_0 \simeq$ the unit matrix of degree 10;

$B_1 \simeq$ the matrix of degree 10 which diagonal elements are all zero and non-diagonal elements are all one, $B_2 \simeq B_1$, $B_3 \simeq B_0$.

Thus the independent quantities of **B** are only B_0 and B_1 in $R_3^{(6)+}$, and they have the multiplicative relations

$$B_1^2 = 9B_0 + 8B_1,$$

whence we get the two pairwise orthogonal idempotents

$$\varepsilon_1 = \frac{1}{10}(B_0 + B_1), \qquad \varepsilon_2 = \frac{1}{10}(9B_0 - B_1).$$

The only proper vector of ε_1 is

$$U = \sum_{i=1}^{10} u_i,$$

which gives rise to the identity representation of **E**, $S=3$, viz. every $P_{ij} \simeq 1$. We next obtain the nine proper vectors of ε_2, as follows:

$$U_i = u_i - u_{10}, \qquad (i = 1, 2, \ldots, 9).$$

From these we derive the representation, $S=1$, of **E**.

$P_{12} \simeq$

1									
	1								
		1							
			1						
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
								1	
							1		
						1			
					1				

,

$P_{13} \simeq$

1									
−1	−1	−1	−1	−1	−1	−1	−1	−1	1
								1	
							1		
						1			
					1				
				1					
			1						
		1							

,

$P_{14} \simeq$

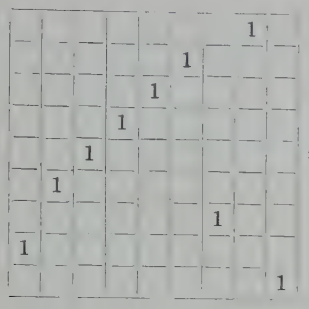
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
	1								
						1			
					1				
				1					
			1						
		1							
							1		
								1	

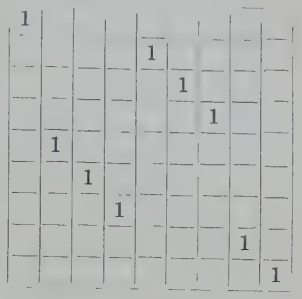
,

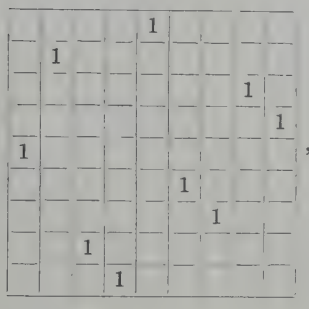
$P_{15} \simeq$

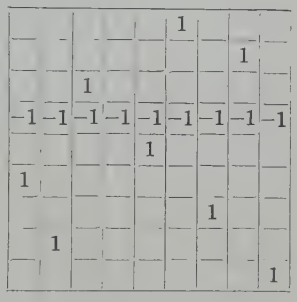
									1
							1		
			1						
					1				
				1					
						1			
	1								
								1	
1									

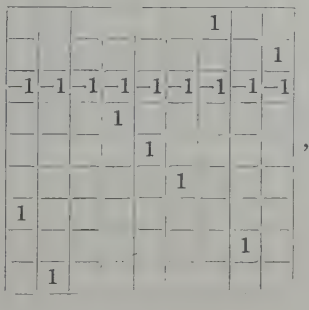
,

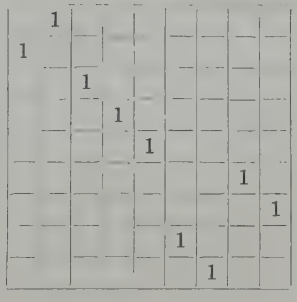
$P_{16} \simeq$  ,

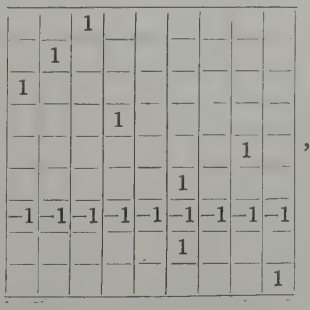
$P_{23} \simeq$  ,

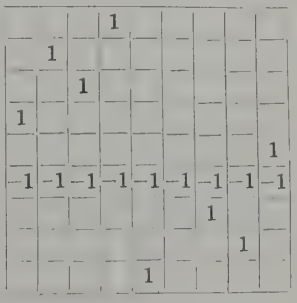
$P_{24} \simeq$  ,

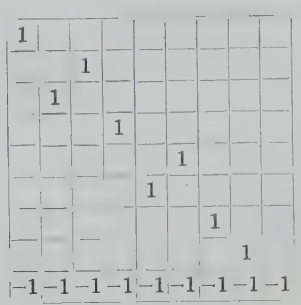
$P_{25} \simeq$  ,

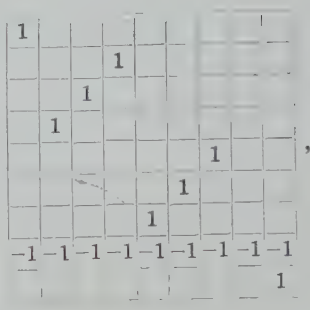
$P_{26} \simeq$  ,

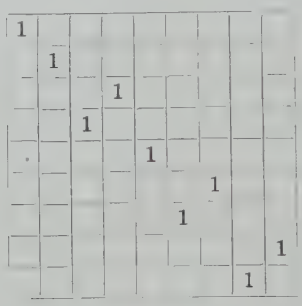
$P_{34} \simeq$  ,

$P_{35} \simeq$  ,

$P_{36} \simeq$  ,

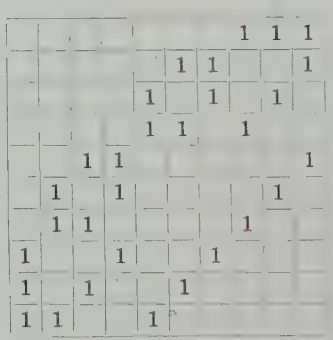
$P_{45} \simeq$  ,

$P_{46} \simeq$  ,

$P_{56} \simeq$  . (6·12)

Next in $R_3^{(6)-}$ the independent quantities in **B** are as follows,

$B_0 \simeq -B_3 \simeq$ unit matrix of degree 10,

$-B_1 \simeq B_2 \simeq$  .

Between these there exists the relation,

$$B_1^2 = 9B_0,$$

whence we derive the following two pairwise orthogonal idempotents

$$\varepsilon_1 = \frac{1}{6}(3B_0 + B_1), \qquad \varepsilon_2 = \frac{1}{6}(3B_0 - B_1).$$

The proper vectors of ε_1 are

$$U_1 = u_1 - u_3 - u_9 - u_{10},$$

$$U_2 = u_2 - u_6 - u_7 - u_{10},$$

$$U_3 = u_3 + u_6 + u_8 + u_{10},$$

$$U_4 = u_4 + u_7 + u_9 + u_{10},$$

$$U_5 = u_5 + u_6 + u_7 + u_8 + u_9 + u_{10}.$$

In this vector space we find the following representation, $S=2$, of **E**.

$$P_{12} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline 1 & 1 & -1 & -1 & -1 \\ \hline \end{array}, \qquad P_{13} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline 1 & 1 & -1 & -1 & -1 \\ \hline 1 & & & -1 & -1 \\ \hline 1 & & -1 & & -1 \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{14} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & -1 & -1 & -1 \\ \hline & 1 & & & \\ \hline & 1 & & -1 & -1 \\ \hline & 1 & -1 & & -1 \\ \hline & & & & 1 \\ \hline \end{array},$$

$$P_{15} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & -1 & -1 \\ \hline & 1 & & -1 & -1 \\ \hline & & 1 & & \\ \hline & & & & -1 \\ \hline & & & -1 & \\ \hline \end{array}, \qquad P_{16} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & -1 & & -1 \\ \hline & 1 & -1 & & -1 \\ \hline & & & & -1 \\ \hline & & & 1 & \\ \hline & & -1 & & \\ \hline \end{array}, \qquad P_{23} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & & & 1 \\ \hline & -1 & 1 & & 1 \\ \hline & -1 & & 1 & 1 \\ \hline & 1 & & & \\ \hline \end{array},$$

$$P_{24} \simeq \begin{array}{|c|c|c|c|c|}\hline & & & & 1 \\ \hline & 1 & & & \\ \hline -1 & & 1 & & 1 \\ \hline -1 & & & 1 & 1 \\ \hline -1 & & & & \\ \hline \end{array}, \qquad P_{25} \simeq \begin{array}{|c|c|c|c|c|}\hline & -1 & 1 & & 1 \\ \hline -1 & & 1 & & 1 \\ \hline & & 1 & & \\ \hline -1 & -1 & 1 & 1 & 1 \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{26} \simeq \begin{array}{|c|c|c|c|c|}\hline & -1 & & 1 & 1 \\ \hline -1 & & & 1 & 1 \\ \hline -1 & -1 & 1 & 1 & 1 \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array},$$

$$P_{34} \simeq \begin{array}{|c|c|c|c|c|}\hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{35} \simeq \begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & & 1 & \\ \hline -1 & & 1 & & 1 \\ \hline \end{array}, \qquad P_{36} \simeq \begin{array}{|c|c|c|c|c|}\hline & & & 1 & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline 1 & & & & \\ \hline -1 & & & 1 & 1 \\ \hline \end{array},$$

$$P_{45} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline & & & 1 & \\ \hline & -1 & 1 & & 1 \\ \hline \end{array}, \qquad P_{46} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline & -1 & & 1 & 1 \\ \hline \end{array}, \qquad P_{56} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & 1 & & & \\ \hline & & & 1 & \\ \hline & & 1 & & \\ \hline & & & & 1 \\ \hline \end{array}.$$

(6·13)

The proper vectors of ε_2 are

$$U_1 = u_1 - u_4 - u_7 - u_8 + u_9 + u_{10},$$

$$U_2 = u_2 - u_4 - u_8 + u_{10},$$

$$U_3 = u_3 - u_4 - u_8 + u_9,$$

$$U_4 + u_5 - u_7 - u_8 + u_{10},$$

$$U_5 = u_6 - u_7 - u_8 + u_9.$$

This space bears the following representation, $S=0$,

$$P_{12} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline -1 & -1 & & -1 & \\ \hline -1 & & -1 & & -1 \\ \hline \end{array}, \qquad P_{13} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline -1 & -1 & & -1 & \\ \hline -1 & & -1 & & -1 \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{14} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & -1 & & -1 & \\ \hline & 1 & & & \\ \hline 1 & & & 1 & 1 \\ \hline & & & 1 & \\ \hline 1 & 1 & 1 & & \\ \hline \end{array},$$

$$P_{15} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & & -1 & & -1 \\ \hline 1 & & & 1 & 1 \\ \hline & & 1 & & \\ \hline 1 & 1 & 1 & & \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{16} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline & & & & -1 \\ \hline & & & -1 & \\ \hline & & -1 & & \\ \hline & -1 & & & \\ \hline \end{array}, \qquad P_{23} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline \end{array},$$

$$P_{24} \simeq \begin{array}{|c|c|c|c|c|}\hline & & & 1 & \\ \hline & 1 & & & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline 1 & & & & \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{25} \simeq \begin{array}{|c|c|c|c|c|}\hline & & & & 1 \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline 1 & & & & \\ \hline \end{array}, \qquad P_{26} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & & & -1 & -1 \\ \hline 1 & & 1 & & 1 \\ \hline 1 & 1 & & 1 & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array},$$

$$P_{34} \simeq \begin{array}{|c|c|c|c|c|}\hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline \end{array}, \qquad P_{35} \simeq \begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{36} \simeq \begin{array}{|c|c|c|c|c|}\hline -1 & -1 & -1 & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline 1 & & 1 & & 1 \\ \hline 1 & 1 & & 1 & \\ \hline \end{array},$$

$$P_{45} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline & & & & 1 \\ \hline & & & 1 & \\ \hline \end{array}, \qquad P_{46} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline -1 & -1 & -1 & & \\ \hline & & 1 & & \\ \hline -1 & & & -1 & -1 \\ \hline & & & & 1 \\ \hline \end{array}, \qquad P_{56} \simeq \begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline & 1 & & & \\ \hline -1 & -1 & -1 & & \\ \hline & & & 1 & \\ \hline -1 & & & -1 & -1 \\ \hline \end{array}.$$

(6·14)

$n=7$ The algebraic table of $R_3^{(7)}$ requires too tremendously large space to be written down, and we are forced to do without it. We can derive the four pairwise orthogonal idempotents ε_1, ε_2, ε_3, ε_4 of the linear combinations of B_μ in $R_3^{(7)}$. These give us the irreducible representation of $\mathbf{E}$ of degree 1, 6, 14 and 14. The first is the identity representation, $S=7/2$: every $P_{ij} \simeq 1$. The second, $S=5/2$, is as follows,

Table 10

$a_4'a_2a_5'a_3a_6'$ $\pm b_4b_2b_5b_3b_6$	$b_5b_2b_6a_4a_3'$ $\pm a_5'a_2a_6'b_4b_3$	$b_4b_2b_6a_5a_3'$ $\pm a_4'a_2a_6'b_5b_3$	$b_4b_2b_5a_6a_3'$ $\pm a_4'a_2a_5'a_6b_3$	$b_5b_3b_6a_4a_2'$ $\pm a_5'a_3a_6'b_4b_2$	$b_4b_3b_6a_5a_2'$ $\pm a_4'a_3a_6'b_5b_2$	$b_4b_3b_5a_6a_2'$ $\pm a_4'a_3a_5'b_6b_2$	$b_6a_4a_2'a_5a_3'$ $\pm a_6'b_4b_2b_5b_3$	$b_5a_4a_2'a_6a_3'$ $\pm a_5'b_4b_2b_6b_3$	$b_4a_5a_2'a_6a_3'$ $\pm a_4'b_5b_2b_6b_3$
$b_5b_2b_6a_3a_4'$ $\pm a_5'a_2a_6'bb$	$a_3'a_2a_5'a_4a_6$ $\pm b_3b_2b_5b_4b_6$	$b_3b_2b_6a_5a_4'$ $\pm a_3'a_2a_6'b_5b_4$	$b_3b_2b_5a_6a_4'$ $\pm a_3'a_2a_5'b_6b_4$	$b_5b_4b_6a_3a_2'$ $\pm a_5'a_4a_6'b_3b_2$	$b_6a_3a_2'a_5a_4'$ $\pm a_6'b_3b_2b_5b_4$	$b_5a_3a_2'a_6a_4'$ $\pm a_5'b_3b_2b_6b_4$	$b_3b_4b_6a_5a_2'$ $\pm a_3'a_4a_6'b_5b_2$	$b_3b_4b_5a_6a_2'$ $\pm a_3'a_4a_5'b_6b_2$	$b_3a_5a_2'a_6a_4'$ $\pm a_3'b_5b_2b_6b_4$
$b_4b_2b_6a_3a_5'$ $\pm a_4'a_2a_6'b_3b_5$	$b_3b_2b_6a_4a_5'$ $\pm a_3'a_2a_6'b_4b_5$	$a_3'a_2a_4'a_5a_6'$ $\pm b_3b_2b_4b_5b_6$	$b_3b_2b_4a_6a_5'$ $\pm a_3'a_2a_4'b_6b_5$	$b_6a_3a_2'a_4a_5'$ $\pm a_6'b_3b_2b_4b_5$	$b_4b_5b_6a_3a_2'$ $\pm a_4'a_5'a_6'b_3b_2$	$b_4a_3a_2'a_6a_5'$ $\pm a_4'b_3b_2b_6b_5$	$b_3b_5b_6a_4a_2'$ $\pm a_3'a_5'a_6'b_4b_2$	$b_3a_4a_2'a_6a_5'$ $\pm a_3'b_4b_2b_6b_5$	$b_3b_5b_4a_6a_2'$ $\pm a_3'a_5'a_4'b_6b_2$
$b_4b_2b_5a_3a_6'$ $\pm a_4'a_2a_5'b_3b_6$	$b_3b_2b_5a_4a_6'$ $\pm a_3'a_2a_5'b_4b_6$	$b_3b_2b_4a_5a_6'$ $\pm a_3'a_2a_4'b_5b_6$	$a_3'a_2a_4'a_6a_5'$ $\pm b_3b_2b_4b_6b_5$	$b_5a_3a_2'a_4a_6'$ $\pm a_5'b_3b_2b_4b_6$	$b_4a_3a_2'a_5a_6'$ $\pm a_4'b_3b_2b_5b_6$	$b_4b_6b_5a_3a_2'$ $\pm a_4'a_6a_5'b_3b_2$	$b_3a_4a_2'a_5a_6'$ $\pm a_3'b_4b_2b_5b_6$	$b_3b_6b_5a_4a_2'$ $\pm a_3'a_6a_5'b_4b_2$	$b_3b_6b_4a_5a_2'$ $\pm a_3'a_6a_4'b_5b_2$
$b_5b_3b_6a_2a_4'$ $\pm a_5'a_3a_6'b_2b_4$	$b_5b_4b_6a_2a_3'$ $\pm a_5'a_4a_6'b_2b_3$	$b_6a_2a_3'a_5a_4'$ $\pm a_6'b_2b_3b_5b_4$	$b_5a_2a_3'a_6a_4'$ $\pm a_5'b_2b_3b_6b_4$	$a_2'a_3a_5'a_4a_6'$ $\pm b_2b_3b_5b_4b_6$	$b_2b_3b_6a_5a_4'$ $\pm a_2'a_3a_6'b_5b_4$	$b_2b_5b_5a_6a_4'$ $\pm a_2'a_3a_5'b_6b_4$	$b_2b_4b_6a_5a_3'$ $\pm a_2'a_4a_6'b_5b_3$	$b_2b_4b_5a_6a_3'$ $\pm a_2'a_4a_5'b_6b_3$	$b_2a_5a_3'a_6a_4'$ $\pm a_2'b_5b_3b_6b_4$
$b_4b_3b_6a_2a_5'$ $\pm a_4'a_3a_6'b_2b_5$	$b_6a_2a_3'a_4a_5'$ $\pm a_6'b_2b_3b_4b_5$	$b_4b_5b_6a_2a_3'$ $\pm a_4'a_5a_6'b_2b_3$	$b_4a_2a_3'a_6a_5'$ $\pm a_4'b_2b_3b_6b_5$	$b_2b_3b_6a_4a_5'$ $\pm a_2'a_3a_6'b_4b_5$	$a_2'a_3a_5'a_5a_6'$ $+b_2b_3b_4b_5b_6$	$b_2b_3b_4a_6a_5'$ $\pm a_2'a_3a_4'b_6b_5$	$b_2b_5b_6a_4a_3'$ $\pm a_2'a_5a_6'b_4b_3$	$b_2a_4a_3'a_6a_5'$ $\pm a_2'b_4b_3b_6b_5$	$b_2b_5b_4a_6a_3'$ $\pm a_2'a_5a_4'b_6b_3$
$b_4b_3b_5a_2a_6'$ $\pm a_4'a_3a_5'b_2b_6$	$b_5a_2a_3'a_4a_6'$ $\pm a_5'b_2b_3b_4b_6$	$b_4a_2a_3'a_5a_6$ $\pm a_4'b_2b_3b_5b_6$	$b_4b_6b_5a_2a_3'$ $\pm a_4'a_6a_5'b_2b_3$	$b_2b_3b_5a_4a_6'$ $\pm a_2'a_3a_5'b_4b_6$	$b_2b_3b_4a_5a_6'$ $\pm a_2'a_3a_5'b_5b_6$	$a_2'a_3a_4'a_6a_5'$ $\pm b_2b_3b_4b_6b_5$	$b_2a_4a_3'a_5a_6'$ $\pm a_2'b_4b_3b_5b_6$	$b_2b_6b_5a_4a_3'$ $\pm a_2'b_4b_3b_6b_5$	$b_2b_6b_4a_5a_3'$ $\pm a_2'a_6a_4'b_5b_3$
$b_6a_2a_4'a_3a_5'$ $\pm a_6'b_2b_4b_3b_5$	$b_3b_4b_6a_2a_5'$ $\pm a_3'a_4a_6'b_2b_5$	$b_3b_5b_6a_2a_4'$ $\pm a_3'a_5a_6'b_2b_4$	$b_3a_2a_4'a_6a_5'$ $\pm a_3'b_2b_4b_6b_5$	$b_2b_4b_6a_3a_5'$ $\pm a_2'a_4a_6'b_3b_5$	$b_2b_5b_6a_3a_4'$ $\pm a_2'a_5a_6'b_3b_4$	$b_2a_3a_4'a_6a_5'$ $\pm a_2'b_3b_4b_6b_5$	$a_2'a_4a_3'a_5a_6'$ $\pm b_2b_4b_3b_5b_6$	$b_2b_4b_3a_6a_5'$ $\pm a_2'a_4a_3'b_6b_5$	$b_2b_5b_3a_6a_4'$ $\pm a_2'a_5a_3'b_6b_4$
$b_5a_2a_4'a_3a_6'$ $\pm a_5'b_2b_4b_3b_6$	$b_3b_4b_5a_2a_6'$ $\pm a_3'a_4a_5'b_2b_6$	$b_3a_2a_4'a_5a_6'$ $\pm a_3'b_2b_4b_5b_6$	$b_3b_6b_5a_2a_4'$ $\pm a_3'a_6a_5'b_2b_4$	$b_2b_4b_5a_3a_6'$ $\pm a_2'a_4a_5'b_3b_6$	$b_2a_3a_4'a_5a_6'$ $\pm a_2'b_3b_4b_5b_6$	$b_2b_6b_5a_3a_4'$ $\pm a_2'a_6a_5'b_3b_4$	$b_2b_4b_3a_5a_6'$ $\pm a_2'a_4a_3'b_5b_6$	$a_2'a_4a_3'a_6a_5'$ $\pm b_2b_4b_3b_6b_5$	$b_2b_6b_3a_5a_4'$ $\pm a_2'a_6a_3'b_5b_4$
$b_4a_2a_5'a_3a_6'$ $\pm a_4'b_2b_5b_3b_6$	$b_3a_2a_5'a_4a_6'$ $\pm a_3'b_2b_5b_4b_6$	$b_3b_5b_4a_2a_6'$ $\pm a_3'a_5a_4'b_2b_6$	$b_3b_6b_4a_2a_5'$ $\pm a_3'a_6a_4'b_2b_5$	$b_2a_3a_5'a_4a_6'$ $\pm a_2'b_3b_5b_4b_6$	$b_2b_5b_4a_3a_6'$ $\pm a_2'a_5a_4'b_3b_6$	$b_2b_6b_4a_3a_5'$ $a_2'a_6a_4'b_3b_5$	$b_2b_5b_3a_4a_6'$ $\pm a_2'a_5a_3'b_4b_6$	$b_2b_6b_3a_4a_5'$ $\pm a_2'a_6a_3'b_4b_5$	$a_2'a_5a_3'a_6a_4'$ $\pm b_2b_5b_3b_6b_4$

* Every element in this table must be multiplied by the factor 1/2. This factor is quite omitted in order to save the space.

$$P_{12} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline -1 & -1 & 2 & 2 & 2 & -1 \\ \hline \end{array},$$

$$P_{13} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 1 & & \\ \hline 2 & 2 & -1 & -1 & & -1 \\ \hline 2 & 2 & -1 & -1 & -1 & \\ \hline \end{array},$$

$$P_{14} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & & -\frac{1}{3} \\ \hline \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & & -\frac{1}{3} \\ \hline \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline & & & & 1 & \\ \hline 2 & -1 & -1 & -1 & & \\ \hline \end{array},$$

$$P_{15} \simeq \begin{array}{|c|c|c|c|c|c|}\hline \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & & -\frac{1}{3} \\ \hline & 1 & & & & \\ \hline -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & & \frac{1}{3} \\ \hline -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline & & & & 1 & \\ \hline -1 & 2 & -1 & -1 & & \\ \hline \end{array},$$

$$P_{16} \simeq \begin{array}{|c|c|c|c|c|c|}\hline \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & & -\frac{1}{3} \\ \hline -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & & -\frac{1}{3} \\ \hline & & 1 & & & \\ \hline -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline -1 & -1 & 2 & -1 & 1 & -1 \\ \hline -1 & -1 & 2 & -1 & & \\ \hline \end{array},$$

$$P_{17} \simeq \begin{array}{|c|c|c|c|c|c|}\hline \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & & -\frac{1}{3} \\ \hline & & & 1 & & \\ \hline -1 & -1 & -1 & 2 & 1 & -1 \\ \hline -1 & -1 & -1 & 2 & & \\ \hline \end{array},$$

$$P_{23} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline & & & 1 & & \\ \hline 3 & 3 & -3 & -3 & -1 & 1 \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{24} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline 1 & 1 & -1 & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline 1 & & & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline 1 & & -1 & & -\frac{1}{3} & \frac{1}{3} \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{25} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & -1 & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline & 1 & & & & \\ \hline & 1 & & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline & 1 & -1 & & -\frac{1}{3} & \frac{1}{3} \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{26} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline & 1 & & -1 & -\frac{1}{3} & \frac{1}{3} \\ \hline & & 1 & & & \\ \hline & & & & -\frac{1}{3} & \frac{1}{3} \\ \hline & & & -3 & & 1 \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{27} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & -1 & & -\frac{1}{3} & \frac{1}{3} \\ \hline & 1 & -1 & & \frac{1}{3} & \frac{1}{3} \\ \hline & & & & -\frac{1}{3} & \frac{1}{3} \\ \hline & & & 1 & & \\ \hline & & -3 & & & 1 \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{34} \simeq \begin{array}{|c|c|c|c|c|c|}\hline 1 & & & & & \\ \hline & & & & \frac{1}{3} & \\ \hline & -1 & 1 & & \frac{1}{3} & \\ \hline & -1 & & 1 & \frac{1}{3} & \\ \hline & 3 & & & & \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{35} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & & & & \frac{1}{3} & \\ \hline & 1 & & & & \\ \hline -1 & & 1 & & \frac{1}{3} & \\ \hline -1 & & & 1 & \frac{1}{3} & \\ \hline 3 & & & & & \\ \hline & & & & & 1 \\ \hline \end{array},$$

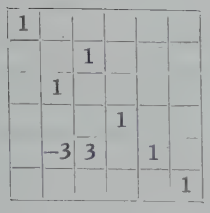
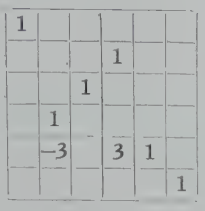
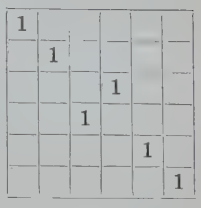
$$P_{36} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & -1 & 1 & & \frac{1}{3} & \\ \hline -1 & & 1 & & \frac{1}{3} & \\ \hline & & 1 & & & \\ \hline -1 & -1 & 1 & 1 & \frac{1}{3} & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{37} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & -1 & & 1 & \frac{1}{3} & \\ \hline -1 & & & 1 & \frac{1}{3} & \\ \hline -1 & -1 & 1 & 1 & \frac{1}{3} & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

$$P_{45} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & 1 & & & & \\ \hline 1 & & & & & \\ \hline & & 1 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

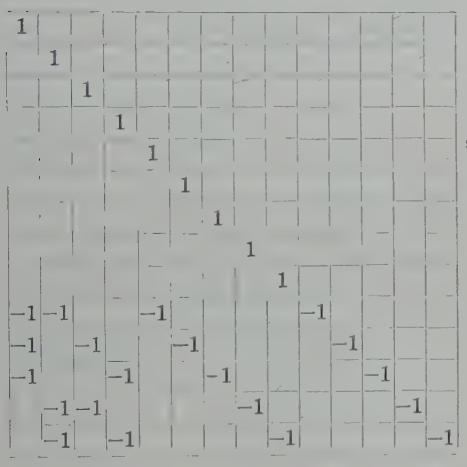
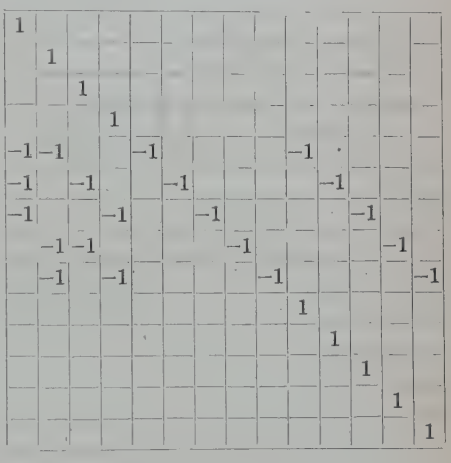
$$P_{46} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & & 1 & & & \\ \hline & 1 & & & & \\ \hline 1 & & & & & \\ \hline & & & 1 & & \\ \hline -3 & & 3 & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

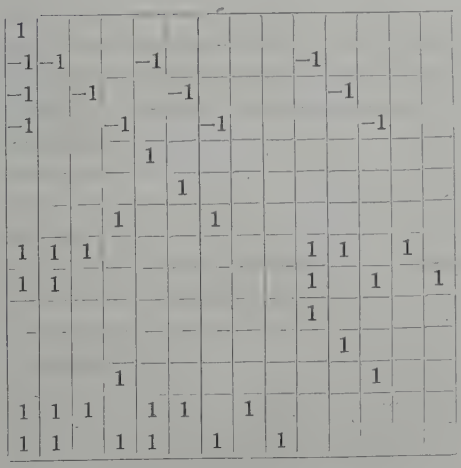
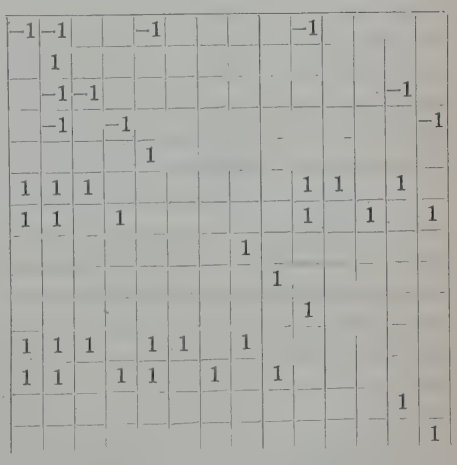
$$P_{47} \simeq \begin{array}{|c|c|c|c|c|c|}\hline & & & 1 & & \\ \hline & 1 & & & & \\ \hline & & 1 & & & \\ \hline 1 & & & & & \\ \hline -3 & & & 3 & 1 & \\ \hline & & & & & 1 \\ \hline \end{array},$$

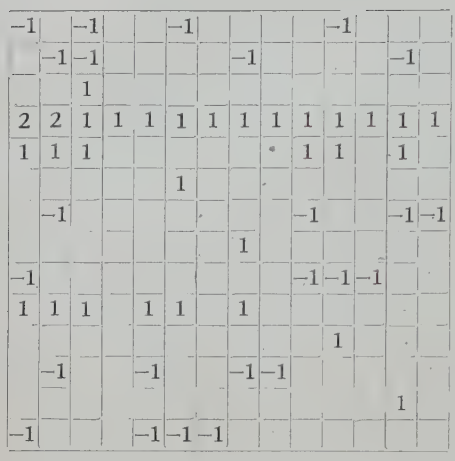
$P_{56} \simeq$  , $P_{57} \simeq$  , $P_{67} \simeq$  .

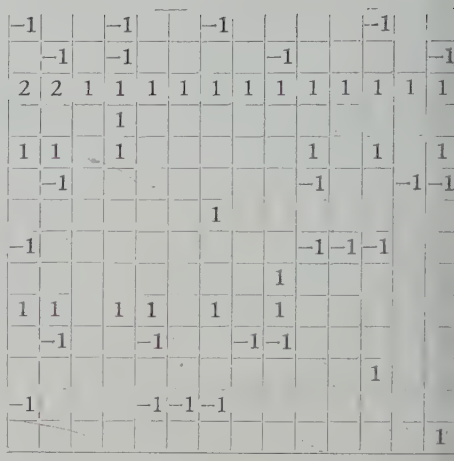
(6·15)

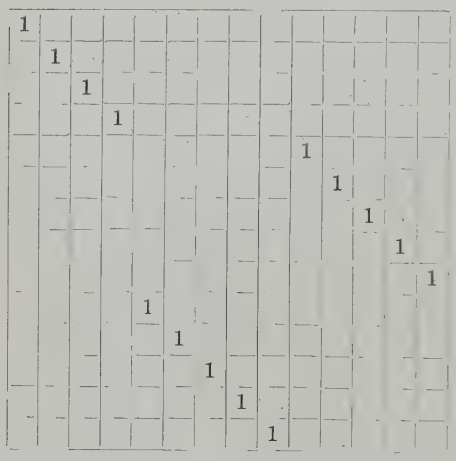
ε_3 brings to us the following representation, $S=1/2$.

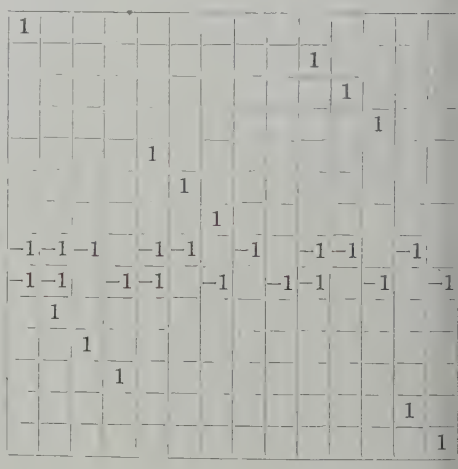
$P_{12} \simeq$  , $P_{13} \simeq$  ,

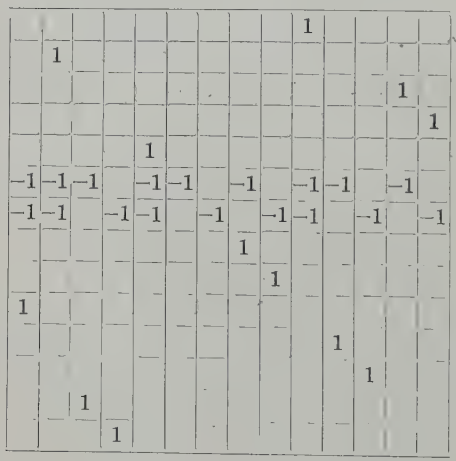
$P_{14} \simeq$  , $P_{15} \simeq$  ,

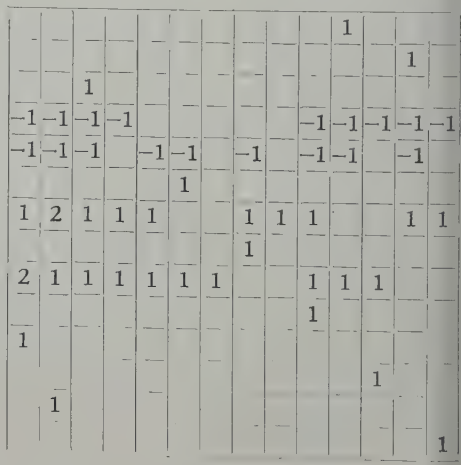
$P_{16} \simeq$ 

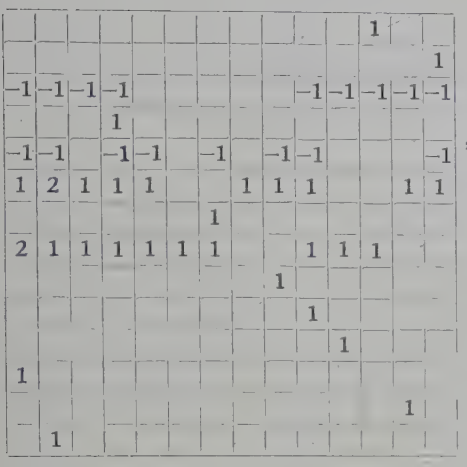
$P_{17} \simeq$ 

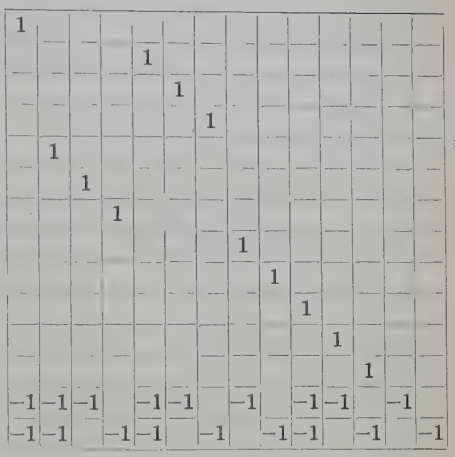
$P_{23} \simeq$ 

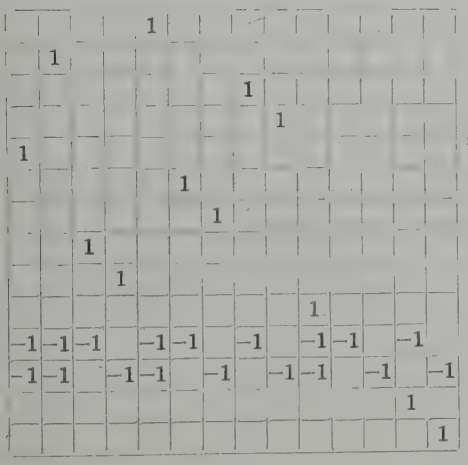
$P_{24} \simeq$ 

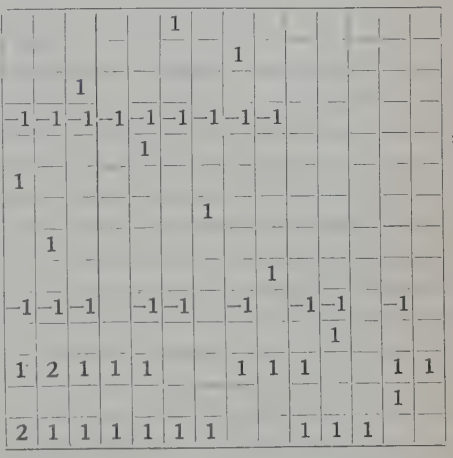
$P_{25} \simeq$ 

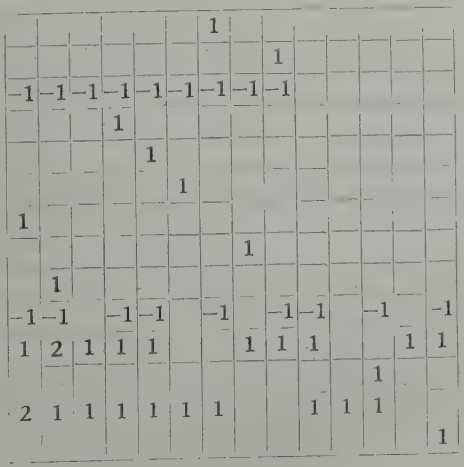
$P_{26} \simeq$ 

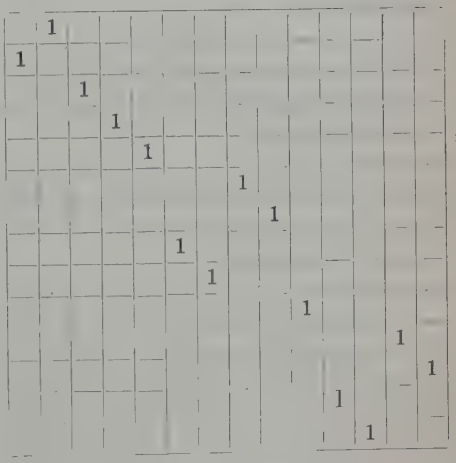
$P_{27} \simeq$  ,

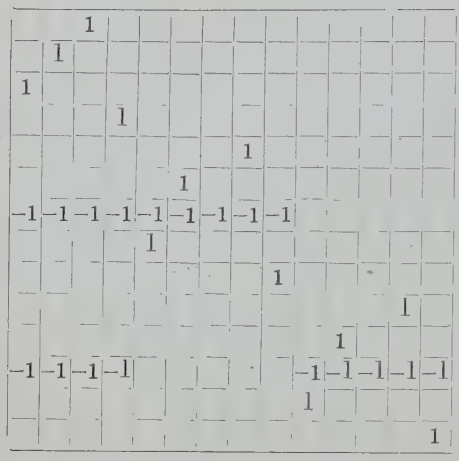
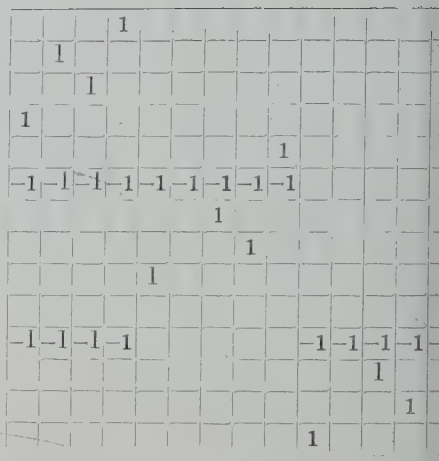
$P_{34} \simeq$  ,

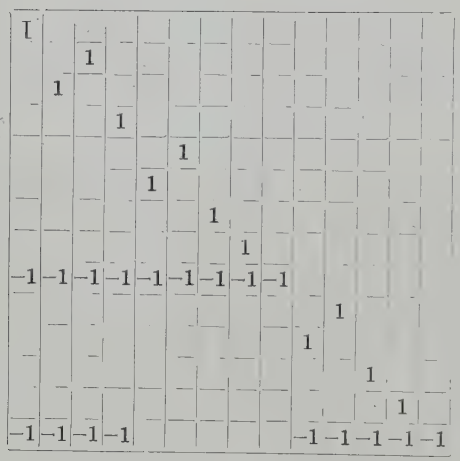
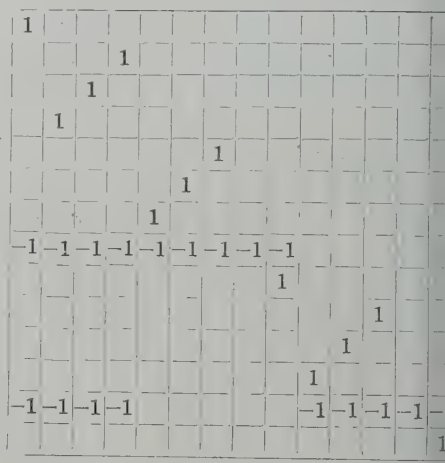
$P_{35} \simeq$  ,

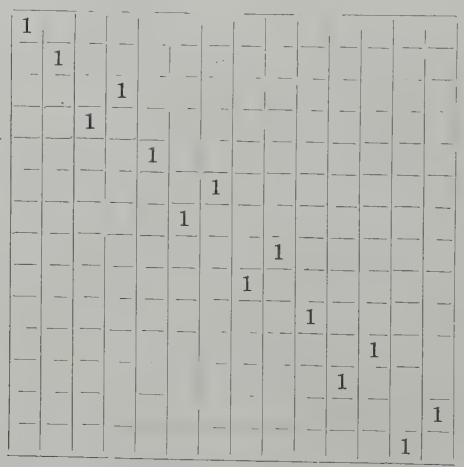
$P_{36} \simeq$  ,

$P_{37} \simeq$  ,

$P_{45} \simeq$  ,

$P_{46} \simeq$  , $P_{47} \simeq$ 

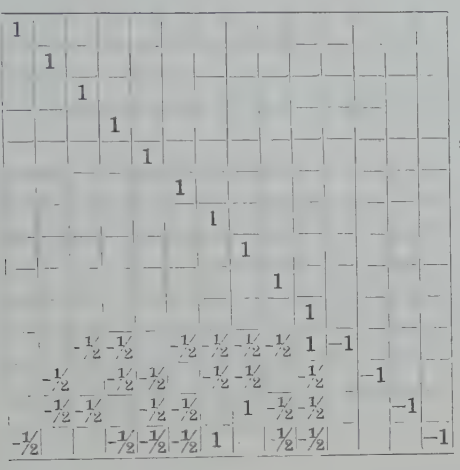
$P_{56} \simeq$  , $P_{57} \simeq$ 

$P_{67} \simeq$  .

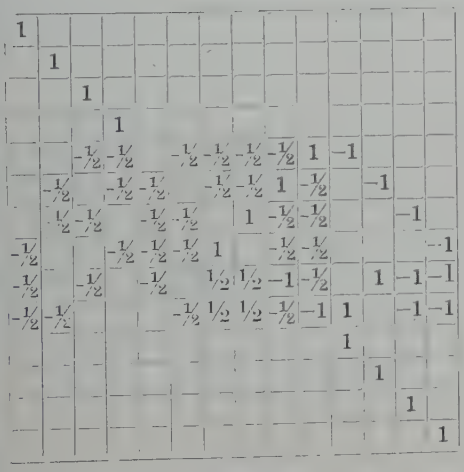
(6·16)

We derive lastly the following representation, $S=3/2$, from ε_4.

$$P_{12} \simeq$$

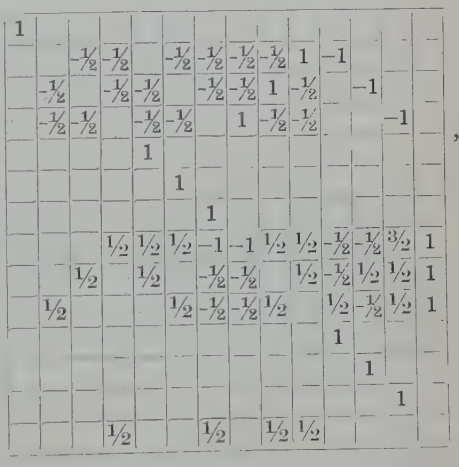


$$\simeq$$



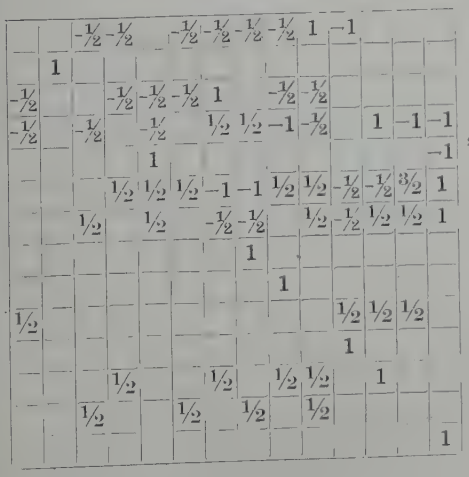
,

$$P_{14} \simeq$$



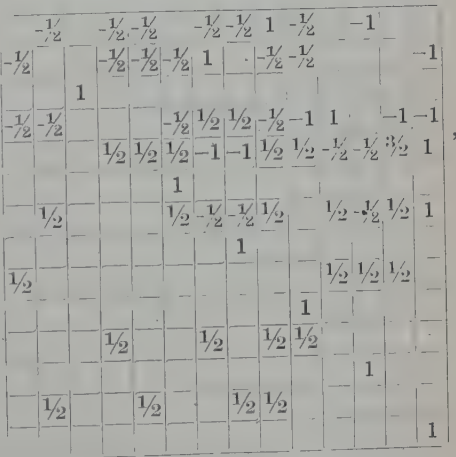
,

$$\simeq$$

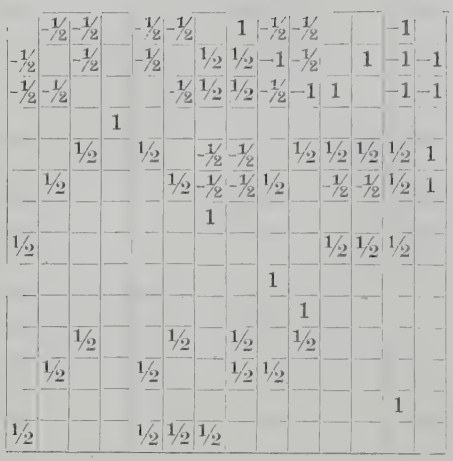


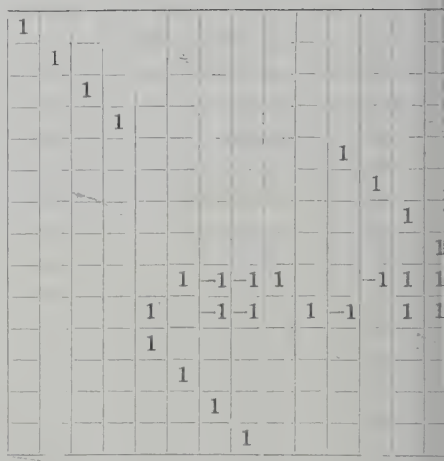
,

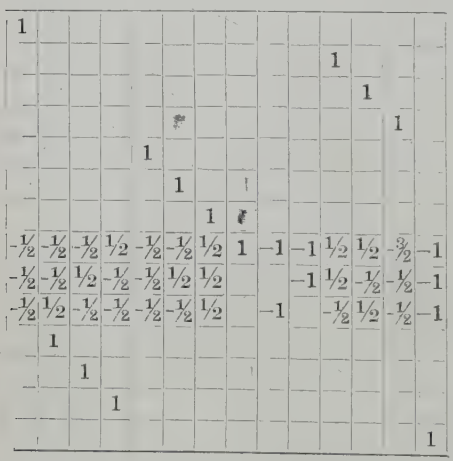
$$P_{16} \simeq$$

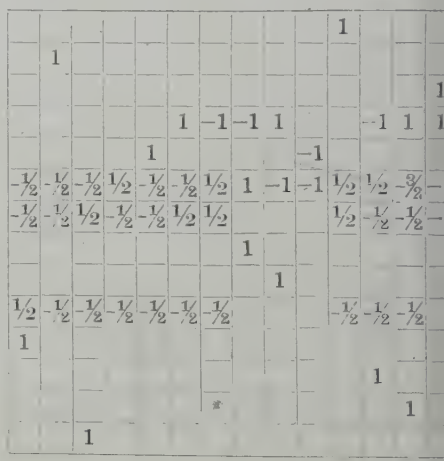


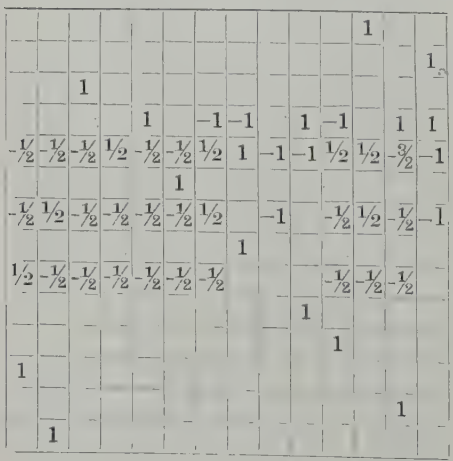
,

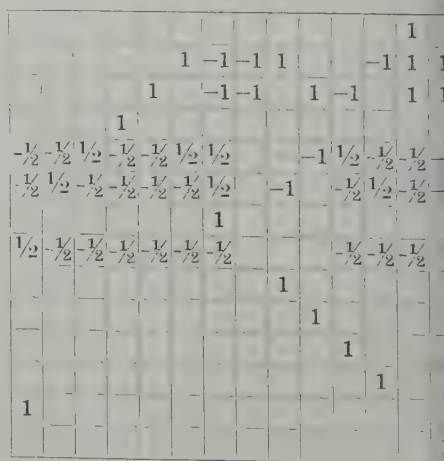
$P_{17} \simeq$ 

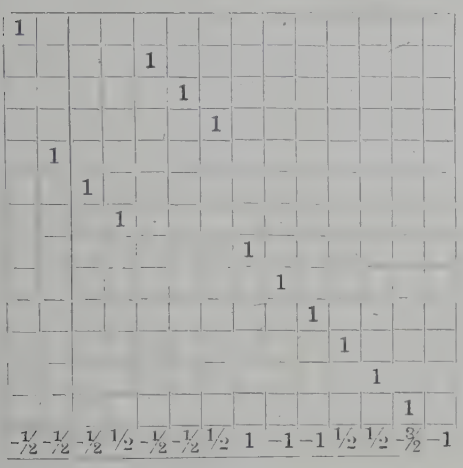
$P_{23} \simeq$ 

$P_{24} \simeq$ 

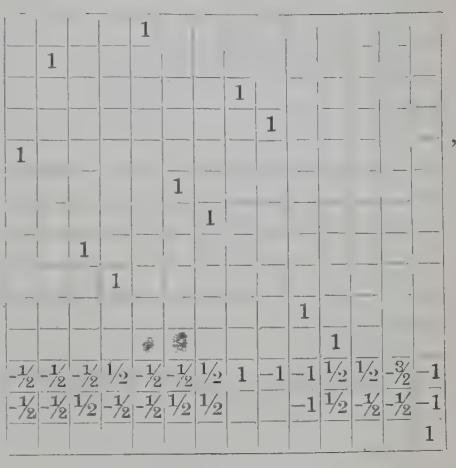
$P_{25} \simeq$ 

$P_{26} \simeq$ 

$P_{27} \simeq$ 



$\simeq$, $P_{35} \simeq$



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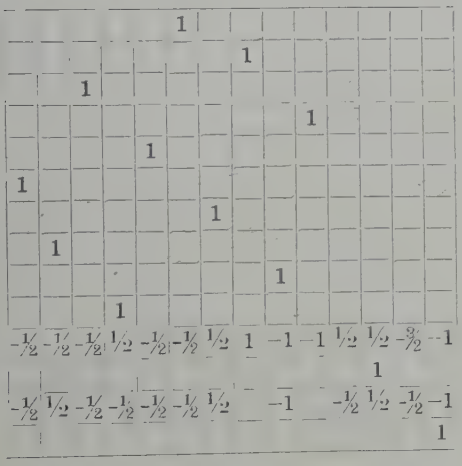
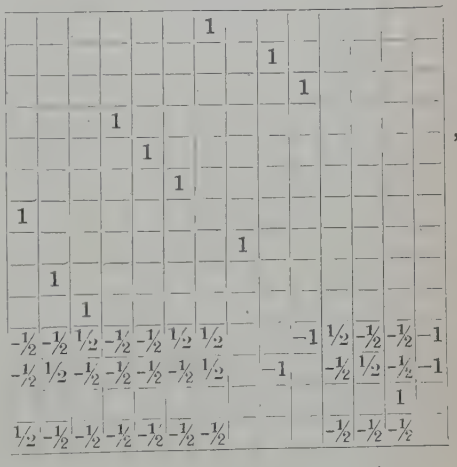


$\simeq$, $P_{37} \simeq$



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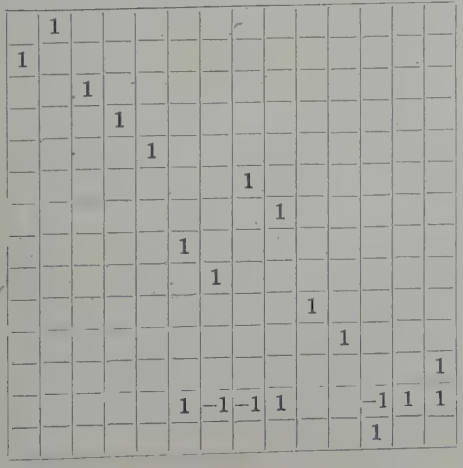
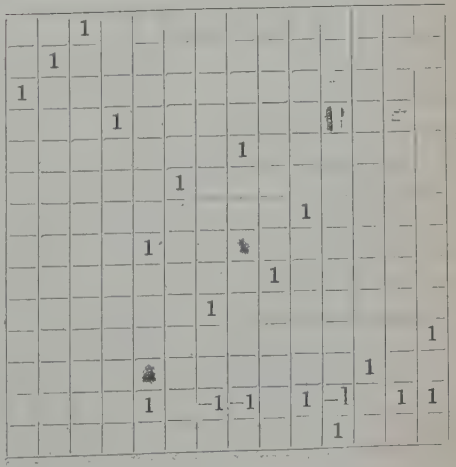
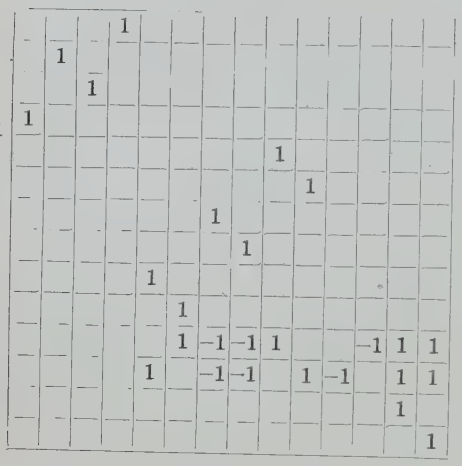
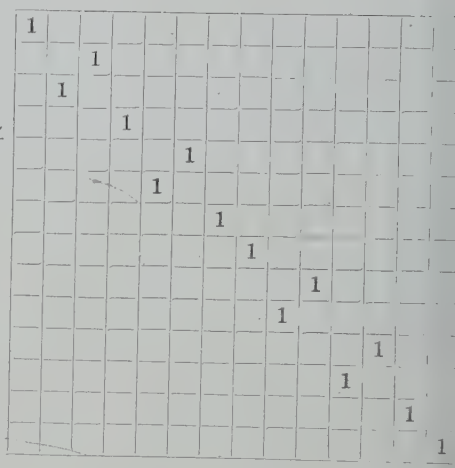


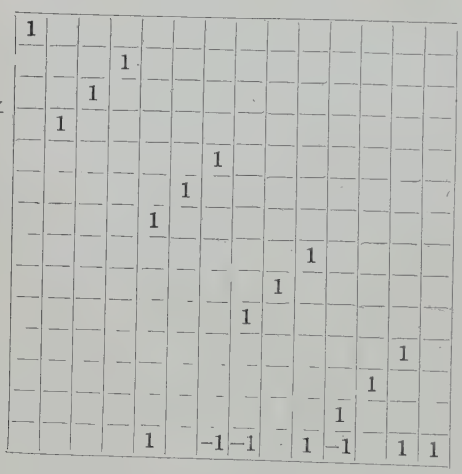
$\simeq$, $P_{46} \simeq$

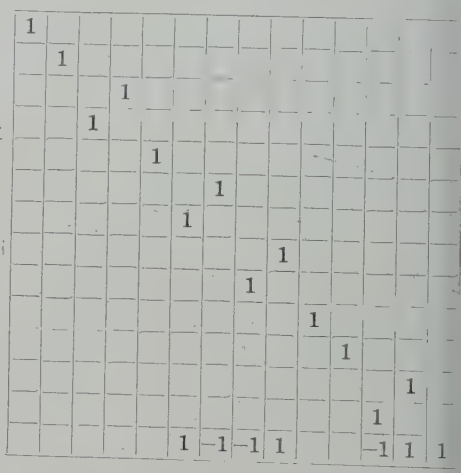


,

$P_{47} \simeq$ 

$P_{56} \simeq$ 

$P_{57} \simeq$ 

$P_{67} \simeq$ 

(6·17)

§ 7. The case of degeneracy

From the beginning we have assumed no degenracies other than the spin one in the eigenvalue problem in the Hartree-Fock field, but this condition is often broken; for example, there appear the azimuthal degeneracies in the problem of isolated atoms. In these cases the computation of $H_1' = S(\lambda_1, \ldots, \lambda_K) H_1 S(\lambda_1, \ldots, \lambda_K)$ only is not sufficient, and it is required to study also $S(\lambda_1, \ldots, \lambda_K) H_1 S(\mu_1, \ldots, \mu_K)$, where $\{\lambda_i\} \neq \{\mu_i\}$. We must enlarge $\mathbf{M}$ appropriately, where our guiding principle in order to carry out this are the followings.

(1) The primitivity of any primitive idempotent of $\mathbf{M}$ should not be broken.

(2) The enlargement should be as small as possible.

Thus we introduce the quantity,

$$A_{p,q}(\{ij\},\{kl\}) = \sum_{\rho=1}^{2} \prod_{(ij)}^{p} a_{i\rho}^{+} a_{j\rho} \prod_{(kl)}^{q} a_{k\bar{\rho}}^{+} a_{l\bar{\rho}} \qquad (7\cdot1)$$

where $\bar{\rho}$ takes the values 1, 2 respectively corresponding to the values 2, 1 of ρ, and the number of the factors $a^{+}a$ bearing the pair of suffices (ij) and (kl) are respectively p and q. All these quantities constitute an algebra, and this is just what we required.

In this algebra we must construct the algebraic tables embodying the connection between two idempotents e_1 and e_2 each representing the single configuration of electrons, such as $e^{(n)}{}_{\mu}(i_1,\ldots,i_\mu\,;\,k_1\ldots k_{n-\mu-1})\cdot\prod_j N_j(2)\prod_l N_l(0)$ or $e^{(n)}{}_{\mu}^{\pm}(i_1\ldots i_{\frac{n}{2}}\,;\,k_1\ldots k_{\frac{n}{2}-1})\cdot$ $\cdot\prod_j N_j(2)\prod_l N_l(0)$. These tables have the forms of three types shown in Table 11 (a) (b), (c), where the electrons occupying the i-th and j-th levels in the configuration represented by e_1 are transferred to the k-th and j-th levels in that of e_2, and where e represents that part of factors independent of these levels and common to the both configurations.

Table 11.

e_1	$\sum_{\rho} a_{i\rho}^{+}a_{j\rho}^{+}a_{k\rho}a_{l\rho}\cdot e$
$\sum_{\rho} a_{l\rho}^{+}a_{k\rho}^{+}a_{j\rho}a_{i\rho}\cdot e$	e_2

(a)

e_1	$\sum_{\rho} a_{i\rho}^{+}a_{j\bar{\rho}}^{+}a_{k\bar{\rho}}a_{l\rho}\cdot e$
$\sum_{\rho} a_{l\rho}^{+}a_{k\bar{\rho}}^{+}a_{j\bar{\rho}}a_{i\rho}\cdot e$	e_2

(b)

e_1	$\sum_{\rho}\sum_{\rho'} a_{i\rho}^{+}a_{j\rho'}^{+}a_{k\rho'}a_{l\rho}\cdot e$
$\sum_{\rho}\sum_{\rho'} a_{l\rho}^{+}a_{k\rho'}^{+}a_{j\rho'}a_{i\rho}\cdot e$	e_2

(c)

It is not required to study tables other than those above described because $S(\lambda_1,\ldots,\lambda_K)\cdot H_1 S(\mu_1,\ldots,\mu_K)$ vanishes if $\sum_i|\lambda_i-\mu_i|>4$. On writing down all possible pairs of number-sets $\{\lambda_1,\ldots,\lambda_K\},\{\mu_1,\ldots,\mu_K\}$ which are different among themselves, we can classify the cases into three categories as in Table 12, (a), (b), (c).

Table 12.

λ_i	λ_j	λ_k	λ_l	μ_i	μ_j	μ_k	μ_l
2	2	1	1	1	1	2	2
2	2	1	0	1	1	2	1
2	2	0	0	1	1	1	1
2	1	1	0	1	2	0	1
2	1	0	0	1	0	1	1
1	1	0	0	0	0	1	1

(a) different in four levels.

λ_i	λ_j	λ_k	μ_i	μ_j	μ_k
2	2	0	1	1	2
2	1	0	0	1	2
2	0	0	0	1	1

(b) different in three levels.

λ_i	λ_j	μ_i	μ_j
2	0	0	2
2	0	1	1
1	0	0	1

(c) different in two levels.

It is sufficient to write down the algebraic tables in these cases. These tables in total will occupy too large space to be shown here one by one, but all of these belong to one of the three types shown in table 11 (a) (b) (c).

We study now the problem of the complex spectra of atoms. In this case we have

$$H_{ik;i'k'}=\delta(m_i+m_k,\ m_{i'}+m_{k'})\cdot\sum_{\nu}c^{\nu}(l_im_i\,;\,l_{i'}m_{i'})\cdot$$
$$\cdot c^{\nu}(l_km_k\,;\,l_{k'}m_{k'})\cdot R_{\nu}(n_il_in_kl_k\,;\ n_{i'}l_{i'}\,;\ n_{i'}l_{i'}n_{k'}l_{k'}),\qquad(7\cdot2a)$$

$$R_{\nu}(n_il_in_kl_k\,;\,n_{i'}l_{i'}n_{k'}l_{k'})=e^2\int_0^{\infty}\int_0^{\infty}\frac{r_<^{\nu}}{r_>^{\nu+1}}\cdot R(n_il_i|r_1)R(n_kl_k|r_2)\cdot$$
$$\cdot R(n_{i'}l_{i'}|r_1)R(n_{k'}l_{k'}|r_2)dr_1dr_2,\qquad(7\cdot2b)$$

$$C^{\nu}(lm\,;\,l'm')=\left(\frac{2}{2\nu+1}\right)^{1/2}\cdot\int_{\theta}^{\pi}\Theta(\nu,\ m-m'|\theta)\,\Theta(lm|\theta)\,\Theta(l'm'|\theta)\sin\theta\,d\theta,\quad(7\cdot2c)$$

according to Ufford and Shortley[6], where $R(n,\,l|r)$ represents the radial part of the atomic wave function having the principal and azimuthal quantum number n and l, $\Theta(l,\,m|\theta)$ represent the associated Legendre function, viz.

$$\Theta(l,\,m|\theta)=\left[\frac{2l+1}{2}\,\frac{(l-|m|)!}{(l+|m|)!}\right]\sin^{|m|}\theta\,\frac{d^{|m|}P_l(\cos\theta)}{d(\cos\theta)^{|m|}},$$

and lastly $r_>$, $r_<$ represent the greater and the smaller of r_1, r_2 under the integrand in (7·2b). We can now carry out the practical calculation. Here is discussed a problem of complex spectra which lies outside the range of application of Slater's trace method[7]. We shall take the configuration d^3 as an example. This gives rise to the terms: ^2H, ^2G, ^4F, ^2F, ^2D, $^2D'$, ^4P, ^2P. Put $\sum m_i=M$, let the suffices of N indicate the values of the magnetic quantum numbers of energy eigenstates of one electron, and make use of Slater's symbol[7] $F_{\nu}(n,\,l)$, viz.

$$F_{\nu}(n,\,l)=(4\pi e)^2\int_0^{\infty}\int_0^{\infty}R(n,\,l|r)^2R(nl|r')^2\frac{r_<^{\nu}}{r_>^{\nu+1}}r^2r'^2drdr'.\qquad(7\cdot3)$$

We are allowed to limit ourselves in each subspace of the states having some fixed values of M, on account of the Kronecker δ symbol in (7·2a).

$M=5$ This subspace contains only the configuration represented by $N_2(2)N_1(2)$. From this, by making use of table 1, we derive the term

$$^2H=J(2,\,2)+2J(2,\,1)-K(2,\,1)$$
$$=F_0(n,\,2)-\frac{6}{49}F_2(n,\,2).$$

$M=4$ We have the configurations, $N_2(2)N_0(1)$, $N_2(1)N_1(2)$. Table 1 and the algebraic tables tell us that H_1 has the following matric form.

$$\begin{pmatrix} H(2)+2J(2,\,0)-K(2,\,0) & H_{2011} \\ H_{2011} & H(1)+2J(1,\,2)-K(1,\,2) \end{pmatrix}$$

whence we can find ${}^2H$ to be the same as before, and the second term to be

$$^2G=3F_0(n,2)-\frac{11}{49}F_2(n,2)+\frac{13}{441}F_4(n,2).$$

$M=3$ There exist the configurations, $N_2(1)N_1(1)$ and $N_2(1)N_1(1)N_0(1)$. On writing down the matrix elements of H_1 and applying the transformation (6·5) (6·6) to the part corresponding to the configuration $N_2(1)N_1(1)N_0(1)$ we attain the following matrix.

$$\begin{pmatrix}
\{J(2,1)+J(2,0)-K(2,1)\} & 0 & 0 & 0 \\
\quad -K(2,0)-K(1,0)\} & & & \\
0 & \{J(2,1)+J(2,0)+J(1,0)\} & \{-K(2,1)+K(2,0)\} & (H_{011-2}-H_{012-1}) \\
 & \quad +K(2,0)-K(1,0)\} & & \\
0 & \{-K(2,1)+K(1,0)\} & \{J(2,1)+J(2,0)-J(1,0)\} & -H_{01-12} \\
 & & \quad -K(2,0)+K(1,0)\} & \\
0 & (H_{01-12}-2H_{012-1}) & (-H_{01-12}-H_{01-12}) & \{J(2,2)+2J(2,-1)\} \\
 & & & \quad -K(2,-1)\}
\end{pmatrix}$$

The first diagonal element gives rise to the term,

$$^4F=3F_0(n,2)-\frac{15}{49}F_2(n,2)-\frac{72}{441}F_4(n,2).$$

The three rowed matrix beneath this gives the expression for ${}^2H$, ${}^2G$ in the same form as before and of ${}^2F$, as follows,

$$^2F=3F_0(n,2)+\frac{9}{49}F_2(n,2)-\frac{87}{441}F_4(n,2).$$

$M=2$ This contains the configurations $N_2(1)N_1(1)N_{-1}(1)$, $N_2(1)N_0(2)$, $N_1(2)N_0(1)$ and $N_2(2)N_{-2}(1)$. By applying the transformation (6·5) (6·6) to the matrix form of H_1 in this space, we obtain the expression for the term ${}^4F$ just as before, and the matrix

$$\begin{pmatrix}
\{J(1,-1)+J(1,2)+J(-1,0)\} & \{(K(1,2)-K(-1,2)\} \\
\quad -K(1,-1)+K(1,2)\} & \{J(1,-1)+J(1,2)+J(-1,2)\} \\
\{K(1,-1)-K(-1,2)\} & \quad +K(1,-1)-K(1,2)\} \\
H_{1-100} & 2H_{1-100} \\
\{2H_{2-110}-H_{2-101}\} & \{H_{2-110}-2H_{2-101}\} \\
\{H_{1-12-2}-H_{1-1-22}\} & \{H_{1-12-2}+H_{1-1-22}\}
\end{pmatrix}$$

$$\begin{pmatrix} 0 & H_{2-110} & \{H_{1-12-2}-H_{1-1-22}\} \\ H_{1-100} & -H_{2-101} & H_{1-1-22} \\ \{J(0,0)+2J(2,0) \\ \quad -K(2,0)\} & H_{2011} & H_{2-200} \\ H_{2011} & \{J(1,1)+2J(1,0) \\ & \quad -K(1,0)\} & 0 \\ H_{2-200} & 0 & \{J(2,2)+2J(2,-2) \\ & & \quad -K(2,-2)\} \end{pmatrix}$$

By taking advantage of the known values of ${}^2H$, ${}^2G$, ${}^2F$, we attain to the results,

$$\left.\begin{matrix} {}^2D \\ {}^2D' \end{matrix}\right\} = 3F_0(n,2)+\frac{5}{49}F_2(n,2)+\frac{3}{441}F_4(n,2)$$
$$\pm\sqrt{\frac{193}{492}F_0(n,2)^2-\frac{1650}{49441}F_0(n,2)F_4(n,2)+\frac{8325}{4413}F_4(n,2)^2}\ .$$

§ 8. Conclusion

We have studied the purely algebraic treatment of the many electron system, and found out the general method of obtaining the irreducible representations required. Though we have had in mind an algebraic treatment of statistical mechanics of electrons, we have not made any progress in this direction and are now of the opinion that this intention almost hopeless and using of the instrument of modern algebra seems not to bring so much economy of thought in this region.

The author wishes to acknowledge with sincerest thanks the kind encouragement and the useful discussions of Prof. K. Husimi who also suggested this problem and Mr. I. Syozi, and also expresses his thanks to all other members of the 'Bussei-ron' group of this Institute.

Appendices

I. The proof of the primitivity of the idempotents.

It is sufficient[5] to show that there exists no idempotent u such that $cu=uc=u\neq c$ in order to prove that primitivity of the idempotent c. Among the idempotents we have in mind first to consider $c_\mu^{(n)}(i_1,\ldots,i_\mu;k_1,\ldots,k_{n-\mu-1})=\prod_{\sigma=1}^{\mu}a'(1i_\sigma)\prod_{\tau=1}^{n-\mu-1}a(1k_\tau)$, where either n is odd or $\mu<n/2$ and n is even. An idempotent u is required to contain some number of b factors in order to be unequal to $c_\mu^{(n)}$, and those b factors may be regarded to have one of their two suffices equal to 1 on account of the relation, $b_{ij}=b_{1i}b_{1j}+b_{1j}b_{1i}$. Hence u must have the factor of the form $\prod_\sigma b(1i_\sigma)\ldots$, but it is quite evident on account of the relation (3·3a) that there is no product of this form which does not vanish when multiplied by $c_\mu^{(n)}$. And the primitivity of $c_\mu^{(n)}$ is proved except when n is even and $\mu=n/2$. In this exceptional case we can invent the quantity,

$$f_{\frac{n}{2}}^{(n)}(i_1, \ldots, i_{\frac{n}{2}}\,;\ k_1, \ldots, k_{\frac{n}{2}-1}) = \prod_{\sigma=1}^{\frac{n}{2}-1} b(1i_\sigma)b(1k_\sigma)b(1i_{\frac{n}{2}})$$

having the relation to $e_{\frac{n}{2}}^{(n)}(i_1, \ldots, i_{\frac{n}{2}}\,;\ k_1, \ldots, k_{\frac{n}{2}-1})$,

$$e_{\frac{n}{2}}^{(n)} f_{\frac{n}{2}}^{(n)} = f_{\frac{n}{2}}^{(n)} e_{\frac{n}{2}}^{(n)} = f_{\frac{n}{2}}^{(n)}.$$

This quantity is not idempotent, but has the property, $(f_{\frac{n}{2}}^{(n)})^2 = e_{\frac{n}{2}}^{(n)}$, whence we can derive the pairwise orthogonal primitive idempotents, as follows,

$$e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_{\frac{n}{2}}\,;\ k_1, \ldots, k_{\frac{n}{2}-1}) \qquad (\text{I}\cdot 1)$$

$$= \frac{1}{2}\{e_{\frac{n}{2}}^{(n)}(i_1, \ldots, i_{\frac{n}{2}}\,;\ k_1, \ldots, k_{\frac{n}{2}-1}) \pm f_{\frac{n}{2}}^{(n)}(i_1, \ldots, i_{\frac{n}{2}}\,;\ k_1, \ldots, k_{\frac{n}{2}-1})\}$$

$$= \frac{1}{2}\left\{\prod_{\sigma=1}^{\frac{n}{2}-1} a'(1i_\sigma)a(1k_\sigma)a'(1i_{\frac{n}{2}}) \pm \prod_{\sigma=1}^{\frac{n}{2}-1} b(1i_\sigma)b(1k_\sigma)b(1i_{\frac{n}{2}})\right\}$$

of which the primitivity is assured if we recognize that no idempotent u can have the property $eu = ue = u \neq e$ for $e = e_{\frac{n}{2}}^{(n)\pm}$.

II. The non-diagonal quantities.

Because every quantity containing only a and a' factors is diagonal, non-diagnal quantities are required to have b factors, of which one of suffices may be taken equal to 1 by the reason explained in appendix I. Hence any non-diagonal quantity can be expressed as a linear combination of products of some $a(1i)$, $a'(1k)$, $b(1l)$. We consider separetely the following two cases.

(1) The quantity containing the product of even factors of $b(1i)$. We take the quantity as an example,

$$g_e = \prod_{\sigma=1}^{\nu} b(1i_\sigma)b(1k_\sigma) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\rho=\nu+1}^{n-\mu-1} a(1k_\rho) \qquad (\text{II}\cdot 1)$$

At first we find the primitive idempotent $e_\mu^{(n)}$ having the property, $e_\mu^{(n)} g_e = g_e$. Such an idempotent is of the following form.

$$e_\mu^{(n)}(i_1, \ldots, i_\mu\,;\ k_1, \ldots, k_{n-\mu-1}) = \prod_{\sigma=1}^{\mu} a'(1i_\sigma) \prod_{\tau=1}^{n-\mu-1} a(1k_\tau).$$

We next find the required form of $e_\mu^{(n)}$ such that $g_e e_\mu^{(n)} = g_e$. This is found to be

$$e_\mu^{(n)}(k_1, \ldots, k_\nu, i_{\nu+1}, \ldots, i_\mu\,;\ i_1, \ldots, i_\nu, k_{\nu+1}, \ldots, k_{n-\mu-1})$$

$$= \prod_{\sigma=1}^{\nu} a'(1k_\sigma)a(1i_\sigma) \prod_{\tau=\nu+1}^{\mu} a'(1i_\tau) \prod_{\rho=\nu+1}^{n-\mu-1} a(1k_\rho),$$

and we have the relation

$$e_\mu^{(n)}(i_1, \ldots, i_\mu\,;\ k_1, \ldots, k_{n-\mu-1}) \cdot g_e = g_e \cdot e_\mu^{(n)}(k_1, \ldots, k_\nu, i_{\nu+1}, \ldots, i_\mu\,;\ i_1, \ldots, i_\nu, k_{\nu+1}, \ldots, k_{n-\mu-1})$$

$$= e_\mu^{(n)}(i_1, \ldots, i_\mu\,;\ k_1, \ldots, k_{n-\mu-1}) \cdot g_e \cdot e_\mu^{(n)}(k_1, \ldots, k_\nu, i_{\nu+1}, \ldots, i_\nu, k_{\nu+1}, \ldots, k_{n-\mu-1}) = g_e.$$

Quantities of the sort of g_e are the non-diagonal quantities connecting certain two idempotents both belonging to the $e_\mu^{(n)}$ type in the algebraic table.

(2) The quantity containing the product of odd factors of $b(1i)$. Such a quantity as

$$g_0 = \prod_{\sigma=1}^{\nu-1} b(1i_\sigma) b(1k_\sigma) b(1i_\nu) \prod_{\tau=\nu+1}^{\mu} a(1i_\tau) \prod_{\rho=\nu}^{n-\mu-1} a'(1k_\rho) \qquad (\text{II}\cdot 2)$$

connects the two idempotents,

$$e_\mu^{(n)}(i_1, \ldots, i_\mu ; k_1, \ldots, k_{n-\mu-1}),$$

and

$$e_{\mu'}^{(n)}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{n-\mu-1} ; k_1, \ldots, k_{\nu-1}, i_{\nu+1}, \ldots, i_\mu)$$

$$= \prod_{\sigma=1}^{\nu-1} a'(1i_\sigma) a(1k_\sigma) a'(1i_\nu) \prod_{\tau=\nu+1}^{\mu} a(1i_\tau) \prod_{\rho=\nu}^{n-\mu-1} a'(1k_\rho),$$

where the number, μ', of a' factors in the latter is related to μ as follows,

$$\mu' = \nu + \{(n+\mu-1) - \nu + 1\} = n - \mu.$$

Therefore these quantities connects one idempotent of the $e_\mu^{(n)}$ type with one of the $e_{n-\mu}^{(n)}$ type. After all we can obtain the following relation,

$$\begin{aligned} & e_\mu^{(n)}(i_1, \ldots, i_\mu ; k_1, \ldots, k_{n-\mu-1}) \cdot g_0 \\ & = g_0 \cdot e_{n-\mu}^{(n)}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{n-\mu-1} ; k_1, \ldots, k_{\nu-1}, i_{\nu+1}, \ldots, i_\mu) \\ & = e_\mu^{(n)}(i_1, \ldots, i_\mu ; k_1, \ldots, k_{n-\mu-1}) \cdot g_0 \\ & \qquad \cdot e_{n-\mu}^{(n)}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{n-\mu-1} ; k_1, \ldots, k_{\nu-1}, i_{\nu+1}, \ldots, i_\mu) \\ & = g_0. \end{aligned}$$

Lastly we consider the quantity connecting the two idempotents of the type $e_{\frac{n}{2}}^{(n)\pm}$,

$$e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots i_{\frac{n}{2}} ; k_1, \ldots, k_{\frac{n}{2}-1})$$

$$= \frac{1}{2} \left\{ \prod_{\sigma=1}^{\frac{n}{2}-1} a'(1i_\sigma) a(1k_\sigma) a'(1i_{\frac{n}{2}}) \pm \prod_{\sigma=1}^{\frac{n}{2}-1} b(1i_\sigma) b(1k_\sigma) b(1i_{\frac{n}{2}}) \right\},$$

and

$$e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{\frac{n}{2}-1} ; k_1, \ldots, k_{\nu-1}, i_{\nu-1}, \ldots, i_{\frac{n}{2}})$$

$$= \frac{1}{2} \left\{ \prod_{\sigma=1}^{\nu-1} a'(1i_\sigma) a(1k_\sigma) a'(1i_\nu) \prod_{\tau=\nu}^{\frac{n}{2}-1} a(1i_{\tau+1}) a'(1k_\tau) \right.$$

$$\left. \pm \prod_{\sigma=1}^{\nu-1} b(1i_\sigma) b(1k_\sigma) b(1i_\nu) \prod_{\tau=\nu}^{\frac{n}{2}-1} b(1i_{\tau+1}) b(1k_\tau) \right\}.$$

This quantity is given below as g^+ and g^-.

$$e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_{\frac{n}{2}}; k_1, \ldots, k_{\frac{n}{2}-1}) \cdot g^{\pm}$$
$$= g^{\pm} \cdot e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{\frac{n}{2}-1}; k_1, \ldots k_{\nu-1}, i_{\nu+1}, \ldots, i_{\frac{n}{2}})$$
$$= e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_{\frac{n}{2}}; k_1, \ldots, k_{\frac{n}{2}-1}) \cdot g^{\pm}$$
$$\cdot e_{\frac{n}{2}}^{(n)\pm}(i_1, \ldots, i_\nu, k_\nu, \ldots, k_{\frac{n}{2}-1}; k_1, \ldots, k_{\nu-1}, i_{\nu+1}, \ldots, i_{\frac{n}{2}}) \qquad (\text{II}\cdot 3)$$
$$= g^{\pm},$$

$$g^{\pm} = \frac{1}{2}\left\{ \prod_{\sigma=1}^{\nu-1} b(1i_\sigma) b(1k_\sigma) b(1i_\nu) \prod_{\tau=\nu}^{\frac{n}{2}-1} a(1i_{\tau+1}) a'(1k_\tau) \right.$$
$$\left. \pm \prod_{\sigma=1}^{\nu-1} a'(1i_\sigma) a(1k_\sigma) a'(1i_\nu) \prod_{\tau=\nu}^{\frac{n}{2}-1} b(1i_{\tau+1}) b(1k_\tau) \right\}.$$

We can now conclude that all idempotents of $e_\mu^{(n)}$ type and those of $e_{n-\mu}^{(n)}$ type constitute the totality of the quantities of the two-sided ideal of **M** together with the quantities of the kinds of η_e and η_0, except in the case that n is even, and $\mu = n/2$, and in this exceptional case all idempotents, $e_{\frac{n}{2}}^{(n)\pm}$ together with the quantities of the sort of $g^{\pm}$ constitute the two-sided ideals of **M**.

III. The exchange property of the elements in **E** and the commutativity of **E** and **B**.

We pay attention at first to the properties of P_{ij} which derive themselves from the relation, $P_{ij} = a(ij) + b(ij)$, as follows,

$$P_{ij}a(ij)P_{ij} = a(ij), \quad P_{ij}a'(ij)P_{ij} = a'(ij), \quad P_{ij}b(ij)P_{ij} = b(ij).$$
$$P_{ij}a(ik)P_{ij} = a(jk), \quad P_{ij}a'(ik)P_{ij} = a'(jk), \quad P_{ij}b(ik)P_{ij} = b(jk),$$
$$P_{ij}a(ik)a(jl)P_{ij} = a(jk)a(il), \quad P_{ij}a'(ik)a'(jl)P_{ij} = a'(jk)a'(il),$$
$$P_{ij}b(ik)b(jl)P_{ij} = b(jk)b(il), \quad P_{ij}a(ik)a'(jl)P_{ij} = a(jk)a'(il),$$
$$P_{ij}a(ik)b(jl)P_{ij} = a(jk)b(il), \quad P_{ij}a'(ik)b(jl)P_{ij} = a'(jk)b(il).$$

In words P_{ij} play the rôle of exchanging the suffices i and j, which property is the origin of the name 'exchange algebra'. Because the quantity B_μ is the symmetric sum with respect to the suffices of its constituent terms, it is in the relation to P_{ij};

$$P_{ij}B_\mu P_{ij} = B_\mu.$$

By virtue of the relation $P_{ij}^2 = 1$, we see the commutativity,

$$P_{ij}B_\mu = B_\mu P_{ij}. \qquad (\text{III}\cdot 1)$$

IV. The commutativity in **B**.

We shall take any two elements B_μ, B_ν belonging to **B**;

$$B_\mu = \sum b(i_1 j_1) b(i_2 j_2) \ldots b(i_\mu j_\mu),$$
$$B_\nu = \sum b(k_1 l_1) b(k_2 l_2) \ldots b(k_\nu l_\nu),$$

where we assume $\mu \leqq \nu$. We pick out one representative term in the product $B_\mu B_\nu$,

$$b(i_1 j_1)\dots b(i_\mu j_\mu)\cdot b(k_1 l_1)\dots b(k_\nu l_\nu). \qquad (\text{IV}\cdot 1)$$

When the suffices k_σ or $l_\tau(\sigma, \tau = 1, 2, \dots, \nu)$ appear in the series of suffices $i_1, j_1, i_2, j_2, \dots, i_\mu, j_\mu$, we can transfer $b(k_\sigma l_\sigma)$ or $b(k_\tau l_\tau)$ to one of the last μ positions in (IV·1), because we can arbitrarily commute b factors having no common suffices. If (IV·1) itself be the result of this rearrangement, the first ν factors and the last μ factors may be regarded as one term in the summation in B_ν and B_μ respectively. In other words any term in the product $B_\mu B_\nu$ appears necessarily also as a term in $B_\nu B_\mu$. We can prove similarly that any term in $B_\nu B_\mu$ appears also as a term in $B_\mu B_\nu$. These facts permit us to conclude that the product $B_\mu B_\nu$ is a part of the sumand of $B_\nu B_\mu$ and inversely that $B_\nu B_\mu$ is a part of the summand of $B_\mu B_\nu$. Thus we see the commutativity,

$$B_\mu B_\nu = B_\nu B_\mu. \qquad (\text{IV}\cdot 2)$$

V. The simple and mechanical rule of constructing the matrices of P_{ij} and B_μ.

We can derive these by using the relations (3·3a) and the algebraic table in each case.

(1) For the exchange quantity $P_{ij} = a(ij) + b(ij)$. If one of the suffices i, j, for example i, is 1, the desired matrix can be obtained by putting 1 at the diagonal places of the primitive idempotents containing $a(1j)$ and hence not $a'(1j')$ and also at the non-diagonal places of the quantities containing the factor $b(1j)$ and no other $b(1k)\,(k \neq j)$ factors, and furthermore putting 0 at all other places in the algebraic table.

In the case of i, j both differing from 1, the matrix for $P_{ij} = a(ij) + b(ij) = a(1i)a(1j) + a'(1i)a'(1j) + b(1i)b(1j) + b(1j)b(1i)$, is derived by putting 1 at the diagonal places of the primitive idempotents containing the factor $a(1i)a(1j)$ or $a'(1i)a'(1j)$ and also at the non-diagonal places of the quantities containing the factor $b(1i)b(1j)$ or $b(1j)b(1i)$ and no other $b(1k)\,(k \neq i,\ k \neq j)$ factors, and furthermore putting 0 at all other places in the algebraic table.

(2) For the quantity B_μ.

The matrix, derived by putting 1 at the places of all quantities containing $(2\mu - 1)$ b factors and all those containing 2μ such factors and furthermore putting 0 at all other places in the algebraic table, is the desired.

Note

A simpler method of distinguishing the 'Multiplicity' of the representation.

Mr. I. Syôzi recently pointed out to the author a simpler method of determining the multiplicity of the representation, which fits more properly in our scheme.

The fact shown in the footnote in § 3 leads to the relation

$$B_1 = \sum b(ij) = \sum P_{ij} - \sum a(ij)$$

$$= \frac{1}{2}\sum\{1 + 4(\boldsymbol{S}_i, \boldsymbol{S}_j)\} - \sum a(ij)$$

$$= (\sum \boldsymbol{S}_i)^2 - \sum a(ij) + \frac{n^2}{4} - n. \tag{1}$$

If the multiplicity of the representation, derived from the idempotent ε belonging to $R_{\frac{n-1}{2}}^{(n)}$ in the case odd n, or belonging to either $R_{\frac{n}{2}}^{(n)+}$ or $R_{\frac{n}{2}}^{(n)-}$ in the case of even n, is $2S+1$, the characteristic value of B_1, given by (1), in this representation is

$$S(S+1) - \begin{Bmatrix} \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} \\ \frac{n^2}{4} - \frac{n}{2} \end{Bmatrix} + \frac{n^2}{4} - n = \begin{cases} S(S+1) - \frac{n}{2} - \frac{1}{4}, \\ S(S+1) - \frac{n}{2}, \end{cases} \tag{2}$$

where the upper and lower line refer to the cases of odd and even n.

On the other hand the characteristic value of B_1 in this representation can be known directly from the relation, derived by using the multiplicative formulae among B_μ's of the form,

$$B_1\varepsilon = \varepsilon B_1 = \beta \cdot \varepsilon \tag{3}$$

where β is a positive or negative integer. By comparing the value of β and that of (2), we can decide the multiplicity of the derived representation.

For example, we try to apply this method to the case $n=3$, where the characteristic values of B_1 given by the expression (2) are 2 and -1 respectively for the multiplicities $S=3/2$ and $S=1/2$. If we use eq. (6·1), (6·2) and (6·3), we can obtain the relations,

$$B_1\varepsilon_1 = \varepsilon_1 B_1 = 2 \cdot \varepsilon_1,$$

$$B_1\varepsilon_2 = \varepsilon_2 B_1 = -\varepsilon_1.$$

From these we can conclude the correspondence between the idempotents and the multiplicities of the representations engendered by them;

$$\varepsilon_1 \longleftrightarrow S=3/2,$$

$$\varepsilon_2 \longleftrightarrow S=1/2.$$

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On the Equivalency for Observers in the Special Theory of Relativity

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Laying stress on the concept of the equivalency of the observers, the properties of the coordinate transformations connecting the observers relatively in uniform motion are investigated. These transformations form the 'uniform motion group' together with the frame-transformations of the special forms. The relations between the uniform motion group and the Lorentz group are investigated.

§ 1. The special theory of relativity and the equivalent observers in uniform motion

The special theory of relativity is ordinarily regarded as based on the following two fundamental principles: the principle of special relativity and the principle of the constancy of the light-velocity. The former states that the laws of nature are in concordance for all inertial systems, and the latter states that in a vacuum light is propagated with the constant velocity c. The validity of the second principle for all inertial systems must also be assumed in consequence of the first principle. Einstein derived the Lorentz transformation as satisfying the postulates of these fundamental principles. In deriving the Lorentz transformation he used the conditions that (1) the relative velocity of the observers is constant, (2) the transformation equation between the observers is linear and (3) the velocity of light is constant[1]. It was afterwards shown that the Lorentz transformation could be derived from the postulate that the transformation should make the metric $ds^2=dx^2+dy^2+dz^2-c^2dt^2$ in Minkowski space-time form-invariant.[2]* Consequently the special theory of relativity may be regarded as based on a single postulate that the mathematical expressions of the laws of nature must be form-invariant under the transformation group which keeps the metric $ds^2=dx^2+dy^2+dz^2-c^2dt^2$ form-invariant. And it is also shown that Lorentz transformation can be derived from the postulate that the wave equation of light, $\Box\varphi=0$, must be form-invariant under the transformation.[3]** In such a manner there exist several ways to derive the Lorentz transformation from the different postulates.[4] Therefore if the essential contents of the special theory of relativity are nothing but the Lorentz scheme of the theory, it is theoretically equivalent whatever

* From this postulate the rotation of the space axes and the translations of the time and space axes are also obtained, and these transformations form the Lorentz group together with special Lorentz transformation.

** From this postulate the transformations forming the Lorentz group L and the dilatation of the space and time axes are obtained.

principles are chosen as the fundamentals of the special theory of relativity so long as from these principles the Lorentz transformation can be derived.

But it is hard to believe that the essential character of the special theory of relativity is only such a formal one. It is rather appropriate to consider that the most fundamental idea of the special theory of relativity is the concept of the equivalency of the observers relatively in uniform translatory motion. Therefore in this paper the relation between the fundamental principles of the special theory of relativity and the Lorentz transformation is investigated from the point of view based on the theory of the equivalent observers proposed by the present writer and Takeno.[5]* The theory of the equivalent observers is based on the ideas that the transformations connecting the equivalent observers must form a group together with ' frame-transformations ', and conversely if any transformation group is given the equivalent observers can be defined by this group. Namely. if an arbitrary transformation group is given, we can derermine (1) the kinematic properties of the equivalent observers and (2) the physical properties of the equivalent observers concerning the invariant laws for the equivalent observers.

We can consider the following three kinds of the equivalent observers relating to the fundamental principles of the special theory of relativity :

(a) *The equivalent observers defined by the Lorentz group L, (the group composed of the Lorentz transformation, rotations of space-frames and translations of space- and time-frames.)*

(b) *The equivalent observers defined by the transformation group under which the expression of the propagation of light in a vacuum is invariant, and*

(c) *The equivalent observers in uniform translatory motion relative to each other.* (Hereafter we shall call them simply ' the equivalent observers in uniform motion '.)

The equivalent observers of (a)-type are treated in the special theory of relativity. As is easily seen, the relative velocity between these equivalent observers is constant and the expressions of the propagation of light in a vacuum are invariant under the Lorentz group.

Milne's kinematic relativity[6] and Page's new relativity,[7] in which the observers of (b)-type play important roles, have somewhat different contents from those of the special relativity. It seems to the writer that this comes from the fact that there exists besides the Lorentz transformation another transformation which keeps $ds^2=dx^2+dy^2+dz^2-c^2dt^2=0$ form-invariant and connects the observers in accelerated motion.[8]

We shall call the transformation group defining the equivalent observers of (c)-type ' uniform motion group '. The Galilean group and the Lorentz group are both uniform motion group, but it is unknown whether or not there exist some other transformation groups which define the equivalent observers of (c)-type. Accordingly in the following the general form of the uniform motion group will be searched for and the relation between the uniform motion group and the Lorentz group will be investigated.

In searching for the general form of the uniform motion group it would seem natural

* This paper will be quoted as (**EO**)

to take the following two processes because of the general properties of the equivalent observers.

Process (1) *The investigation of the properties of the uniform motion group by taking account of the kinematic properties of the equivalent observers.*

By the kinematic conditions only, the transformation group cannot be determined uniquely, though we can obtain some knowledge about the properties of the uniform motion group. Therefore in the next step the conditions concerning the physical properties of the equivalent observers must be added.

Process (2) *The investigation of the invariancy of the mathematical expression of some physical law or laws under this transformation group.*

In accordance with the physical law or laws which we choose in the Process (2), different kinds of the uniform motion groups may be obtained. If we choose, however, the physical laws ordinarily used as the invariant relations in the Process (2), then we obtain the Lorentz group as the uniform motion group, e. g., in the following Case (i) as well as in the Case (ii).

Case (i) : *the case where* $ds^2=0$ *or the Maxwell equation is adopted as the mathematical expression of the invariant relation.*[4), 8)*]

Case (ii) : *the case where the Dirac equation of the electron or Yukawa's meson equation is assumed to be invariant.*[4)]

In the Case (ii), the group can be determined without using the kinematic conditions required in the Process (1).[4)]

From the above considerations it becomes clear that the Lorentz group L can be obtained as the uniform motion group by choosing the various laws in physics as the invariant relations in the Process (2). Therefore we need not necessarily adopt the principle of the constancy of the light-velocity as the fundamental principle of the special theory of relativity. Concerning this circumstance Stiegler has recently proposed a new scheme of the principles of the special theory of relativity.

Here it is to be mentioned that if we use in the Process (2) the Newton's equation of motion as the invariant relation, we cannot obtain the Lorentz transformation. In the scope of the special theory of relativity the Newton's equation of motion is regarded as an approximate equation, therefore by using it we obtain the Galilean transformation which can be regarded as an approximate equation to the Lorentz transformation.

In the next step the following problem will arise, whether the fundamental equations of the elementary particles, such as Maxwell, Dirac or Yukawa's equation, are to be the accurate equations or not, because for the present the applicability of these equations in the very high energy region has been an open question. And this problem suggests that there may exist such equations as have wider applicability than the above equations which can be regarded as approximate in the low energy region. If there exist such equations, a new special theory of relativity which does not take the Lorentz scheme can be constructed. Therefore if we wish to extend the expression of any physical law, it may be

* In this case the dilatation of the space and time axes are also obtained.

necessary to seek another possibility to take as the uniform motion group any transformation group other than the Lorentz group.

Accordingly in the following sections by using the kinematic conditions for the observers the general form of the uniform motion group is searched for and then physical properties of this transformation group and the relation between this group and the Lorentz group are investigated.

§ 2. Coordinate transformation connecting the equivalent observers in uniform motion

Here we shall carry out the investigations of the Process (1) in the last section, i. e., the investigations will be made to what extent the form of the uniform motion group can be determined, if the kinematic conditions of the observers only are taken into account.

(1) *Observers, space- and time-frames and coordinate transformations.*

To determine the transformation group defining the equivalent observers in uniform motion an axiomatic method is taken.* In the following we shall give our postulates for obtaining the transformation group. Let us start with the folllowing postulates as in (**EO**).

(P.I) *Any observer M is furnished with a three dimensional space-frame S with origin M and a one dimensional time-frame T, and can give one and only one set of space coordinates (x, y, z) and time coordinate (t) to any point event E to within frame-transformations.***

Let M and M' be any observers, and E be any event. If (x_μ) and (x'_μ), where $\mu=1, \ldots, 4$, and $x, y, z, t \equiv x_1, \ldots, x_4$, are coordinates of E given by M and M', we have

$$x'_\mu = f_\mu(x_\nu), \quad \left| \partial x'_\mu / \partial x_\nu \right| \neq 0, \quad (\mu, \nu = 1, \ldots, 4.). \tag{2.1}$$

Here we do not assume that the transformations form a group in the usual sense, that is, the functional form of the transformation may depend on the relative velocity of the observers.

As was stated in (**EO**), the elements of the transformation group are classified into the following two kinds : (1) the transformations connecting the frames relatively in motion and (2) those connecting the frames relatively at rest, i. e., the frame-transformations. We assume that the frame-transformations have the following property.

(P. II) *The frame-transformations are assumed to have the following form :*

$$x'_i = h_i(x_j), \quad (i, j = 1, 2, 3.), \quad t' = g(t). \tag{2.2}$$

If the two frames are connected by these transformations, any point at rest in one space-frame is also at rest in the other frame, and all simultaneous point events in one time-frame are also simultaneous in the other frame.

(2) *Expression of the motions with constant velocity. Normalized frame.*

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* See (**EO**) § 4.

** See also (**EO**) § 4.

In general, the expression of the velocity of any moving point event depends on the nature of the space- and time-frames adopted by the observers. Accordingly it is convenient to adopt the frames in which the expression of velocity becomes the simplest. Therefore each observer is expected to adopt such frames as have the characters expressed by the following postulates.

(P. III) *Any observer uses the 'normalized space- and time-frames'*

We call the space- and time-frames normalized when the coordinates of any point event moving with a constant velocity have the following relations.

$$x_i = v_i t + c_i, \quad (i=1, 2, 3.) \tag{2.3}$$

where v_i and c_i are arbitrary constants, v_i being the velocity component and c_i the initial value of the space coordinate. In this case the existence of the point events moving with constant velocity is assumed. How to construct the normalized frames we shall discuss in the next section.

As is easily seen, if any normalized space- and time-frame (ST) is given, $(S'T')$ obtained from (ST) by any frame-transformation is normalized when and only when this frame-transformation has the following form,

$$x'_i = \sum_j a_{ij} x_j + d_i, \quad (i, j=1, 2, 3.), \quad t' = at + d, \tag{2.2'}$$

where a_{ij}, d_i, a and d are constants. The frame-transformation (2.2') gives translation, dilatation and change of the directions of the axes of space-frames, and translation and dilatation of axis of time-frame.

(3). *Relative motion of the frames. Transformation between coordinate systems in uniform motion.*

Under the assumption of (P. III) the concept of the relative motion of the frame is introduced. Two arbitrary frames are called relatively at rest, when any point at rest relatively to one frame is also at rest to the other frame and vice versa. When different frames are not relatively at rest, they are moving relatively to each other. In these cases the uniform translatory motion of the frames is the most important, in which all points at rest to one frame are moving with the same constant velocity components relatively to the other frame. Concerning the uniform translatory motion of the frame the following postulate is assumed because of the group property of the transformations.

(P. IV) *The postulate for the uniform translatory motion of the frame:*

(*i*) *When* $(S'T')$ *moves uniformly relative to* (ST), (ST) *also moves uniformly relative to* $(S'T')$.

(*i'*) *When* $(S'T')$ *is at rest relatively to* (ST), (ST) *is also at rest to* $(S'T')$.

(*ii*) *When* (ST) *and* $(S'T')$ *are in uniform motion (including the case of zero velocity, namely of being relatively at rest), and* $(S'T')$ *and* $(S''T'')$ *are also in uniform motion, then* (ST) *and* $(S''T'')$ *are also in uniform motion.*

Now using the conditions postulated in (P. IV) the nature of the coordinate transformation between the equivalent observers in uniform motion will be investigated. Let

(ST) and $(S'T')$ be two arbitrary frames relatively in uniform motion. If another arbitrary frame $(S''T'')$ is also in uniform motion relatively to $(S'T')$, all points relatively at rest to $(S''T'')$ describe the parallel straight lines in (x', y', z', t') space. (The term 'parallel line' is borrowed from euclidean geometry.) And from (ii) in (P. IV), (ST) and $(S''T'')$ are also relatively in uniform motion so that all points at rest relative to $(S''T'')$ also describe the parallel straight lines in (x, y, z, t) space. Then the transformation between (ST) and $(S'T')$ transforms any arbitrary parallel lines into another parallel lines. It is well-known that the most general transformation having this property is the following affine transformation :

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu + c_\mu, \quad (\mu, \nu = 1, \ldots, 4.) \tag{2.4}$$

where $a_{\mu\nu}$ and c_μ are arbitrary constants.*

And conversely two arbitrary systems connected by the transformation of this type are relatively in uniform motion and satisfy the above postulates concerning the systems moving with uniform velocity.

(4) *The special transformation.* (*The transformation corresponding to the special Lorentz transformation.*)

It is easily seen that (2.4) can be reduced into the following form, if (ST) and $(S'T')$ are transformed by suitably chosen frame-transformations of the type of (2.3) :

$$x'_1 = a'_{11} x_1 + a'_{14} x_4, \quad x'_2 = a'_{22} x_2, \quad x'_3 = a'_{33} x_3, \quad x'_4 = a'_{41}\ x_1 + a'_{44} x_4, \tag{2.5}$$

where $a'_{\mu\nu}$ is constant. By a dilatation of the time axis the velocity components of (ST) to $(S'T')$ can be made equal to those of $(S'T')$. After this regraduation, if we designate (x_1, x_2, x_3, x_4) and (x'_1, x'_2, x'_3, x'_4) as (x, y, z, t) and (x', y', z', t') anew, it is easily seen that (2.5) have the following form :

$$x' = k(v)(x - vt), \quad y' = a_2(v) y, \quad z' = a_3(v) z, \quad t' = k(v)(t - b(v) x). \tag{2.5'}$$

In (2.5') $k(v)$, $b(v)$, $a_2(v)$ and $a_3(v)$ are assumed to be the one-valued analytic functions of v, the relative velocity between (ST) and $(S'T')$.

Then the scale of (ST) and $(S'T')$ must be regraduated by the dilatations, so that these transformations may tend to the identical transformations, if the relative velocity v tends to zero. (These frames are to be called after this regraduation (ST) and $(S'T')$ again.) The limiting values of the functions in (2.5') must be then as follows :

$$\lim_{v \to 0} k = 1, \quad \lim_{v \to 0} a_2 = \lim_{v \to 0} a_3 = 1. \tag{2.6}$$

Next let the coordinates of a point event expressed by the third frame $(S''T'')$ be (x'', y'', z'', t''). If these coordinates are obtained by transforming the coordinates (x', y', z', t') of $(S'T')$ frame by the equations of the type (2.5'), the following equations are obtained by the calculation of the relation between (ST) and $(S'T')$, under the assumption that $k(v) = k(-v)$, $a_2(v) = a_2(-v)$ and $a_3(v) = a_3(-v)$. (This assump-

* The functional form of the transformation is thus determined from our postulates.

tion is a natural one because these functions give the rate of the moving clock and scale.) (See Note.)

$$k(v)=1/\sqrt{|1-av^2|},\ b=av,\ a_2=a_3=1, \tag{2.7}$$

where a is an undetermined constant. Using the results of (2.7), (2.5′) can be reduced into the following form:

$$x'=(x-vt)/\sqrt{|1-av^2|},\ y'=y,\ z'=z,\ t'=(t-avx)/\sqrt{|1-av^2|}. \tag{2.8}$$

We shall denote this transformation by U_x. These equations have been obtained by several authors[5) 10)] by taking different standpoints and by different ways.

Further we can define U_y and U_z by cyclic interchanges of (xyz) from U_x. In the same way as the special Lorentz transformation, these U_x, U_y and U_z form a group together with the rotation of (x, y, z) axes and the translation of (x, y, z, t) axes. This is the most general form of the uniform motion group under the above postulates.

§ 3. Discussions

The most important result obtained in the last section is that the transformation equation between the equivalent observers in uniform motion can be reduced into (2.8) under our postulates. Here we shall examine the characteristics of this formula and then investigate the nature of the postulates in the last section.

To examine the character of (2.8) we must consider the significance of the constant a and determine the value of a. Accordingly we must adopt besides the postulates in § 2 another postulate concerning a certain physical relation invariant under the uniform motion group, namely, the Process (2) in § 1 must be taken.

In constructing the additional principle for determing the unknown constant a in (2.8), we can take the following four standpoints concerning the nature of the uniform motion group:

(A) *The point of view of the special theory of relativity.*

(B) *The standpoint in which the validity of the principle of special relativity is assumed, but the principle of the constancy of the light-velocity is not required to be valid.*

Taking this standpoint it is assumed that all laws of nature are invariant under the uniform motion group, and it is also assumed that the velocity of light is not constant in any inertial system.

(C) *The standpoint in which the truth of the principle of special relativity is not assumed, accordingly it is not all but some laws of nature that are required to be invariant under the uniform motion group. The ordinary expressions of the propagation of light are assumed to be one of the invariant laws of nature.*

(D) *The standpoint in which, as in* (c), *the truth of the principle of special relativity is not assumed, and also the validity of the principle of the constancy of the light-velocity is not assumed.*

If the standpoint (A) is taken, the uniform motion group is nothing but the Lorentz group. In the case of (C) the transformation group also becomes the Lorentz group.[4] In both cases the value of α becomes equal to the inverse square of the light-velocity c in a vacuum, that is, $\alpha=1/c^2$.

When the standpoint (B) or (D) is taken, the Process (2) in § 1 must be used, i. e., some law or laws of nature must be assumed to be invariant under the uniform motion group. In the case of (B) we can choose any law of nature as an invariant law under the uniform motion group. Due to the principle of special relativity the uniform motion group will be determined uniquely, whatever law may be assumed to be true. In this case the invariant law used in the Process (2) must be regarded as one of the fundamental principles which specify the character of the equivalent observers.

When the standpoint (D) is taken, we can take several different values of α corresponding to the law of nature used in the Process (2). In this case we can get different kinds of the uniform motion group in accordance with the different values of α.

As is easily seen the value of α relates directly to the rate of the scale and clock used by each observer. Therefore if the value of α is determined, we shall get the rule for the rate of the moving scale and clock.

Next we shall investigate the nature of the postulates in the last section. Regarding the (P. I) and (P. II) we do not touch here, because these postulates are general concerning the equivalent observers. And it seems natural to take (P. IV) due to the group character of the transformation. The most fundamental postulate of the uniform motion group is (P. III), therefore the nature of the normalized frame shall be investigated in detail. Here we shall investigate the problem of how to construct the normalized space- and time-frames. From the point of view stated in the last section we conclude that the normalized space- and time-frame can be constructed by taking any uniform motion as the standard. Namely, using this standard we can decide the graduation of the scale and clock. We interpret that in the classical mechanics the motion of the free particle, and in the special theory of relativity the propagation of light in a vacuum as well as the motion of the free particles are taken as the standard of the uniform motion. But, strictly speaking, we cannot judge what the free particle is. It is ordinarily believed that the isolated particle situated far enough from the other substances is free. Therefore it is an idealized procedure that we graduate the scale and clock by means of the motion of such free particles. Similar circumstances will occur, if we take the propagation of light as the standard. Nevertheless if the existence of the uniform motion is assumed, a set of the normalized space- and time-frames can be defined to within the frame-transformations of the form (2.2′). Conversely if any normalized space- and time-frame is given previously, we can decide whether any given motion is a uniform motion or not. Consequently we can conclude that the concept of the normalized frame and that of the uniform motion are not independent, and that we can define these two concepts only simultaneously.

In the next place the problem concerning the metric of the space and time is discussed briefly. We can determine the uniform motion group without introducing the metric of the space and time. And this is one of the characteristics of our method. But it

seems convenient to introduce the metric of the four dimensional space-time at least for the treatment of the mathematical formulae. One of the simplest ways to introduce the metric is probably the following, though there are many possible ways: If we adopt the metric of four dimensional space-time which is form-invariant under the uniform motion group, we shall get the metric of the Minkowski type space-time. Nevertheless, contrary to the ordinary point of view that the actual space and time are fused into four dimensional space-time, Milne has taken the standpoint that the space and time are separated from each other. His kinematic relativity is based on such an idea, and this is taken as one of the fundamental postulates of his theory.[6] But from the considerations in the present paper we cannot decide as to which standpoint we must take.

Summarizing the above stated arguments, we can conclude as follows:

(1) If there are the equivalent observers in uniform motion, we can reduce the transformation connecting them into the form (2.8) by introducing the suitable normalized space- and time-frame.

(2) If there is any relation expressing the law of nature which is invariant under the uniform motion group, we can determine the uniform motion group, taking this invariant law as one of the additional fundamental principle.

If in the future another principle may be adopted instead of the principle of the constancy of the light-velocity as the additional principle, the theory thus constructed will have the same nature as that based upon the Lorentz scheme at least with respect to the mathematical properties, so long as $\alpha > 0$. The case in which $\alpha < 0$ will hardly happen even in the future, because the constancy of the velocity of light must be taken into the theory at least as the approximate relation.

Lastly we mention briefly the relations between the results obtained above and the theory of the elementary particles. For the present the theory of the elementary particles which takes the Lorentz scheme has many difficulties. Though it is hard to decide in what points these difficulties originate, it is not meaningless to examine whether the Lorentz scheme is one of these origins or not. By many reasons it is supported that the theory of the elementary particles must be special relativistic. But in the scope of the present theory of the equivalent observers the contents of the concept 'special relativistic' and 'equivalent for the observers in uniform motion' is not the same, therefore these circumstances must be investigated in full detail. And even if it is postulated that the whole contents of the theory of elementary particles must be invariant under the uniform motion group, it will be an open question whether we can take (2.8) as the transformation connecting the equivalent observers in uniform motion. Because in deriving (2.8) the quantum mechanical principles have not been taken into account. The postulates used to derive (2·8) must be examined in detail from the standpoint of the quantum mechanics. Here we do not touch on this problem further.

The writer is grateful to Prof. Y. Mimura and Prof. H. Takeno for their helpful discussions.

Note

Determination of the functional forms of $k(v)$, $b(v)$, $a_2(v)$ and $a_3(v)$.

From the results obtained in § 2 the inverse transformation of (2.5′) becomes as follows :

$$x=k(-v)(x'+vt'),\ t=k(-v)(t'-b(-v)x'),\ y=a_2(-v)y',\ z=a_3(-v)z'. \quad \text{(N1)}$$

From (NI) and (2.5′) we obtain $b(v)=(1-1/k(v)k(-v))/v$.

Next (x'', y'', z'', t'') be connected to (x', y', z', t') by the following transformations.

$$x''=k(w)(x'-wt'),\ t''=k(w)(t'-b(w)x'),\ y''=a_2(w)y',\ z''=a_3(w)z'. \quad \text{(N2)}$$

From (N1) and (N2) we obtain

$$x''=k(w)k(v)(1+wb(v))(x-(v+w)t/(1+wb(v))), \quad \text{(N3)}$$

then if we put $u=(v+w)/(1+wb(v))$, (N3) becomes $x''=k(u)(x-ut)$, where $k(u)=k(v)k(w)(1+wb(v))$.

After some calculations we obtain the following functional form of $k(v)$ and $b(v)$:

$$k(v)=\begin{cases} \left(\dfrac{1+\sqrt{a}\,v}{1-\sqrt{a}\,v}\right)^{\frac{p}{2\sqrt{a}}}\cdot\left(|1-av^2|\right)^{-\frac{1}{2}}, & a>0 \\ \exp\,(pv), & a=0 \\ \exp\left(\dfrac{p}{\sqrt{-a}}\cdot\tan^{-1}\sqrt{-a}\,v\right) & \\ \hline \sqrt{(1-av^2)}, & a<0 \end{cases} \quad \text{(N4)}$$

where $p=dk/dv|_{v=0}$. And $b(v)=av$, where a is an undetermined constant.

In the same way we can obtain

$$a_2(v)=a_3(v)=\begin{cases} \left(\dfrac{1+\sqrt{a}\,v}{1-\sqrt{a}\,v}\right)^{\frac{p}{2\sqrt{a}}}, & a>0 \\ \exp(qv), & a=0 \\ \exp\left(\dfrac{q}{\sqrt{-a}}\tan^{-1}\sqrt{-a}\,v\right), & a<0 \end{cases} \quad \text{(N5)}$$

where $q=da_2/dv|_{v=0}=da_3/dv|_{v=0}$.

If we put $k(v)=k(-v)$, $a_2(v)=a_2(-v)$ and $a_3(v)=a_3(-v)$ in (N4) and (N5) respectively we shall obtain (2.7).

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Letters to the Editor

Transition Factor in Superexchange Interaction Matrix

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Anderson[1] has shown that the superexchange interaction matrix $H_{tt'}$ proposed by Kramers[2] is written down directly by the terms of spin operators under simplifying assumptions. In his theory one of the important expressions is such as

$$H_{tt'}=\sum(u,u')\frac{(t|H^{tr}|u)(u|H^{ex}|u')(u'|H^{tr}|t')}{(E_u-E_t)(E_{u'}-E_{t'})} \qquad (A\cdot 3)$$

$$=b^2(t|V_2'|t'). \qquad (A\cdot 14)$$

$$b=\frac{(a|H^{tr}|a')}{(E_u-E_t)_{\text{lowest}}}. \qquad (A\cdot 15)$$

b-factor is responsible to the transition from t- to u-electronic configuration and a, a' denotes the initial and final wave fn. of transition electron. As our model is oxidization antiferromagnetic substance (e.g. MnO), we may take $2p$ fn. of oxygen as a and $3d$ fn. of metal as a'.

Anderson's theory is successful in the treatment of $u-u'$ interaction term, i.e. $(t|V_2'|t')$, but only a little knowledge is given for b-factor. Here a method of calculation of the factor is introduced

For an electron of coordinate 1 the Hartree-Fock eq. is

$$[H_1+\sum(j)\int A_j^2(2)\,V(\overline{12})d\tau(2)$$
$$-\sum(j)\int A_j(2)A_i(2)V(\overline{12})d\tau(2)]A_i(1)$$
$$=E_iA_i(1). \qquad (1)$$

By the Slater's method[3] exchange potential term may be written down as

$$\sum(j,k,\|Spin)$$
$$\times\frac{\int A_j(1)A_k(2)A_k(1)A_j(2)V(\overline{12})d\tau(2)}{\sum(j,\|Spin)A_j^2(1)}. \qquad (2)$$

Applying this averaged exchange potential we will be aware of an interesting fact showing that the Hartree-Fock potential for plus spin electron is smaller than that for minus spin electron at the atomic sites of plus center spin and *vice versa*. Now we will try to apply this result to the electronic behavior in the theory of superexchange interaction. As the center spin of oxygen atom is zero the Hartree-Fock potentials at this site are same for both electronic spin $(U_0(1))$, but at the metal site the potential for minus spin electron should be greater than for plus spin electron $(U_-(1))$. In present case transition electron has spin of minus by the Hund's law, so we may take the expression

$$U_-(1)-U_0(1)=-\frac{\sum(-)\cdots}{\sum(-)\cdots}$$
$$-(-1)\left[\frac{\sum(+)\cdots}{\sum(+)\cdots}+\frac{\sum(-)\cdots}{\sum(-)\cdots}\right]$$
$$=\sum(j,k,+Spin)\times$$
$$\times\frac{\int A_j(1)A_k(2)A_k(1)A_j(2)V(\overline{12})d\tau(2)}{\sum(j,+Spin)A_j^2(1)} \qquad (3)$$

as perturbation H^{tr} which is responsible to transition, i.e. the perturbation for minus spin electron is the exchange potential for plus spin electron.

The same considerations as above will be possible for the denominator of b-factor, i.e. $(E_u-E_t)_{\text{lowest}}$. Neglecting the difference between coupling energy of two electrons at oxygen site in t-configuration and the one at metal site in u-configuration, $(E_u-E_t)_{\text{lowest}}$ becomes nearly equal to the energy corresponding to the difference of the Hartree-Fock potential for transition electron. Thus we may have a clear knowledge about b-factor and obtain

$$b=\frac{(p|H^{tr}|d)}{(E_u-E_t)_{\text{lowest}}}$$
$$\approx\frac{\int U_+(1)p(1)d(1)d\tau(1)}{-\int U_-(1)d^2(1)d\tau(1)-(-1)\int U_0(1)p^2(1)d\tau(1)}$$
$$=\frac{J_1}{J_2}. \qquad (4)$$

In the case of simple model the density matrix is

$$\sum(j,+Spin)A_j(1)A_j(2)$$
$$=d(1)d(2)+p(1)p(2) \qquad (5)$$

and for the small range of vector **1** we get,

$$J_1 \approx \int\int d\tau(1)\,d\tau(2)\,V(\overline{12})\,[d(1)\phi(1)\,d^2(2) + 2\phi(2)\,d(2)\,d^2(1) + d(1)\phi(1)\phi^2(2)], \quad (6)$$

$$J_2 \approx \int\int d\tau(1)\,d\tau(2)\,V(\overline{12})\,[d^2(1)\,d^2(2) + \phi^2(1)\phi^2(2) + 2\phi(1)\phi(2)\,d(2)\,d(1)]. \quad (7)$$

Strictly speaking, however, the fns. d and ϕ must not be taken as $3d$ fn. of metal and $2p$ fn. of oxygen respectively. This situation is due to the fact that each energy bands would be splitted into two sub-bands one of which is allowed to only plus spins and the another to minus spins[4], so the density matrix has composed by the wave fn. proper to the plus spin sub-bands.

It is expected in further study whether such a treatment of b-factor as above is influential for the general theory of superexchange interaction or not.

The author notes his thanks to Prof. K. Ariyama for his continual encouragements.

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New Description of Field

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One of the methods to resolve the difficulties we meet with in the present field theory will perhaps be the introduction of mathematical formalism which is different from one now in practical use. It is expected that the present theory will be included, as a special case, in the theory to appear in the future. In this paper we have tried to make a description of the field by the differential form E. Cartan created. Below are given the two reasons why we applied the differential form: (i) This differential form is most suitable to the algebraization of differential operation, and is in close connection with the analysis and the theory of Lie group. (ii) We find the possibility that the differential form will allow us to discuss the nonlocality.

In the first paper we will describe the well-known classical field by means of the differential form, and will give clear correspondence with our usual formalism.

The vector or pseudovector field:

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial \boldsymbol{U}}{\partial t} + grad\, U_0 + \varkappa \boldsymbol{F} = 4\pi g_2 \boldsymbol{T}, \\ curl\, \boldsymbol{U} - \varkappa \boldsymbol{G} = -4\pi g_2 \boldsymbol{S}, \end{cases} \quad (1°)$$

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial \boldsymbol{F}}{\partial t} - curl\, \boldsymbol{G} - \varkappa \boldsymbol{U} = -4\pi g_1 \boldsymbol{M}, \\ div\, \boldsymbol{F} + \varkappa U_0 = 4\pi g_1 M_0 \end{cases} \quad (2°)$$

will be described in another formalism here aimed at by the making of differential forms:

$$\omega_2^{(1)} = U_1 dx + U_2 dy + U_3 dz - cU_0 dt, \quad (1')$$

$$\omega_1^{(2)} = (\varkappa G_1 - 4\pi g_2 S_1)\,dydz + (\varkappa G_2 - 4\pi g_2 S_2)\,dzdx + (\varkappa G_3 - 4\pi g_2 S_3)\,dxdy - c(-\varkappa F_1 + 4\pi g_2 T_1)\,dxdt - c(-\varkappa F_2 + 4\pi g_2 T_2)\,dydt - c(-\varkappa F_3 + 4\pi g_2 T_3)\,dzdt, \quad (1'')$$

$$\omega_3^{(2)} = F_1 dydz + F_2 dzdx + F_3 dxdy - cG_1 dxdt - cG_2 dydt - cG_3 dzdt, \quad (2')$$

$$\omega_2^{(3)} = (-\varkappa U_0 + 4\pi g_1 M_0)\,dxdydz - c(-\varkappa U_1 + 4\pi g_1 M_1)\,dydzdt - c(-\varkappa U_2 + 4\pi g_1 M_2)\,dzdxdt - c(-\varkappa U_3 + 4\pi g_1 M_3)\,dxdydt. \quad (2'')$$

The result is:

$$\begin{cases} d\omega_2^{(1)} = \omega_1^{(2)}, & (1) \\ d\omega_3^{(2)} = \omega_2^{(3)}, & (2) \end{cases}$$

where r in $\omega_i^{(r)}$ indicates the degree of differential form and i means only the suffix by which to know the differences found in the differential forms of same degree.

Similarly, the scalar or pseudoscalar field:

$$\begin{cases} \varkappa \boldsymbol{V} - grad\, \phi = 4\pi f_2 \boldsymbol{M}, \\ \varkappa V_0 + \dfrac{1}{c}\dfrac{\partial \phi}{\partial t} = 4\pi f_2 M_0, \end{cases} \quad (3°)$$

$$div\, \boldsymbol{V} + \frac{1}{c}\frac{\partial V_0}{\partial t} - \varkappa\phi = 4\pi f_1 W \quad (4°)$$

are described as follows by way of the differential form:

$$\begin{cases} d\omega_1^{(0)} = \omega_0^{(1)}, & (3) \\ d\omega_4^{(3)} = \omega_3^{(4)}, & (4) \end{cases}$$

where

$$\omega_1^{(0)} = \phi, \quad (3')$$

$$\omega_0^{(1)}=(\varkappa V_1-4\pi f_2 M_1)dx+(\varkappa V_2-4\pi f_2 M_2)dy$$
$$+(\varkappa V_3-4\pi f_2 M_3)dz-c(\varkappa V_0-4\pi f_2 M_0)dt, \quad (3'')$$
$$\omega_4^{(3)}=V_0 dxdydz-cV_1 dydzdt$$
$$-cV_2 dzdxdt-cV_3 dxdydt, \quad (4')$$
$$\omega_3^{(4)}=-c(\varkappa\phi+4\pi f_1 W)dxdydzdt. \quad (4'')$$

Now, the use of differential form in such a way is easily found to lead to a fine unity between eqs. (1), (2), (3), (4).

The electromagnetic field was previously treated by E. Kähler. The Maxwell equations:

$$\begin{cases} curl\,\boldsymbol{E}=-\dfrac{1}{c}\dfrac{\partial \boldsymbol{B}}{\partial t}, \quad div\,\boldsymbol{B}=0, & (5°) \\ curl\,\boldsymbol{H}=\dfrac{1}{c}\dfrac{\partial \boldsymbol{D}}{\partial t}+\dfrac{4\pi}{c}\boldsymbol{I}, \quad div\,\boldsymbol{D}=4\pi\rho & (6°) \end{cases}$$

are described as follows by means of the differential form:

$$\begin{cases} d\omega^{(2)\prime}=0, & (5) \\ d\omega^{(2)\prime\prime}=\omega^{(3)}, & (6) \end{cases}$$

where

$$\omega^{(2)\prime}=B_1 dydz+B_2 dzdx+B_3 dxdy+cE_1 dxdt$$
$$+cE_2 dydt+cE_3 dzdt, \quad (5')$$
$$\omega^{(2)\prime\prime}=D_1 dydz+D_2 dzdx+D_3 dxdy-cH_1 dxdt$$
$$-cH_2 dydt-cH_3 dzdt, \quad (6')$$
$$\omega^{(3)}=4\pi\rho dxdydz-4\pi I_1 dydzdt-4\pi I_2 dzdxdt$$
$$-4\pi I_3 dxdydt. \quad (6'')$$

Moreover, the character of differential form:

$$d(d\omega)=0$$

will lead to

$$d\omega^{(3)}=0, \quad (7)$$

i.e., the equation of continuity

$$\frac{\partial\rho}{\partial t}+div\,\boldsymbol{I}=0 \quad (7°)$$

is formed.

Special consideration will be paid to the free field. Using the above-mentioned character, we shall find that eqs. (1), (2), (3) will bring out the following:

$$d\omega_1^{(2)}=0, \quad (8)$$
$$d\omega_2^{(3)}=0, \quad (9)$$
$$d\omega_0^{(1)}=0. \quad (10)$$

These, when written in usual notation, will be set forth in such familiar equations as this:

$$\frac{1}{c}\frac{\partial \boldsymbol{G}}{\partial t}+curl\,\boldsymbol{F}=0, \quad div\,\boldsymbol{G}=0, \quad (8°)$$
$$\frac{1}{c}\frac{\partial U_0}{\partial t}-div\,\boldsymbol{U}=0, \quad (9°)$$
$$curl\,\boldsymbol{V}=0, \quad \frac{1}{c}\frac{\partial \boldsymbol{V}}{\partial t}+grad\,V_0=0. \quad (10°)$$

Next, we will enter into studies of the interaction of fields, quantization, nonlocality. And it may be that these three should be introduced to the theory at one and the same time.

The author is very grateful to Mr. I. Suzuki for his kind hints on mathematics.

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The Critical Velocity and Frictions in the Flow of Liquid Helium II

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Recent experimental results[1),2)] show that friction acting on superfluid appears abruptly as soon as the velocity of superfluid exceeds a certain critical velocity. In the case of the slit flow, we have the experimental evidence indicating strongly that this friction is of Gorter's type, i.e., proportional to $(V_n-V_s)^3$. In the preceding letter, we have tentatively assumed that the critical velocity in the case of the surface flow should be taken as the critical velocity relative to the wall. In the present letter we take up a more general expression for the friction acting on the superfluid:

$$f_s=-A\rho_n\rho_s(V_s-V_n)^3-(B_3/d)\rho_s V_s^3, \quad \text{(for slit)} \quad (1)$$
$$=-A\rho_n\rho_s(V_s-V_n)^3-2(B_3/r)\rho_s V_s^3, \quad \text{(for capillary)} \quad (1')$$
$$A=0 \quad \text{when} \quad |V_s-V_n|<V_c, \quad (2)$$
$$B_3=0 \quad \text{when} \quad |V_s|<V_c. \quad (3)$$

Though the first term of the right hand side of eq. (1) or (1′) has the form of Gorter's cubic mutual friction, it is to be noted that we here assume A to vanish when $|V_s-V_n|<V_c$. The second term is the friction between the superfluid and the

Table I. Recalculated values of the critical velocity in the case that the slit width is 9~10 μ. (From the paper of *N-T-U*,)[3]

temperature °K	$V_{s,c}$ cm/sec.	$V_s - V_n\|_c$ cm/sec.	V_c, (*B-M*) cm/sec.
1.97	1.5~6	2.5~10	12
1.8	1.5~4.5	5 ~14	18
1.55	1.8~5	17 ~35	23

Table II. Recalculated values of the critical velocity in the case that the slit width is 1.6~1.9 μ. (From the paper of *H-H-W*[1])

temperature °K	$V_{s,c}$ cm/sec.	$V_s - V_n\|_c$ cm/sec.	V_c, (*B-M*) cm/sec.
2.00	13	22	12
1.73	8	28	21
1.58	6	35	23

wall which has already been suggested by *H-H-W*[1]. They have not, however, recognized that there should be a critical velocity for B_3 friction.

On the basis of eqs. (1), (3), we can explain many experimental facts as follows.

A. Thermal conduction (no net mass flow)

In this case, we can put

$$\rho_n V_n = \rho_s V_s. \qquad (4)$$

Therefore, if the temperature is not very near λ-point, $|V_s - V_n|$ becomes fairly larger than $|V_s|$. Accordingly, except for temperatures very near λ-point, B_3 term has no effect, and our previous interpretations are still applicable.

The only, and important, alteration is necessary for the values of the critical velocity. We have recalculated the critical velocities for the various experiments on the basis of eq. (2).

In the tables I and II, we tabulate at the last column the values of V_c which were obtained by *B-M* from the experiments[2] on the mass flow through a narrow slit ($d = 1.2\ \mu$). The new values of the critical velocity are nearly independent of the slit width or the film thickness.

B. Mass flow ($V_s \gg V_n$)

In this case, $|V_s - V_n|$ is nearly equal to $|V_s|$, and the both terms in eq. (1) have effects of the same order of magnitude, and the second term becomes larger than the first term when temperatures are fairly low and the width of the slit is narrower.

By adding the second term, we can explain the results of the fact reported by *H-H-W*[1] in the way suggested by them. (The difficulties pointed out by them do not matter for us, because we consider the critical velocity also for the B_3 term.)

From their results we can estimate the magnitude of B_3 as follows:

$$B_3 = 4 \sim 6 \times 10^{-4}\ \text{sec./cm.}\ (\text{at } 1.7^\circ\text{K}). \qquad (5)$$

Since the experimental error seems fairly large, our estimation is not so certain. Perhaps, B_3 will be change with temperatures of wall.

In the case of surface flow phenomena, as reported in the first letter, the friction of the second term becomes very larger than the first term. This term is also very adequate to explain the experiment of *B-B-M*[4].

The peculiar result reported by *B-M*[2] can also be explained by adding the B_3 term, in the case of the mass flow established by the level difference. In this case, the pressure drop of about $(\rho_s/\rho)\Delta P$ (ΔP is the total pressure difference) occurs in the narrowest part, and this is very satisfactory to account for their results. A small difference would be due to the end effect.

The effect will be more important in thermal conduction, because the velocity of the normal fluid is fairly large in this case. The end effect is now under investigation.

The writer wishes to express his cordial thanks to Associate Prof. S. Nakajima and Mr. M. Shimizu for their valuable guidance and helpful discussions.

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Damped Oscillation in the Surface Film of Liquid Helium II

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There are many experimental data on the flow phenomena of liquid helium II. The most of them are classified in one of the three categories; the heat conduction and mass transfer through narrow slits, and the surface flow. Several parameters have been introduced to account for the experimental results. The systematic analysis of them, however, is not yet complete. Among those parameters, the critical velocity and the mutual friction seem to be most essential, because both of these, in a sense, imply the break down of ideal two fluidity of liquid helium. In the present and following letters we shall clarify their interrelation.

We first take up the oscillation of the surface film observed by Atkins[1]. He has succeeded in explaining the period of oscillation by means of the two fluid theory. Here we concern ourselves with the damping. Simple estimation shows that the normal viscosity is too small and Gorter's cubic mutual friction is too large to account for the observed logarithmic decrement. Gorter's mutual friction contradicts also the fact that the observed decrement has the constant value 1.9×10^{-2}. As for a linear mutual friction, its existence is rather doubtful. Even if we assume its existence and adjust its magnitude to explain some data of heat conduction through slits, the damping arising from this friction is too large.

Consequently we are forced to assume that there exists no friction in the equation of motion for the superfluid, at least, below a certain critical velocity. In fact, as we shall see below, we can explain the observed damping without introducing any kind of friction in the equation of motion.

A part of the superfluid flowing in over the wall of the capillary must transform itself into the normal fluid. This transformation requires an amount of energy, which has to be supplied from the outer bath to the reservoir attached to the capillary. On account of the well-known Kapitza resistance at the boundary, a small but finite temperature difference ΔT is established between the reservoir and the bath. As the period of oscillation is fairly long (27.8 sec.) and thermal conductivity of the copper boundary is very high, we can assume that ΔT is given by

$$\Delta T=(R\pi r^2\rho ST/A)(dX/dt). \tag{1}$$

where R, r, and X are respectively the value of Kapitza resistance, the radius of the vessel, and the level difference between the vessel and the bath, and A is the surface area of the reservoir.

Now the equation of the motion for the superfluid contains the fountain force proportional to ΔT, which represents evidently a damping for the level oscillation. Adding this force to Atkins' equation, we have

$$X=X_0\exp[-kt]\sin\omega t, \tag{2}$$

where

$$\omega=\frac{2\pi}{\tau}\fallingdotseq(2\rho_s gd'/\rho rl')^{1/2}, \tag{3}$$

and

$$k=(\pi R\rho_n\rho_s rd'S^*ST/\rho Al'). \tag{4}$$

l' and d' are effective length and width of the film. They are adjusted to fit the period τ given by eq. (2) to the observed value. Then eq. (3) gives the logarithmic decrement, which agrees well with the observed value, if A has the value of 7 cm^2.[2] Unfortunately we can not find the actual value of A in Atkins' paper. The value of 7 cm^2, however, seems not to be unreasonable, because Atkins has used the other large reservoir of volume 8.45 c.c. and the third small reservoir of volume 0.73 c.c. to test the effect produced by the temperature difference between the reservoir and the bath. If our estimation is not incorrect, we may conclude that the motion of the superfluid is perfectly frictionless in itself.

This conclusion is very important to interpret the saturation of the surface flow rate, which will be discussed in the subsequent letter.

Finally we should like to remark that if we calculate more strictly, ΔT becomes as follows,

$$\Delta T=a\frac{dX}{dt}+bX, \tag{5}$$

where

$$a=(K\pi r^2\rho ST/A(1+x^2)), \tag{6}$$

$$b=(K\pi r^2\rho ST/A(1+x^2))\omega x, \tag{7}$$

$$x=(\omega CR/A) \tag{8}$$

where C is the heat capacity of liquid helium II in the reservoir. The second term in eq. (2) becomes rapidly very small when the temperature becomes lower than the certain value where x becomes nearly unity. By this term we can explain the fact reported by Atkins[1] that there is a minimum in the period of oscillation at 1.5~1.6°K.

At the temperature 1.10°K where the experiment of damping oscillation was performed, the second term becomes very small and then the first term is equal to Eq. (1).

The author wishes to express his sincere thanks to Prof. K. Ariyama and other members of this laboratory. In particular the author is indebted to Associate Prof. S. Nakajima and Mr. M. Shimizu for their valuable guidance and for many helpful discussions.

1) K. R. Atkins, Proc. Roy. Soc. (London) A **203** (1950), 119, 240.
2) In this calculation we take the value of A from the following paper: J. J. M. Beenakker, K. W. Taconis, E. A. Lynton, Z. Dokoupil and G. van Soest; Physica **18** (1952), 433.

Gorter's Friction and Saturation of Surface Flow Rate

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January 4, 1953

Saturation of flow rate in He II film is well known. The critical velocity of superfluid corresponding to this saturated flow rate is comparable to that of slit flow.[1) 2) 3)] The critical velocity in the case of slit flow is interpreted as follows. As soon as the velocity of the superfluid exceeds a certain critical value, there suddenly appears Gorter's friction. In the previous letter we have seen that the motion of the superfluid in the surface film is perfectly frictionless below a certain critical velocity. In the present letter, we assume that above the critical velocity Gorter's friction appears also in the surface flow, and explain the saturation of flow rate on this basis.

In the region of saturated flow rate, we can consider the flow to be stationary. The velocity of the normal fluid and other smaller terms in the equation of motion in the two fluid model are neglected. Then we have the equation of motion for the superfluid:

$$-(\rho_s/\rho)\nabla P+(\rho_n\rho_s/\rho)S^*\nabla T-A\rho_n{}^{\nu}{}_s V_s{}^3, \tag{1}$$

$$A=0 \quad \text{if} \quad V_s<V_c, \tag{2}$$

$$S^*=(\partial S/\partial\xi)_{PT} \quad \text{and} \quad \xi=(\rho_n/\rho), \tag{3}$$

V_c is the critical velocity.

In the stationary flow V_s is inversely proportional to the cross-section of the film. Therefore V_s begins to exceed the critical velocity at the narrowest part, i.e., at the top of the film. Now we consider that V_s exceeds the critical velocity in the region of length l_1, and that in the other region the motion of superfluid is completely frictionless. We neglect the temperature difference between the inner and outer vessels. (surely ΔT is about 10^{-6} deg.), and integrate eq. (1) over the film length l. If we assume that film thickness varies very gradually with height, we have

$$\Delta P=A\rho_n\rho V_c{}^3 l_1, \tag{4}$$

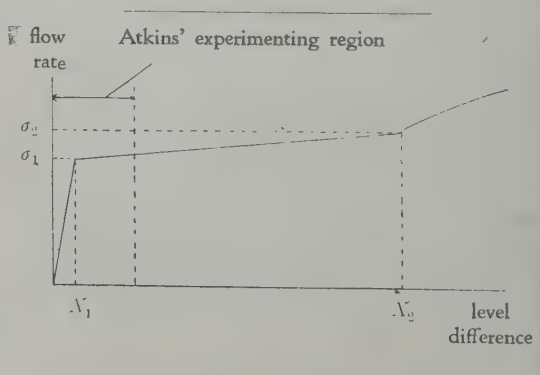
$$\Delta P=-\rho g X, \tag{5}$$

$$\text{flow rate} \equiv\sigma=V_c d_c, \tag{6}$$

where X is the level difference between the inner vessel and the outer bath. d_c is the thickness of the film at the point where Gorter's friction sets in.

From eq. (6) we can see easily that the flow rate remains saturated until l_1 is equal to l. By Atkins[1], $d\propto H^{-0.14}$ (where H is height of film).

For example, if we take the film length l equal to 4 cm, the flow rate at the ending saturation (σ_2 in fig. 1) is larger by about 20% than at the beginning of saturation (σ_1).



The level difference X_2 at which saturation is released is given by eq. (4);

$$(X_2/l)=(A\rho_n/g)\,V_c^3. \qquad (7)$$

If the pressure difference exceeds X_2, we have, from eq. (1)

$$\text{flow rate} \propto (\Delta P)^{1/3}. \qquad (8)$$

If the surface flow is established by means of the temperature difference, we may replace ΔP by $\rho_n S^* \Delta T$.

Finally we should like to remark the following facts:

i) In this paper we have taken up Gorter's mutual friction. It seems, however, preferable to assume a friction between the superfluid and the wall. This type of friction is particularly adequate to account for the anomalous mass flow through very narrow slits.[3)4)] Analysis of this experiment gives, in the case of the surface flow, the friction nearly one hundred times larger than Gorter's. This large value is adequate also in the case of surface flow to explain the experiment of Mendelssohn *et al.*[5)] The detailed discussion will be published later.

ii) The level difference X_2 in fig. 1 has a very large value, so that it is more convenient to establish the pressure head by the temperature difference. Recent work of Long-Mayer[6)] is just of this kind. Their result that the flow rate is roughly proportional to $(\Delta T)^{1/3}$ seems to be the strong support of our theory. Unfortunately the detail of this experiment is not yet published.

In conclusion the writter wishes to express his cordial thanks to Prof. K. Ariyama and other members of this laboratory, particularly to Associate Prof. S. Nakajima and Mr. M. Shimizu for their valuable guidance and for many helpful discussions.

1) K. R. Atkins, Proc. Roy. Soc. A **203** (1950), 119, 240.
2) to be published in the subsequent letter.
3) R. Bower and K. Mendelssohn, Proc. Roy. Soc. A **213** (1952), 158.
4) C. S. Hung, B. Hunt and P. Winkel, Physica **18** (1952), 629.
5) R. Bower, D. F. Brewer and K. Mendelssohn, Intern. Confer. Low Temp. Phys. (Oxford) (1952), 153.
6) E. Long and L. Mayer, Phys. Rev. **87** (1952), 153.

Meson Nucleon Scattering

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January 5, 1953

Recently Kamefuchi and Umezawa[1)] have proposed a non singular theory which can apply to the second kinds of interaction. They have shown that we can get the non singular theory for all kinds of interaction by renormalizing finite number of divergent term, if we correct the propagation functions by higher order effects since field reactions ought to be more strong in the second kinds of interaction than in the renormalizable ones. Hence it will be very interesting to apply this theory to real processes and examine the characteristic features of this theory.

From this point of view, we have calculated the meson nucleon scattering using the nucleon propagation function S_F' corrected S_F by pseudovector coupling of pseudoscalar interaction. S_F' is obtained from S_F by the following relation[2)]

$$S_F'(\not p)=S_F(\not p)/1-S_F(\not p)\cdot\Sigma_c^{(2)}(\not p),$$

where $\Sigma_c^{(2)}(\not p)$ is the finite nucleon self energy part of the second order after performing the renormalization procedure for the diverging parts. Kamefuchi and Umezawa have shown that we can get the non singular theory using the above defined S_F' if we consider only the interaction between pseudoscalar mesons and nucleons.

Now the nucleon self energy part of the second order, $\Sigma^{(2)}(\not p)$, is given by the following expression in the symmetrical theory:

$$\Sigma^{(2)}(\not p)=G^2\int(dk)\,\frac{\gamma_5(\not k\cdot\gamma)\,[i(\not p-\not k\cdot\gamma)-\varkappa]\,\gamma_5(\not k\cdot\gamma)}{[(\not p-k)^2+\varkappa^2]\,[k^2+\mu^2]}$$

where $G^2=3ig^2/(2\pi)^4\mu^2$; $\varkappa$ and μ are the masses of the nucleon and the meson respectively, and g is the coupling constant of the pseudovector interaction. We can separate the finite parts from this divergent integral by expanding it to power series with respect to $A=i(\not p\cdot\gamma)+\varkappa$ and retaining only the terms of A^n of $n\geq 4$, while we renormalize the terms of $n\leq 3$ by adding the counter parts into the Lagrangian. Thus we have

$$\Sigma_c^{(2)}(\not p)=G^2 I_c,$$

where I_c is the finite part of the above integral. Since $S_F(\not p)=-A^{-1}$, $S_F'(\not p)$ becomes as follows:

$$S_F'(p) = -[A + G^2 I_c]^{-1}.$$

When we rationalize the denominator with respect to γ-matrix, we can write

$$S_F'(p) = \frac{-1}{u(ip\cdot\gamma)+v} = \frac{u(ip\cdot\gamma)-v}{u^2p^2+v^2},$$

where u and v are both the functions of $\varkappa$, μ and p^2, and they can be calculated from I_c.

The main features of results calculated for the meson nucleon scattering using this S_F' are as follows:

(a) $\pi^+ + P \to \pi^+ + P \quad (\sigma_+)$
  (i) the angular distribution is forward-up,
  (ii) the pole exists in the denominator and the cross section becomes infinitely large for some energy of meson,

(b) $\pi^- + P \to \pi^- + P \quad (\sigma_-)$
  (i) the angular distribution is isotropic,
  (ii) $\sigma_+ > \sigma_-$ with respect to the total cross sections.

The angular distributions are not the so-called $\cos^2\theta$-type. This fact does not agree with the experimental results[3]. Existence of the pole may be generally presumed from the beginning, but the peculiar behaviour of σ_+ will be due to the imperfect correction to the propagation function, which may be seen from the definition of S_F' for which no other graphs, for instance the "vertex part", are taken into account. Detailed accounts will soon appear in this journal.

In conclusion the author wishes to express the cordial thanks to Prof. S. Sakata, Dr. H. Umezawa and Mr. S. Kamefuchi for their kind interests and discussions on this work.

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2) F. J. Dyson, Phys. Rev. **75** (1949), 1736.
3) H. L. Anderson *et al.*, Phys. Rev. **85** (1952), 936; ibid. **86** (1952), 793.

Photo Positive Pion Production

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Recently, Kamefuchi and Umezawa[1] proposed a new prescription of managing divergencies arising from interactions of the second kind. One special case of this prescription, which belongs to the type (A) in their terminology, was applied to the pion-nucleon scattering by one of the present authors (S.G.)[2]. He used the nucleon propagation function which was corrected by the second order finite proper self-energy part $\sum_c^{(2)}$ of a nucleon in the pseudoscalar meson theory with pseudovector coupling, and could get the unambiguous finite results.

As soon as the electromagnetic field is taken into account, however, this prescription encounters to a serious obstacle. For constructing finite corrected S_F' function, additional counter terms must be introduced into Lagrangian. These require, in order that the Lagrangian should be gauge invariant, new electromagnetic interactions which lead to divergent results again. When using so obtained S_F', moreover, in spite of adopting the gauge invariant Lagrangian we cannot get gauge invariant matrix elements merely by summing up all the Feynman-Dyson graphs for the process in question which are permitted to certain fixed order in the coupling constant[5].

Still, we carried out the calculation for the photo positive[4] pion production by neglecting tentatively non gauge invariant terms from the eg order matrix element and obtained following results (taking $g^2/4\pi = 0.93$): (1) the excitation function in the laboratory system increases as the three second power of incident photon energy at threshold and shows rather better agreement with experimental data; (2) the angular distribution in the barycentric system is lacking in the flatness of curve compared with the ordinary eg order perturbation result. Although these depend on the artificial procedure of dropping non gauge invariant terms and must be read qualitatively, from which we can see the characteristic feature of our prescription that the using of the corrected propagation function only for all internal nucleon lines (different from Hu's method[5]) would diminish the contribution from nucleonic current and would make the effect of the mesonic current impressive. Therefore it might be expected that gauge invariant prescriptions of similar nature (B_1 and B_2 type of Kamefuchi-Umezawa's prescription[1]) would be promissing e.g. for the anomalous magnetic moment of the nucleon.

Detailed explanations will appear in this journal.

The authors express their cordial thanks to Prof. S. Sakata for his continuous encouragement and also

to Dr. H. Umezawa and Mr S. Kamefuchi for the suggestion of this work and their valuable discussions.

1) S. Kamefuchi and H. Umezawa, Prog. Theor. Phys. 8 (1952), 579.
2) S. Goto, Prog. Theor. Phys. 9 (1953), 91.
3) There remains a possibility that one might get the gauge invariant matrix element by adding higher order graphs. See also Z. Koba, N. Mugibayashi and S. Nakai, Prog. Theor. Phys. 6 (1951), 322.
4) For the negative and neutral pion production, there is the same difficulty of the zero point newly appearing in the corrected S_F' function as in the case of reference 2).
5) N. Hu, Phys. Rev. 80 (1950), 1109.

Charge Independence of Nuclear Interaction and High Energy Nucleon-nucleon Scattering

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January 15, 1953

The charge independence of nuclear interaction has been evidenced in the low energy region[1]. However, the high energy nucleon-nucleon scattering data have shown many unexpected properties, which seem at first sight to destroy the charge independence.

1) p-p scattering cross section is almost independent of energy at $E > 100$Mev $\left(\frac{d\sigma}{d\Omega} \sim 4 \text{ mb}\right)$, while n-p cross section decreasing as $1/E$ is small.
2) p-p differential cross section is isotropic at the angle where Coulomb scattering is negligible. In contrast to it n-p cross section shows the strong back and forward maximals and the symmetry about 90° in C. M. system.

To maintain the charge independence property, several authors proposed new potentials.[2)3)4)] Such a charge independent potential must satisfy the following conditions:

1) It must give the anomality of p-p scattering without affecting the n-p scattering so strongly.
2) The large cross section in p-p scattering must appear only at high energy region.

To investigate such potentials, we shall examine some consequences of the charge independent interaction. "Charge independence" of the interaction means that the interaction energy is invariant with respect to the rotation in the isotopic spin space. From this it follows directly that the isotopic spin of two nucleon system must be conserved throughout the scattering process. Here we restrict ourselves to unpolarized incident waves. Then the differential cross section is written as follows (neglecting Coulomb interaction):

$$\sigma_{pp}(\theta) = \sum_{\sigma_z^1,\sigma_z^2} |f^{\text{even}}_{{}^3T_1{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2)|^2 + \sum_i \sum_{\sigma_z^1,\sigma_z^2} |f^{\text{odd}}_{{}^3T_1{}^3S_i}(\theta,\varphi,\sigma_z^1,\sigma_z^2)|^2, \quad (1)$$

$$\sigma_{np}(\theta) = \frac{1}{4}\sum_{\sigma_z^1,\sigma_z^2} |f^{\text{even}}_{{}^3T_0{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2) + f^{\text{odd}}_{{}^1T_0{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2)|^2 + \frac{1}{4}\sum_i \sum_{\sigma_z^1,\sigma_z^2} |f^{\text{odd}}_{{}^3T_0{}^3S_i}(\theta,\varphi,\sigma_z^1,\sigma_z^2) + f^{\text{even}}_{{}^1T_0{}^3S_i}(\theta,\varphi,\sigma_z^1,\sigma_z^2)|^2, \quad (2)$$

where $f_{TS}\frac{e^{ikr}}{r}\chi_T(t_z^1,t_z^2)$ is the asymptotic form of scattered wave, and ${}^mT_i({}^mS_i)$ specifies the isotopic (spatial) spin eigenstate of the incident two nucleon system. m and i denote the multiplicity and the third component of the spin respectively. Following expressions can be obtained easily:

$$2\{\sigma_{np}(\theta)+\sigma_{np}(\pi-\theta)\}-\sigma_{pp}(\theta) = \sum_{\sigma_z^1,\sigma_z^2} |f^{\text{odd}}_{{}^1T_1{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2)|^2 + \sum_i \sum_{\sigma_z^1,\sigma_z^2} |f^{\text{even}}_{{}^1T_1{}^3S_i}(\theta,\psi,\sigma_z^1,\sigma_z^2)|^2 \geqq 0, \quad (3)$$

$$\sigma_{np'}(\theta)-\sigma_{np}(\pi-\theta) = Re\Big[\sum_{\sigma_z^1,\sigma_z^2} f^{\text{even}*}_{{}^3T_1{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2) f^{\text{odd}}_{{}^1T_1{}^1S_0}(\theta,\varphi,\sigma_z^1,\sigma_z^2) + \sum_i \sum_{\sigma_z^1,\sigma_z^2} f^{\text{odd}*}_{{}^3T_0{}^3S_i}(\theta,\varphi,\sigma_z^1,\sigma_z^2) \cdot f^{\text{even}}_{{}^1T_0{}^3S_i}(\theta,\varphi,\sigma_z^1,\sigma_z^2)\Big]. \quad (4)$$

(3) is the counterpart of (1). The former depends only on ${}^1T$-interaction and the latter on ${}^3T$-interaction. And (4) gives the measure of asymmetry of n-p

scattering. From the experimental cross section we see that ${}^3T$-interaction (p-p scattering) gives the large flat cross section, while ${}^1T$-interaction (fictitious scattering) gives the small concave cross section, and that ${}^1T{}^3T$ interference is very small. These are the all information obtained from the high energy scattering data. Now it is to be noted that ${}^3T$-interaction is effective in p-p scattering as four times as in n-p scattering, provided that ${}^1T{}^3T$ interference is neglected (i.e. provided that n-p cross section is symmetric about 90°).

Further informations are obtained:

for ${}^1T{}^3$S-interaction from the properties of deuteron,

for ${}^1T{}^3S$- and ${}^3T{}^1S$-interactions from low energy scattering data,

for ${}^3T{}^3S$- and ${}^3T{}^1S$-interactions from the interference of nuclear force with Coulomb force in p-p scattering.

From this point of view, we can easily understand in the following way the reason why the above-mentioned potentials are suitable.

1) L-S coupling potential by Case and Pais.[2] (Born approximation).

Additional L-S coupling potential is effective only in the triplet states with $L\neq 0$. That L-S potential is zero in S-state and singlet states restricts effectively this potential to ${}^3T$-states (${}^3T{}^3S$-P-waves) at not too high energy (this selection rule becomes exact if one adopts the exchange property $(1-P_x)$ or $(1+P_\tau)$). And singular radial dependence makes this potential effective only at considerably high energy in cooperation with the exclusion of S-state. Flat distribution is obtained from $\sin^2\theta$ distribution of L-S contribution additional to the ordinary distribution. Symmetry of n-p cross section is expected, because Case and Pais used the Serber type potential plus L-S coupling and L-S coupling contribution does not interfere with the central and tensor contributions.

2) Singular tensor potential by Christian and Noyes.[3] (Born approximation).

This potential is first proposed as pp-potential. But it is known that it serves as n-p potential if one chooses the appropriate exchange properties. In this case all the situations are similar to the case of the L-S potential. The difference is that instead of excluding S-state it is necessary to choose the exchange factor $(1-P_x)$ or $(1+P_\tau)$, and that the flat distribution is attributed mainly to P_0 and P_1 angular distributions of the wave scattered by the tensor force.

The potential proposed by Otsuki for p-p scattering which gives good agreement in the excitation function and the angular distribution of p-p scattering, is based on the same reasoning.[5]

3) Hard core in singlet potential by Jastrow.[4] This potential is different from the others essentially. We state about it only that it uses the constructive S-D interference about 90° in consequence of the inversion of the sign of S-wave phase-shift by the presence of the hard core potential. Such an effect is not so strong in n-p scattering where singlet contribution is covered by the triplet contribution.

In this way we obtain some directions to find the type of the charge independent interactions. But, of course, to obtain a good agreement with experiment it is required to adjust the parameters by careful calculation. And some types of the interactions will be unable to give good agreement for any value of parameter and proved to be incorrect.

The author expresses his cordial thanks to Prof. M. Kobayasi for his continual encouragement and to Mr. S. Otsuki for kind discussions.

1) Series of works by G. Breit *et al.*, J. Schwinger, Phys. Rev. **78** (1950), 135.
2) K. M. Case and A. Pais, Phys. Rev. **85** (1952), 939.
3) R. S. Christian and H. P. Noyes, Phys. Rev. **79** (1950), 85.
4) R. Jastrow, Phys. Rev. **81** (1951), 636.
5) S. Otsuki, Soryusiron Kenkyu (Mimeographed circular in Japanese) **4** No. 10 (1952), 46.

The Effect of the Excited States of Nucleons in High Energy Nucleon-nucleon Scattering

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January 15, 1953

On the line of the consideration mentioned in the above letter[1] we propose another explanation of the high energy p-p scattering cross section. It is based on the assumption of the existence of nucleon isobars. Recent investigations on the meson reaction and the anomalous magnetic moment of nucleon seem to provide the evidence of the existence of the excited states of nucleons with $t=3/2$, $s=3/2$ and the excitation energy $E\fallingdotseq 250$ Mev.[2][3] And it is

the well-known conclusion of the theory of strong coupling apart from the value of excitation energy. The existence of isobar with $t=3/2$ provides the possibility to explain the large cross section at high energy as the resonance effect in ${}^3T$-states. We try to examine this possibility qualitatively by the use of the static potential of Pauli-Kusaka type[4], which allows the transition to the excited state of two nucleon system. This potential is of the form

$$V=\sum_{\alpha}\Big[A(r)(\boldsymbol{e}_I^{\alpha}\cdot\boldsymbol{e}_{II}^{\alpha}) + B(r)\left\{\frac{3}{r^2}(\boldsymbol{e}_I^{\alpha}\boldsymbol{x})(\boldsymbol{e}_{II}^{\alpha}\boldsymbol{x})-(\boldsymbol{e}_I^{\alpha}\cdot\boldsymbol{e}_{II}^{\alpha})\right\}\Big].$$

The main properties of this potential are as follows:

1) It changes the isotopic spin and spatial spin of each nucleon at the same time by 1 or 0.

2) It conserves the total isotopic spin $\boldsymbol{T}$

3) It conserves the total angular momentum $\boldsymbol{J}$. And in the case of the central force $\boldsymbol{L}$ and $\boldsymbol{S}$ are conserved separately. In the case of the tensor forces it changes $\boldsymbol{L}$ and $\boldsymbol{S}$ by $2\geqq|\Delta S|$, $|\Delta L|$, $\Delta L_z=-\Delta S_z$.

4) It conserves the parity of the total system and therefore the symmetric property of the spin part about the exchange of two particles too.

Here we consider only one excited state in which the one of nucleons is excited to the state with $t=s=3/2$. From the above selection rules, allowed intermediate states are obtained in table 1.

$$V_c=A\left(-\frac{1}{3}\pm\frac{2}{3}\right) \qquad \text{for} \quad S'=2.$$

The first terms are obtained from the diagonal part of $\boldsymbol{e}_I^{\alpha}$ and $\boldsymbol{e}_{II}^{\alpha}$, the second from the non-diagonal part of $\boldsymbol{e}_I^{\alpha}$ and $\boldsymbol{e}_{II}^{\alpha}$ corresponding to the exchange of the excitation. Plus and minus sign correspond to the states with even and odd L' respectively, which is the consequence of the symmetric properties of the spin part functions. From this we may conclude that the main contribution to resonance comes from the P-states. The tensor potentials are not so large as to alter this conclusion unless $A\ll B$. The situation is very similar to the case of the singular tensor forces. However it is an advantage of this explanation that it can be understood naturally with the plausible assumption that the anomality appears only in p-p scattering and at high energy region.

On the other hand, we must note that the following points are not yet clear.

1) The resonant contribution does not interfere so strongly that it makes n-p cross section too large and asymmetrical.

2) The P-wave contribution makes p-p cross section isotropic at high energy.

3) The resonance-like effect actually contributes in the rather low energy region 150 ~ 400 Mev.

To give any conclusion on this problem, further detailed analysis is required.

Table 1

$T=1$

initial states	L	0	1					2			3			.	.	.	.		
	S	0	1					0			1			.	.	.	.		
	J	0	0	1	2	2	2	2	2	2	2	2	3	.	.	.	.		
intermediate states	L'	2	1	1	1	1	1	0	2	2	1	1	1	.	.	.	.		
	S'	2	1	1	2	1	2	2	1	2	1	2	2	.	.	.	.		
final states	L''	0	1	1	1	1	3	1	3	2	2	2	1	3	1	3	3	.	.	.	.
	S''	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	.	.	.	.

Since the centrifugal force pushes out the wave function, the states with higher L, L'' or L' do not contribute appreciablly and are omitted in the table. Further, speaking qualitatively, the intermediate states with repulsive potentials will contribute little and the states with attractive potentials will contribute to the resonance-like effect at lower energy. By the use of Pauli-Kusaka's potential we obtain the central potential at intermediate states

$$V_c=A\left(\frac{5}{9}\pm\frac{1}{18}\right) \qquad \text{for} \quad S'=1,$$

The author wishes to express his sincere thanks to Prof. M. Kobayasi for the suggestion of this problem and the helpful interest, and to Dr. Y. Yamaguchi for his valuable advices.

1) J. Iwadare, the preceding letter, Prog. Theor. Phys. **9** (1953), 93.
2) For example, Y. Fujimoto and H. Miyazawa, Prog. Theor. Phys. **5** (1950), 1050.
3) M. Sugawara, Prog. Theor. Phys. **8** (1952), 549.
4) W. Pauli and S. Kusaka, Phys. Rev. **63** (1943), 400.

ERRATUM

Photodisintegration of the Deuterons at High Energies

Muneo SASAKI, Prog. Theor. Phys. **8** (1952), 557

We have made a mistake in the normalizing factor of P_2 and thus the expression for D state in eq. (11) must be multiplied by $3/2$. Accordingly eqs. (22), (23) and (25) are also multiplied by the same factor and eq. (26) by $(3/2)^2$.

Further, $1/k$ in eqs. (13), (16), (19) should be dropped.

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Realization of Special Contact Transformations with Static Electromagnetic Fields in Vacuo, II

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(Received December 13, 1952)

Dynamical systems realizing the momentum-coordinate transformation and coordinate-coordinate transformation are enumerated. The stigmatic transformation using an isoenergetic dynamical system is shown to be an orthogonal transformation, so that the design of a stigmatic electron microscope, is impossible. The extended canonical transformation reveals the relativistic stigmatic transformation to be a Lorentz transformation. An example of integrated energy spectrometer is presented.

§ 7. The eiconal matrix

Let $x_r(t, a_s, b_s)$, $y_r(t, a_s, b_s)$ be solutions to the canonical equations

$$\frac{dx_r}{dt}=\frac{\partial H}{\partial y_r}, \qquad \frac{dy_r}{dt}=-\frac{\partial H}{\partial x_r} \tag{11}$$

a_s, b_s denoting initial values of canonical variables x_s, y_s. Further let δx_r, δy_r be small variations of x_r, y_r corresponding to small variations δa_s, δb_s of a_s, b_s. The matrix S that transforms $(\delta a_s, \delta b_s)$ into $(\delta x_r, \delta y_r)$

$$\begin{pmatrix}\delta x_r\\ \delta y_r\end{pmatrix}=S\begin{pmatrix}\delta a_s\\ \delta b_s\end{pmatrix}, \qquad S=\left\|\begin{matrix}\frac{\partial x_r}{\partial a_s} & \frac{\partial x_r}{\partial b_s}\\ \frac{\partial y_r}{\partial a_s} & \frac{\partial y_r}{\partial b_s}\end{matrix}\right\|=\left\|\begin{matrix}S_{xa} & S_{xb}\\ S_{ya} & S_{yb}\end{matrix}\right\|, \quad s\alpha\gamma \tag{12}$$

characterizes the contact transformation $(a_s, b_s)\to(x_r, y_r)$. If S_{xa} or S_{xb} vanishes, the transformation is a mometum-coordinate transformation or coordinate-coordinate one respectively. So the matrix S might be called eiconal matrix. Since the contact transformation leaves unaltered the skew product

$$dx_r\delta y_r-dy_r\delta x_r=da_s\delta b_s-db_s\delta a_s,$$

the matrix S is a symplectic matrix with its determinant 1.

The variational equations of the canonical equations are written

$$\frac{d\delta x_r}{dt}=\frac{\partial^2 H}{\partial y_r\partial x_s}\delta x_s+\frac{\partial^2 H}{\partial y_r\partial y_s}\delta y_s,$$

$$\frac{d\delta y_r}{dt}=-\frac{\partial^2 H}{\partial x_r\partial x_s}\delta x_s-\frac{\partial^2 H}{\partial x_r\partial y_s}\delta y_s$$

or, in matrix form

$$\frac{d}{dt}\begin{pmatrix}\delta x_r\\ \delta y_r\end{pmatrix}=K\begin{pmatrix}\delta x_s\\ \delta y_s\end{pmatrix},\qquad K=\begin{Vmatrix}\dfrac{\partial^2 H}{\partial y_r\partial x_s} & \dfrac{\partial^2 H}{\partial y_r\partial y_r}\\ -\dfrac{\partial^2 H}{\partial x_r\partial x_s} & -\dfrac{\partial^2 H}{\partial x_r\partial y_s}\end{Vmatrix}.$$

Using the eiconal matrix we have

$$\frac{d}{dt}S\begin{pmatrix}\delta a_s\\ \delta b_s\end{pmatrix}-KS\begin{pmatrix}\delta a_s\\ \delta b_s\end{pmatrix}=\left(\frac{dS}{dt}-KS\right)\begin{pmatrix}\delta a_s\\ \delta b_s\end{pmatrix}=0.$$

Since $(\delta a_s, \delta b_s)$ is arbitrary, we get the equation to be satisfied by the eiconal matrix

$$\frac{dS}{dt}=KS, \tag{13}$$

the initial value of S being the unit matrix 1.

That S is symplectic might be seen also as follows. Introducing the matrix J

$$J=\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}$$

we see

$$\begin{aligned}\frac{d}{dt}(S^*JS)&=\frac{dS^*}{dt}JS+S^*J\frac{dS}{dt}\qquad * \text{ transposition}\\ &=S^*K^*JS+S^*JKS\\ &=S^*(K^*J+JK)S.\end{aligned}$$

As is easily verified,

$$K^*J+JK=0.$$

So S^*JS preserves its initial value J. The condition $S^*JS=J$ is nothing but the definition of a symplectic matrix.

When the Hamiltonian H contains no terms of higher degree than 2 in x, y, the elements of the matrix K are all constant, so we have

$$S(t)=e^{Kt}.$$

If the matrix K contains canonical variables x, y, the perturbation method may be applied to.

§ 8. The momentum-coordinate transformation

Resuming the path taken in § 6 of the preceding paper, we shall exhaust dynamical systems which allow the momentum-coordinate transformation. Because the matrix C must satisfy the condition $CC^*=C^*C$, the matrix $\gamma=CC^*$ takes a diagonal form

$$\gamma=\begin{Vmatrix}\lambda^2 & 0 & 0\\ 0 & \mu^2 & 0\\ 0 & 0 & \nu^2\end{Vmatrix}, \qquad \lambda^2+\mu^2+\nu^2=2\omega^2$$

so we have the Hamiltonian

$$H=\frac{1}{2}(y_1^2+y_2^2+y_3^2)-\omega(y_1x_2-y_2x_1)+\frac{1}{2}(\lambda^2x_1^2+\mu^2x_2^2+\nu^2x_3^2)$$

which enables us to separate the motion of a particle into the motion in the x_1-x_2-plane and that in the x_3-direction.

The motion in the x_1-x_2-plane will be described by the Hamiltonian H'

$$H'=\frac{1}{2}(y_1^2+y_2^2)-\omega(y_1x_2-y_2x_1)+\frac{1}{2}(\lambda^2x_1^2+\mu^2x_2^2)$$

which gives the matrix K

$$K=\begin{Vmatrix}0 & -\omega & 1 & 0\\ \omega & 0 & 0 & 1\\ -\lambda^2 & 0 & 0 & -\omega\\ 0 & -\mu^2 & \omega & 0\end{Vmatrix}.$$

Since the rank of the matrix is 4, the eiconal matrix $S(t)=e^{Kt}$ can be expressed as

$$S(t)=e^{Kt}=d_0 1+d_1K+d_2K^2+d_3K^3$$

d_0, d_1, d_2, d_3 being functions of t.

The small matrix s_{xa} of the eiconal matrix $S(t)$ is computed to be

$$s_{xa}=\begin{Vmatrix}d_0-(\omega^2+\lambda^2)d_2 & -\omega d_1+\omega(\omega^2+\lambda^2+2\mu^2)d_3\\ \omega d_1-\omega(\omega^2+\mu^2+2\lambda^2)d_3 & d_0-(\omega^2+\mu^2)d_2\end{Vmatrix}.$$

The momentum-coordinate transformation is not realizable unless the matrix s_{xa} vanishes at a certain instant. The vanishing of s_{xa} requires 4 conditions

$$d_0-(\omega^2+\lambda^2)d_2=0, \qquad d_1-(\omega^2+\mu^2+2\lambda^2)d_3=0,$$
$$d_0-(\omega^2+\mu^2)d_2=0, \qquad d_1-(\omega^2+\lambda^2+2\mu^2)d_3=0$$

which entail, under the assumption $\lambda^2\neq\mu^2$, the vanishing of d_i

$$d_1=d_2=d_3=d_4=0, \quad \text{so} \quad S=0.$$

For the sake of $det\,S=1$, the assumption $\lambda^2\neq\mu^2$ must be put aside. Therefore there remains only the case $\lambda^2=\mu^2$ in which we perform the change of variables as in § 6 and get the Hamiltonian

$$\frac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)+\frac{1}{2}(\lambda^2\xi_1^2+\lambda^2\xi_2^2+\nu^2\xi_3^2).$$

The eiconal matrix has its small matrix s_{xq}

$$s_{xa}=\begin{Vmatrix} \cos\lambda t & 0 & 0 \\ 0 & \cos\lambda t & 0 \\ 0 & 0 & \cos\nu t \end{Vmatrix}$$

the vanishing of which requires

$$\cos\lambda t=\cos\nu t=0$$

or

$$\lambda:\nu=2m+1:2n+1$$

m, n being integers. Among these cases falls the case $m=n$ or $\lambda^2=\nu^2$ which was already mentioned in § 6.

§ 9. The coordinate-coordinate transformation

The momentum-coordinate transformation realizable with static electromagnetic field is seen to be of the form

$$b_r=c_{rs}x_s,$$

$$y_r=-c_{sr}a_s$$

in § 5. This transformation repeated twice gives a coordinate-coordinate transformation.

Further we shall exhaust all dynamical systems allowing the coordinate-coordinate transformation under the assumptions:

1) the magnetic field is uniform, or the vector potentials are linear functions of the coordinates,

2) the scalar potential is a quadratic form of the coordinates.

As in § 6, we can put

$$A_1(x)=\omega x_2,\quad A_2(x)=-\omega x_1,\quad A_3(x)=0$$

then the condition (4″) may be rewritten

$$dx_1\delta x_2-dx_2\delta x_1=da_1\delta a_2-da_2\delta a_1. \tag{14}$$

Denoting the coordinate-coordinate transformation, which is orthogonal, as

$$x_r=c_{rs}a_s,$$

we have from (14)

$$\begin{aligned} dx_1\delta x_2-dx_2\delta x_1&=c_{1i}c_{2j}da_i\delta a_j-c_{2i}c_{1j}da_i\delta a_j \\ &=(c_{1i}c_{2j}-c_{2i}c_{1j})da_i\delta a_j \\ &=da_1\delta a_2-da_2\delta a_1. \end{aligned}$$

Therefore we have

$$\left.\begin{aligned} &c_{11}c_{22}-c_{12}c_{21}=1, \\ &c_{11}c_{23}-c_{13}c_{21}=0 \\ &c_{12}c_{23}-c_{13}c_{22}=0 \end{aligned}\right\}\rightarrow c_{13}=c_{23}=0,$$

Because of $c_{3i}c_{3i}=c_{i3}c_{i3}=1$, we see

$$c_{33}{}^2=1, \quad c_{31}=c_{32}=0.$$

Consequently the transformation takes the form

$$\left.\begin{aligned} x_1&=a_1\cos\phi-a_2\sin\phi\ ,\\ x_2&=a_1\sin\phi+a_2\cos\phi\ ,\\ x_3&=\varepsilon a_3, \qquad\qquad \varepsilon^2=1. \end{aligned}\right\} \tag{15}$$

I) $\sin\phi\neq 0$ (or $\cos^2\phi\neq 1$). In the case the quadratic form $B(x)=1/2\cdot\gamma_{ij}x_ix_j$ is simplified in the form

$$B(x)=\frac{1}{2}(\lambda^2x_1{}^2+\lambda^2x_2{}^2+\nu^2x_3{}^2), \quad 2\lambda^2+\nu^2=2\omega^2$$

because of the invariance of $B(x)$ under the transformation (15). The Hamiltonian is identical to that in § 6,

$$H=\frac{1}{2}(y_1{}^2+y_2{}^2+y_3{}^2)-\omega(y_1x_2-y_2x_1)+\frac{1}{2}(\lambda^2x_1{}^2+\lambda^2x_2{}^2+\nu^2x_2{}^2)$$

which can be transformed into

$$H'=\frac{1}{2}(\eta_1{}^2+\eta_2{}^2+\eta_3{}^2)+\frac{1}{2}(\lambda^2\xi_1{}^2+\lambda^2\xi_2{}^2+\nu^2\xi_3{}^2)$$

by a coordinate-coordinate transformation as was already seen in § 6. The small matrix s_{xb} of the eiconal matrix belonging to the Hamiltonian H' is computed to be

$$s_{xb}=\begin{Vmatrix} \dfrac{\sin\lambda t}{\lambda} & 0 & 0\\ 0 & \dfrac{\sin\lambda t}{\lambda} & 0\\ 0 & 0 & \dfrac{\sin\nu t}{\nu} \end{Vmatrix}$$

so we have the condition for the coordinate-coordinate transformation

$$\sin\lambda t=\sin\nu t=0$$

or

$$\lambda:\nu=l:n\ ,$$

l, n being integers.

II) $\sin\phi=0$.

IIa) $\cos\phi=\varepsilon$. In the case the coordinate-coordinate transformation is written as

$$x_r=\varepsilon a_r, \qquad y_r=\varepsilon b_r$$

so that the eiconal matrix is $\varepsilon 1$. Therefore

$$S(t)=e^{Kt}=\varepsilon 1. \tag{16}$$

Since the eigenvalues of the matrix K can be denoted by $\varkappa_1, \varkappa_2, \varkappa_3, -\varkappa_1, -\varkappa_2, -\varkappa_3$, we have

$$S(t) = TD(t)T^{-1},$$

T being a non-singular matrix, $D(t)$ being a diagonal matrix with its elements $\exp \varkappa_r t$, $\exp(-\varkappa_r t)$, $r=1, 2, 3$. Therefore the condition (16) is satisfied by

$$\exp \varkappa_r t = \varepsilon, \qquad r=1, 2, 3.$$

So we have

$$\varkappa_1, \varkappa_2, \varkappa_3 \quad \text{purely imaginary numbers,}$$

$$\varepsilon=1, \quad \varkappa_1 : \varkappa_2 : \varkappa_3 = l : m : n,$$

$$\varepsilon=-1, \quad \varkappa_1 : \varkappa_2 : \varkappa_3 = 2l+1 : 2m+1 : 2n+1,$$

l, m, n being integers.

IIb) $\cos\phi = -\varepsilon$. In this case we can assume

$$B(x) = \frac{1}{2}(\gamma_{11}x_1^2 + 2\gamma_{12}x_1x_2 + \gamma_{22}x_2^2 + \gamma_{33}x_3^2),$$

$$\gamma_{11}+\gamma_{22}+\gamma_{33} = 2\omega^2$$

and separate the Hamiltonian into the sum of H' and H'':

$$H' = \frac{1}{2}(y_1^2+y_2^2) - \omega(y_1x_2 - y_2x_1) + \frac{1}{2}(\gamma_{11}x_1^2 + 2\gamma_{12}x_1x_2 + \gamma_{22}x_2^2),$$

$$H'' = \frac{1}{2}y_3^2 + \frac{1}{2}\gamma_{33}x_3^2.$$

Using the preceding calculation we have the conditions for the coordinate-coordinate transformation

$$\exp \varkappa_1 t = \exp \varkappa_2 t = -\exp \varkappa_3 t = \varepsilon.$$

$\varkappa_1, \varkappa_2$ being the eigenvalues of the matrix K associated to the Hamiltonian H', $\varkappa_3$ to H''. So we get

$$\varkappa_1, \varkappa_2, \varkappa_3 \quad \text{purely imaginary numbers,}$$

$$\varepsilon=1, \quad \varkappa_1 : \varkappa_2 : \varkappa_3 = 2l+1 : 2m+1 : 2n,$$

$$\varepsilon=-1, \quad \varkappa_1 : \varkappa_2 : \varkappa_3 = 2l : 2m : 2n+1,$$

l, m, n being integers.

§ 10. The isoenergetic stigmatic transformation

The equation of motion of a particle can be cast in the variation principle

$$\delta\int \omega_d(x) = \delta\int y_r dx_r - H dt.$$

For a particle of energy h, the variation principle can be put in the form

$$\delta\int y_r dx_r = 0\,.$$

Using the relations

$$y_r = \frac{dx_r}{dt} + \frac{e}{mc} A_r(x),$$

$$H = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 + \frac{e}{m} A(x) = h,$$

$$ds^2 = dx_r dx_r,$$

we have

$$y_r dx_r = \frac{dx_r}{dt} dx_r + \frac{e}{mc} A_r(x)\, dx_r$$

$$= \left(2h - \frac{2e}{m} A(x)\right)^{\frac{1}{2}} ds + \frac{e}{mc} A_r(x) dx_r = \mathfrak{W}_a(x), \text{ say}\,.$$

So we get the variation principle for an isoenergetic system

$$\delta\int \mathfrak{W}_a(x) = 0. \tag{17}$$

If a stigmatic transformation $(a_1, a_2, a_3) \to (x_1, x_2, x_3)$, $x_r = x_r(a)$ is realizable by a dynamical system with its Lagrange form $\mathfrak{W}_a(x)$, the transformation must satisfy the condition

$$\mathfrak{W}_a(x) = \mathfrak{W}_a(a) + d\psi$$

$d\psi$ being a perfect differential.

Therefore we have a necessary condition for a stigmatic transformation

$$\left(2h - \frac{2e}{m} A(x)\right)^{\frac{1}{2}} (dx_r dx_r)^{\frac{1}{2}} + \frac{e}{mc} A_r(x) dx_r$$

$$= \left(2h - \frac{2e}{m} A(a)\right)^{\frac{1}{2}} (da_s da_s) + \frac{e}{mc} A_s(a) da_s + d\psi.$$

Since the differentials da_s are arbitrary and those dx_r are to be replaced by $(\partial x_r/\partial a_s) da_s$, we have, following the line of reasoning of Caratheodory,

$$\left(h - \frac{e}{m} A(x)\right) dx_r dx_r = \left(h - \frac{e}{m} A(a)\right) da_s da_s\,, \tag{18}$$

$$\frac{e}{mc} A_r(x)\, dx_r = \frac{e}{mc} A_s(a) dx_s + d\psi. \tag{19}$$

While (19) is a gauge transformation, (18) is a conformal transformation. A general conformal transformation $(a_r) \to (x_r)$ was proved to be

$$dx_r dx_r = \frac{k^2}{[(a_i-\alpha_i)(a_i-\alpha_i)]^2} da_s da_s$$

by Liouville a century ago, k, α_i being constants.

If we put

$$\xi_r = \frac{k(a_r-\alpha_r)}{(a_i-\alpha_i)(a_i-\alpha_i)}$$

we have

$$\xi_r \xi_r = \frac{k^2}{(a_i-\alpha_i)(a_i-\alpha_i)}$$

and

$$d\xi_r d\xi_r = \frac{k^2}{[(a_i-\alpha_i)(a_i-\alpha_i)]^2} da_s da_s$$

$$= \frac{\xi_k \xi_k}{(a_i-\alpha_i)(a_i-\alpha_i)} da_s da_s$$

or

$$\frac{d\xi_r d\xi_r}{\xi_k \xi_k} = \frac{da_s da_s}{(a_i-\alpha_i)(a_i-\alpha_i)}.$$

Putting $\xi_r = x_r - \alpha_r$, we have an invariant form for a conformal transformation

$$\frac{dx_r dx_r}{(x_k-\alpha_k)(x_k-\alpha_k)} = \frac{da_s da_s}{(a_i-\alpha_i)(a_i-\alpha_i)}. \tag{20}$$

The condition that (20) be equivalent to (18) is

$$h - \frac{e}{m} A(x) = \frac{C}{(x_k-\alpha_k)(x_k-\alpha_k)}$$

C being a constant. The left member, however, should satisfy the Laplacian equation

$$\frac{\partial}{\partial x_r} \frac{\partial}{\partial x_r} \left(h - \frac{e}{m} A(x) \right) = 0,$$

so that the constant C must be zero if the constants α_i be finite. When α_i become infinite, (20) reduces to

$$dx_r dx_r = da_s da_s, \tag{21}$$

whence we have

$$A(x) = A(a). \tag{22}$$

These conditions (21), (22), (19) are those obtained in § 4. Consequently the necessary conditions obtained so far for the coordinate-coordinate transformation (or stigmatic transformation) are indifferent to whether the dynamical system is isoenergetic or not. It is inferred at least that to construct a stigmatic electron microscope is impossible since a stigmatic contact transformation is an orthogonal transformation.

§ 11. An integrated energy spectrometer

It results from the reasoning in § 10 that the condition obtained for stigmaticity (anastigmat) is seemingly indifferent to whether the dynamical system is isoenergetic or not. Although the condition is necessary and not sufficient, there remain few hopes of obtaining the dispersion of energy using a stigmatic transformation. The dynamical system found in § 6, 1, however, may give us a hope to devise an integrated energy spectrometer.

The motion of a particle there is given by

$$\xi_1=\alpha_1\cos\nu t+\frac{\beta_1}{\nu}\sin\nu t\,,$$

$$\xi_2=\alpha_2\cos\nu t+\frac{\beta_2}{\nu}\sin\nu t\,,$$

$$\xi_3=\alpha_3\cos\nu t+\frac{\beta_3}{\nu}\sin\nu t\,,$$

α_i, β_i being initial coordinates and momenta respectively. We look for the enveloping surface of the paths to be described by particles emitted from the point $(0, 0, \alpha)$ on the ξ_3-axis with the velocity $\beta=(\beta_1^2+\beta_2^2+\beta_3^2)^{\frac{1}{2}}$. So we have

$$\left.\begin{aligned}\xi_1&=\frac{\beta_1}{\nu}\sin\nu t\,,\\ \xi_2&=\frac{\beta_2}{\nu}\sin\nu t\,,\\ \xi_3&=\frac{\beta_3}{\nu}\sin\nu t+\alpha\cos\nu t.\end{aligned}\right\}\qquad(23)$$

The time when two adjacent paths meet again is to be determined by

$$\delta\xi_1=\frac{\sin\nu t}{\nu}\delta\beta_1+\beta_1\cos\nu t\,\delta t=0,$$

$$\delta\xi_2=\frac{\sin\nu t}{\nu}\delta\beta_2+\beta_2\cos\nu t\,\delta t=0,$$

$$\delta\xi_3=\frac{\sin\nu t}{\nu}\delta\beta_3+(\beta_3\cos\nu t-\nu\alpha\sin\nu t)\,\delta t=0$$

with $\beta_1^2+\beta_2^2+\beta_3^2=\beta^2$.

Multiplying $\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$ by β_1, β_2, β_3 respectively and adding them, we get

$$\tan\nu t=\frac{\beta^2}{\nu\alpha\beta_3}$$

whence

$$\sin\nu t=\frac{\beta^2}{(\beta^4+\nu^2\alpha^2\beta_3^2)^{\frac{1}{2}}},\qquad\cos\nu t=\frac{\nu\alpha\beta_3}{(\beta^4+\nu^2\alpha^2\beta_3^2)^{\frac{1}{2}}}.$$

Inserting these values in (23), we have

$$\xi_1=\frac{\beta_1\beta^2}{\nu(\beta^4+\nu^2\alpha^2\beta_3^2)^{\frac{1}{2}}},\quad \xi_2=\frac{\beta_2\beta^2}{\nu(\beta^4+\nu^2\alpha^2\beta_3^2)^{\frac{1}{2}}},\quad \xi_3=\frac{(\nu^2\alpha^2+\beta^2)\beta_3}{\nu(\beta^4+\nu^2\alpha^2\beta_3^2)^{\frac{1}{2}}}.$$

Eliminating β_1, β_2, β_3, we have

$$\frac{\xi_1^2}{\beta^2}+\frac{\xi_2^2}{\beta^2}+\frac{\xi_3^2}{\beta^2+\nu^2\alpha^2}=\frac{1}{\nu^2}.$$

This equation represents an ellipsoid of revolution having its 3 semi-axes β/ν, β/ν, $(\beta^2+\nu^2\alpha^2)^{\frac{1}{2}}/\nu$ and its two foci at the points $(0, 0, \alpha)$, $(0, 0, -\alpha)$ which are the source point of particles and its image point. If we can so arrange that all the particles which start from one focus and collide against this ellipsoid may be trapped, all the particles with the velocity less than β will be centered at the other focus. This apparatus, if realized, will present a conspicuous parallelism to an ellipsoidal mirror of revolution in which all the rays emitted from one focus meet at the other focus. Returning to the fixed coordinate system does not alter the shape of ellipsoid and the velocity.

§ 12. The relativistic stigmatic transformation

The relativistic equation of motion of a particle moving in the electromagnetic field is cast in the variation principle

$$\delta\int\Omega_a(x)=0 \tag{24}$$

with

$$\Omega_a(x)=-mc(dx_0^2-dx_1^2-dx_2^2-dx_3^2)^{\frac{1}{2}}+\frac{e}{c}(A_0dx_0+A_1dx_1+A_2dx_2+A_3dx_3),$$

x_0, A_0 denoting ct, $-A$ respectively.

Introducing an arbitrary timelike parameter $\tau=\tau(x_0, x_1, x_2, x_3)$, we define the momentum y_σ $(\sigma=0, 1, 2, 3)$ conjugate to x_σ by

$$y_\sigma=\frac{\partial L}{\partial\dfrac{dx_\sigma}{d\tau}},\qquad L(x)d\tau=\Omega_a(x)$$

and get the equation of motion

$$\frac{dy_\sigma}{d\tau}=\frac{\partial L}{\partial x_\sigma}.$$

Since the momenta y_σ are of degree 0 in dx_σ, they are independent from the parameter τ and satisfy an equation

$$F(x_\sigma, y_\sigma)=0.$$

If we evaluate the variation of the integral

$$W=\int_1^2 \Omega_a(x)=\int_{\tau_1}^{\tau_2} L(x)\,d\tau\,,$$

we have

$$\begin{aligned}\delta W&=\delta\int_{\tau_1}^{\tau_2} L(x)\,d\tau\\&=\int_{\tau_1}^{\tau_2}\left[\frac{\partial L}{\partial x_\sigma}\delta x_\sigma+y_\sigma\delta\frac{dx_\sigma}{d\tau}\right]d\tau\\&=\int_{\tau_1}^{\tau_2}\left[\frac{d}{d\tau}(y_\sigma\delta x_\sigma)+\left(\frac{\partial L}{\partial x_\sigma}-\frac{dy_\sigma}{d\tau}\right)\delta x_\sigma\right]d\tau\\&=[y_\sigma\delta x_\sigma]_1^2\end{aligned}$$

with the aid of the equation of motion.

Therefore we have

$$y_\sigma dx_\sigma-b_\sigma da_\sigma=dW$$

dW being the perfect differential, a_σ, b_σ denoting the initial values of x_σ, y_σ with the condition

$$F(a_\sigma, b_\sigma)=0.$$

If we eliminate y_0, b_0 using the relations that y_0, b_0 are equal to negative Hamiltonians $-H(x, y)$, $-H(a, b)$ respectively, we obtain

$$y_r dx_r-H(x, y)\,dx_0=b_r da_r-H(a, b)\,da_0+dW\,.$$

This is nothing but the extended canonical transformation, which can be replaced by

$$\Omega_a(x)=\Omega_a(a)+dW. \tag{25}$$

If a relativistic stigmatic transformation $(a_\sigma)\rightarrow(x_\sigma)$ is realizable, as in § 10 the condition (25) is to be satisfied identically for arbitrary differentials da_σ and $dx_\sigma=(\partial x_\sigma/\partial a_\rho)da_\rho$. So we get the conditions

$$dx_0^2-dx_1^2-dx_2^2-dx_3^2=da_0^2-da_1^2-da_2^2-da_3^2, \tag{26}$$

$$A_\sigma(x)dx_\sigma=A_\sigma(a)da_\sigma+\frac{c}{c}dW. \tag{27}$$

The relation (26) implies that the relativistic stigmatic transformation is a Lorentz transformation. On the other hand the relation (27) represents nothing but a gauge transformation. In other words a Lorentz transformation plus a gauge transformation turns out to be no more than a special case of the contact transformation, so that there appears a possibility of setting up a unified theory of fields on the basis of the contact transformation.

Whether there exists a dynamical system realizing the relativistic stigmatic transformation or not is a question to be answered in the future.

References are cited in the preceding paper which appeared in this journal Vol. 8, No. 2 (1952), 183 – 192.

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The Scattering of Gamma-ray by Nucleon and Nucleon Isobars

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Assuming the nucleon isobar with spin $J=3/2$, isotopic spin $I=3/2$ and mass $M=1210$ Mev, we have studied the scattering of gamma ray by nucleon by the covariant perturbation method. As is expected, the total cross section varies strongly with incident gamma-ray i.e., the correction for Klein-Nishina formula at the threshold energy for meson production is only of about 30 percent, but at the excitation energy of isobar a remarkable resonance scattering is found. The angular distribution has a tendency to become predominant in the forward direction with the increase of incident energy.

§ 1. Introduction

In order to examine the features of meson cloud about a nucleon, we previously investigated in detail the photomeson production[1)2)]. With the same object in mind, the problem of the scattering of gamma-ray by nucleon has been studied in this paper.

As the calculation of the lowest order perturbation may not always lead to correct results, we carried out the calculation of higher order for γ-π process[1)] and obtained a fairly good one. Such a procedure, however, is too laborious to be pursued. By the introduction of a nucleon isobar which has spin $J=3/2$, isotopic spin $I=3/2$ and follows the Rarita-Schwinger equation, we could explain fairly well both π-meson scattering by nucleon and photomeson production[2)]*. Here, Compton scattering by nucleon will be studied from the same standpoint.

From the consideration on modes of disturbance of meson cloud by gamma-ray, we may suppose in advance the following circumstances to occur for the energy region in which the experiments have been done. (1) Since the property of a meson cloud should vary exceptionally at the threshold energy of meson production, the cross section of Compton scattering is expected to show some anomaly at this energy. (2) The fact that the experimental results of π-meson scattering by nucleon[3)] and of photomeson production[4)5)] have pointed out the resonance phenomena when the energy of π-meson and gamma-ray is of about 140 Mev and 280 Mev respectively at the center of mass system suggests us a similar anomaly at this energy as well as that of (1).

There are already several attempts to clarify this problem. The effect of anomalous magnetic moment of nucleon for 100 Mev gamma-rays gives approximately 25 percent

* We refer to the paper "Pion Reactions in One Nucleon System and Nucleon Isobars" by paper [1] throughout this paper.

increase in the scattering cross section of proton over that predicted by the Klein-Nishina formula[6)7)]. But, if we recall the result that, for the higher order calculation of photomeson production at the energy of 250 Mev, the contributions from the non-static part of anomalous magnetic moment (that which depends on energy) are larger than those from the static part, it seems difficult to account adequately for the virtual meson effect only by a Pauli-type interaction in the energy region of our case.

A calculation has carried out on this problem by Sachs and Foldy[8)] in the non-relativistic, weak coupling approximation, neglecting nucleon recoil. According to their result, the anomaly of cross section mentioned in (1) really reveals itself, to the effect that, in pseudoscalar theory, the cross section has a peak value as high as 15 σ_0 at the threshold energy of meson production.

There are of course at present no experimental data to be compared with our results. However, the predictions based on various theories will necessarily prepare for the future possibility.

In order that the theoretical magnitude of cross section for meson scattering by nucleon may agree with the measurement, we have ascribed a pseudospinor character to the isobar[2)]. The coupling constant between isobar and electromagnetic field ϵ_P (or ϵ_N) has been so chosen as to lead to the experimental magnitude of photomeson production. In our Compton scattering by nucleon, the cross section is proportional to ϵ_P^4, so that the value of ϵ_P will be decided upon clearly in a future experiment.

§ 2. The method of calculation

As to the character of nucleon isobar adopted in our theory, we have explained in detail in paper [1], so only necessary equations are listed below.

The wave equation of isobar is

$$D_{\mu\nu}(\partial)\Psi_\nu=0,$$

$$D_{\mu\nu}(\partial)=(\gamma\partial+M)\delta_{\mu\nu}-\frac{1}{3}(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)+\frac{1}{3}\gamma_\mu(\gamma\partial-M)\gamma_\nu, \tag{1}$$

the propagation function is

$$\begin{aligned}\mathfrak{S}_F(t)_{\mu\nu}=\frac{1}{(1+A_1-B_1)^2t^2+(M-mA_1)^2}\Big[&\delta_{\mu\nu}(i\gamma t-M)+\frac{2}{3M^2}t_\mu(i\gamma t)t_\nu\\&-\frac{1}{3M}\{it_\mu(i\gamma t)\gamma_\nu+\gamma_\mu(i\gamma t)it_\nu\}-\frac{4}{3M}t_\mu t_\nu\\&-\frac{1}{M^2}\left(\frac{2}{3}t^2+M^2\right)\{-\gamma_\mu(i\gamma t)\gamma_\nu+\gamma_\mu(it_\nu)+(it_\mu)\gamma_\nu-M\gamma_\mu\gamma_\nu\}\Big],\end{aligned} \tag{2}$$

for $t^2+(m+\mu)^2<0$

$$A_1=\frac{i}{4\mu^2}\frac{G^2}{4\pi}\frac{P^2Q}{t_0},\qquad B_1=\frac{i}{4\mu^2}\frac{G^2}{4\pi}\frac{P^2Q^2}{-t^2},$$

$$P=\sqrt{\frac{[t^2+(m+\mu)^2][t^2+(m-\mu)^2]}{-4t^2}},\qquad Q=\sqrt{\frac{(t^2+m^2-\mu^2)^2}{-4t^2}},\tag{3}$$

for $t^2+(m+\mu)^2>0$

$$A_1=B_1=0,$$

where M, m and μ is the mass of isobar, nucleon and meson respectively.

Interaction Hamiltonian H_{int} is

$$H_{int}=H_1+H_2,$$

$$H_1=-ie\bar{\psi}\gamma_\mu\tau_P\psi A_\mu,$$

$$H_2=\left[i\frac{\epsilon_P}{m}\bar{\Psi}_\lambda\gamma_5\gamma_\mu\psi+i\frac{\epsilon_N}{m}\bar{\Psi}_\lambda\gamma_5\gamma_\mu\psi\right]F_{\lambda\mu}+\text{comp. conj.},$$

$$F_{\lambda\mu}=\frac{\partial A_\mu}{\partial x_\lambda}-\frac{\partial A_\lambda}{\partial x_\mu}.\tag{4}$$

Feynman-Dyson diagrams with regard to the Compton scattering by nucleon are following four ones. (Fig. 1～Fig. 4)

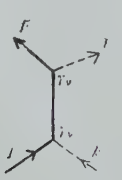
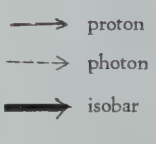


Fig. 1

Fig. 2

Fig. 3

Fig. 4



Here are four momenta I, F and k, l which are initial, final momenta of nucleon and incident, scattered photon momenta respectively; from them we eliminate one (in our case, F) making use of the relation $I+k=F+l$.

In rearranging the transition matrix elements, it should be borne in mind that the gauge invariance with respect to both k and l must be guaranteed. Though theoretically there are only ten independent gauge-invariant fundamental forms[9] in our matrix elements, we have avoided some complications in reduction and made use of the following redundant fourteen forms:

$$U_1=\frac{1}{m^3}\bar{\psi}(F)\gamma_\mu(i\gamma l)(i\gamma k)\gamma_\nu\psi(I),$$

$$U_2=\frac{1}{m^4}\bar{\psi}(F)[(kl)\gamma_\mu-k_\mu(\gamma l)](i\gamma k)\gamma_\nu\psi(I),$$

$$U_3=\frac{1}{m^4}\bar{\psi}(F)[(Il)\gamma_\mu-I_\mu(\gamma l)](i\gamma k)\gamma_\nu\psi(I),$$

$$U_4=\frac{1}{m^4}\bar{\psi}(F)\gamma_\mu(i\gamma l)[(Ik)\gamma_\nu-I_\nu(\gamma k)]\psi(I),$$

$$U_5=\frac{1}{m^4}\bar{\psi}(F)\gamma_\mu(i\gamma l)[(kl)\gamma_\nu-l_\nu(\gamma k)]\psi(I),$$

$$U_6=\frac{1}{m^5}\bar{\psi}(F)[(kl)\gamma_\mu-k_\mu(\gamma l)][(kl)\gamma_\nu-l_\nu(\gamma k)]\psi(I),$$

$$U_7=\frac{1}{m^5}\bar{\psi}(F)[(kl)\gamma_\mu-k_\mu(\gamma l)][(Ik)\gamma_\nu-I_\nu(\gamma k)]\psi(I),$$

$$U_8=\frac{1}{m^5}\bar{\psi}(F)[(Il)\gamma_\mu-I_\mu(\gamma l)][(Ik)\gamma_\nu-I_\nu(\gamma k)]\psi(I),$$

$$U_9=\frac{i}{m^6}\bar{\psi}(F)[(kl)\gamma_\mu-k_\mu(\gamma l)][(kl)I_\nu-l_\nu(Ik)]\psi(I),$$

$$U_{10}=\frac{1}{m^5}\bar{\psi}(F)[(Il)\gamma_\mu-I_\mu(\gamma l)][(kl)\gamma_\nu-l_\nu(\gamma k)]\psi(I),$$

$$U_{11}=\frac{1}{m^5}\bar{\psi}(F)[(Il)k_\mu-I_\mu(kl)][(Ik)l_\nu-I_\nu(kl)]\psi(I),$$

$$U_{12}=\frac{i}{m^6}\bar{\psi}(F)[(kl)I_\mu-k_\mu(Il)][(kl)\gamma_\nu-l_\nu(\gamma k)]\psi(I),$$

$$U_{13}=\frac{1}{m^3}\bar{\psi}(F)[k_\mu l_\nu-\delta_{\mu\nu}(kl)]\psi(I),$$

$$U_{14}=\frac{i}{m^4}\bar{\psi}(F)[k_\mu l_\nu-\delta_{\mu\nu}(kl)](\gamma l)\psi(I). \tag{5}$$

§ 3. Matrix elements

The matrix element R_A of ordinary Compton scattering (Fig. 1 and Fig. 2) can be expressed as (6) in the notation of (5).

$$R_A=\frac{ie^2}{4(2\pi)^2}\frac{1}{2k_0l_0}\left[m^2U_3-m^2U_4-\frac{2m^4}{(kl)}U_9+\frac{2(Ik)}{(kl)}m^2U_{14}\right]. \tag{6}$$

Where k_0 and l_0 stand for the energy of incident and scattered photon respectively. Also, the explicit expressions for the matrix elements R_B and R_C corresponding to Fig. 3 and Fig. 4 have the following forms:

$$R_B=\frac{i\epsilon_P^2}{4(2\pi)^2}\frac{1}{D+iE}\sum_{j=1}^{14}[G_j(1)U_j],$$

$$R_C=\frac{i\epsilon_P^2}{4(2\pi)^2}\frac{1}{C}\sum_{j=1}^{14}[G_j(2)U_j]. \tag{7}$$

$$G_1(1)=\frac{2(-Ik)}{m^2}\left(3\frac{m}{M}+\frac{8}{3}\frac{m^2}{M^2}\right)-4\left(1-\frac{2}{3}\frac{m^2}{M^2}\right)\left(1+\frac{M}{m}\right),$$

$$G_2(1)=\frac{2(-Ik)}{m^2}\left(-\frac{m^2}{M^2}\right)+\left(1+\frac{2}{3}\frac{m}{M}-\frac{4}{3}\frac{m^2}{M^2}\right),$$

$$G_3(1)=\frac{2(-Ik)}{m^2}\left(-\frac{m^2}{M^2}\right)+\left(2+\frac{2}{3}\frac{m}{M}-\frac{4}{3}\frac{m^2}{M^2}\right),$$

$$G_4(1)=\frac{8(-Ik)}{m^2}\left(\frac{m^2}{M^2}\right)+\left(-6-\frac{2}{3}\frac{m}{M}+4\frac{m^2}{M^2}\right),$$

$$G_5(1)=0,\quad G_6(1)=-m(m+M)/(kl),\quad G_7(1)=G_8(1)=(-2/3)(m^2/M^2),$$

$$G_9(1)=-2\frac{m^2}{(kl)},\quad G_{10}(1)=G_{11}(1)=G_{12}(1)=0,\quad G_{13}(1)=2\left(1+\frac{M}{m}\right)\frac{l(I+k)}{(kl)},$$

$$G_{14}(1)=-2\left[1+\frac{M}{m}+\frac{(-Ik)}{m^2}\right]\frac{m^2}{(kl)}, \tag{8}$$

and

$$G_1(2)=\frac{2(Il)}{m^2}\frac{m}{M}\left(3+\frac{8}{3}\frac{m}{M}\right)-4\left(1-\frac{2}{3}\frac{m^2}{M^2}\right)\left(1+\frac{M}{m}\right),\qquad G_2(2)=0,$$

$$G_3(2)=-8\frac{(Il)}{m^2}\frac{m^2}{M^2}+6+\frac{2}{3}\frac{m}{M}-4\frac{m^2}{M^2},$$

$$G_4(2)=2\frac{(Il)}{m^2}\frac{m^2}{M^2}-2-\frac{2}{3}\frac{m}{M}+\frac{4}{3}\frac{m^2}{M^2},$$

$$G_5(2)=-2\frac{(Il)}{m^2}\frac{m^2}{M^2}+1+\frac{2}{3}\frac{m}{M}-\frac{4}{3}\frac{m^2}{M^2},$$

$$G_6(2)=\frac{m^2}{(kl)}\left[\frac{2(Il)}{m^2}\left(6\frac{m}{M}+\frac{16}{3}\frac{m^2}{M^2}\right)-7\left(1+\frac{M}{m}\right)+\frac{16}{3}\left(\frac{m}{M}+\frac{m^2}{M^2}\right)\right],$$

$$G_7(2)=0,\quad G_8(2)=(2/3)(m^2/M^2),$$

$$G_9(2)=-\frac{4}{3}\frac{m^2}{(kl)}\left[-3\frac{(Il)}{m^2}\frac{m^2}{M^2}+3+\frac{m}{M}-2\frac{m^2}{M^2}\right],$$

$$G_{10}(2)=(-2/3)(m^2/M^2),\qquad G_{11}(2)=(-4m^2/3M^2)m^2/(kl),$$

$$G_{12}(2)=2\frac{m^2}{(kl)}\left[-8\frac{(Il)}{m^2}\frac{m^2}{M^2}+5+\frac{2}{3}\frac{m}{M}-4\frac{m^2}{M^2}\right],$$

$$G_{13}(2)=2\frac{(Il)}{(kl)}\left\{\left(1+\frac{M}{m}\right)\left(7-\frac{16}{3}\frac{m^2}{M^2}\right)+\frac{2}{3}\frac{(Ik)}{m^2}\frac{m^2}{M^2}-2\frac{(Il)}{m^2}\left(6\frac{m}{M}+\frac{16}{3}\frac{m^2}{M^2}\right)\right\}$$
$$+2+2\frac{M}{m}-\frac{4}{3}\frac{(Il)}{m^2}\frac{m^2}{M^2},$$

$$G_{14}(2)=2\frac{m^2}{(kl)}\left\{\left(1+\frac{M}{m}\right)\left(-7+\frac{16}{3}\frac{m^2}{M^2}\right)+\frac{(Il)}{m^2}\left(-5+\frac{34}{3}\frac{m}{M}+\frac{44}{3}\frac{m^2}{M^2}\right)\right.$$
$$+\frac{(Ik)}{m^2}\left(2+\frac{2}{3}\frac{m}{M}-\frac{4}{3}\frac{m^2}{M^2}\right)+8\frac{(Il)^2}{m^4}\frac{m^2}{M^2}-2\frac{(Ik)(Il)}{m^4}\left.\frac{m^2}{M^2}\right\}$$
$$+2\left(-1-\frac{2}{3}\frac{m}{M}+\frac{4}{3}\frac{m^2}{M^2}\right)+4\frac{(Il)}{m^2}\frac{m^2}{M^2}, \qquad (9)$$

where

$$D=-1+2(Ik)/m^2+(M^2/m^2)+[-(A_1+B_1)^2\{1+2(-Ik)/m^2\}+A_1^2],$$
$$E=2(A_1+B_1)\{1+2(-Ik)/m^2\}+2A_1(M/m),$$
$$C=-1+2(-Fk)/m^2+(M^2/m^2). \qquad (10)$$

We hitherto have talked about the problem of the scattering of gamma-ray by a proton; in the case of a neutron we have only to put $R_A=0$ and to replace ϵ_P by ϵ_N in R_B and R_C.

§ 4. Remarks on the calculation of cross section

Making use of the matrix elements obtained in § 3, we can compute straightforwardly the cross section of scattering, namely by

$$\sigma=\frac{4\pi^2}{m^2}\int\frac{l_0^2}{k_0^2}|(l,F|R|k,I)|^2d\Omega, \qquad (11)$$
$$R=R_A+R_B+R_C.$$

However, there are some points to be remarked. $G_j(1)$ and $G_j(2)$ in (6), ..., (9) contain the factor $1/(kl)$ in their terms. Because of $(kl)=-m(k_0-l_0)$ which is easily obtained in the laboratory system* as an expression for the conservation of momentum-energy, the situation : $(kl)=0$ takes place either at angle $\theta=0^\circ$ or in the limit of $k_0\to 0$. The values of fundamental forms U_j and their coefficients G_j cannot be computed separately. In this connection we take advantage of the following relation, though the treatment is somewhat troublesome.

After averaging over the polarizations of the incident photon, summing over those of the scattered one and making use of the equations of

$$\cos\theta=1+\frac{(kl)}{k_0l_0}, \qquad \sin^2\theta=-2\frac{(kl)}{k_0l_0}-\frac{(kl)^2}{k_0^2l_0^2}$$

derived from the well-known relation : $l_0/m=(k_0/m)/\{1+(k_0/m)(1-\cos\theta)\}$, we express $\sin^2\theta$, $\sin^4\theta$ and $\sin^2\theta\cos\theta$ in terms of (kl) and k_0l_0.

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* We use the laboratory system throughout this paper.

The final results run as follows. All the terms derived from $R_B{}^*R_B$, $R_C{}^*R_C$ and $(R_B{}^*R_C+R_C{}^*R_B)$ respectively can be arranged to the forms : $\frac{k_0{}^p l_0{}^q (kl)^r}{m^{p+q+2r}}$ $(\,;\,p+q+2r\geq 1)$* and in a similar fashion those derived from $(R_A{}^*R_B+R_B{}^*R_A)$ or $(R_A{}^*R_C+R_C{}^*R_A)$ can be reduced to the forms : $\frac{k_0{}^p l_0{}^q (kl)^r}{m^{p+q+2r}}$ $(\,;\,p+q+2r\geq 1)$, $\frac{k_0{}^p (kl)^r}{l_0 m^{p-2r-1}}$ $(\,;\,r\geq 1,\ p\geq 0$ or $r=0,\ p\geq 2)$, $\frac{l_0{}^q (kl)^r}{k_0 m^{q+2r-1}}$ $(\,;\,r\geq 1,\ q\geq 0$ or $r=0,\ q\geq 2)$ and $\frac{(kl)^r}{k_0 l_0 m^{2r-2}}$ $(\,;\,r=2,\ 3)$.

Thus, the defect mentioned above has been removed and moreover in the limit of $k_0 \to 0$, all the contributions from the nucleon isobar disappear. Consequently in such limit the cross section for our process is expressed only by the term $R_A{}^*R_A$ which is the same as Klein-Nishina formula and certainly agrees with that of the classical Thomson scattering.

§ 5. Results

We show the angular distribution and the energy dependence of the total cross section in Fig. 5 and Fig. 6.

Although the cross section at the energy of 160 Mev (near meson production threshold) gives the correction of only 30 percent to the one predicted from the Klein-Nishina formula, at the energy of 280 Mev (a little this side of resonance) it increases to about twenty times as large as σ_0. This is due to the fact that the propagation function of nucleon isobar contains a high derivative and that the denominator of R_B which gives the largest contribution is very small at this point.

There is no particular difference between the angular distribution of ours and that of Klein-Nishina formula. As is seen from Fig. 5, there appears a

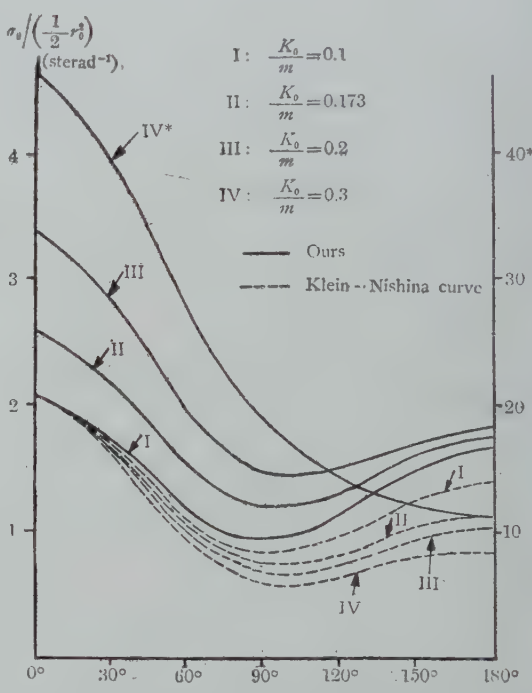

Fig. 5. Differential cross section for the scattering of gamma-ray by a proton. ($r_0=e^2/m$, $\epsilon_P=1.5e$) Only the curve IV* should be read on the scale of righthand side.

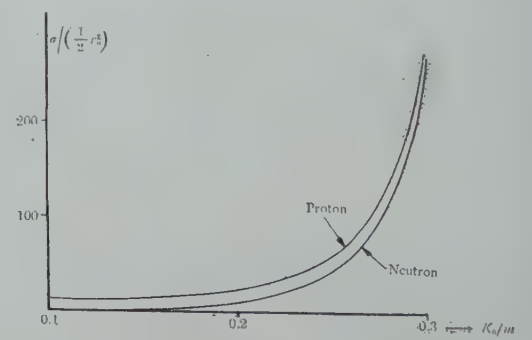

Fig. 6. Energy dependence of the total cross section ($\epsilon_P=-\epsilon_N=1.5e$)

* p, q and r are zero or positive integer.

minimum in the neighbourhood of 90° at lower energies, while the differential cross section gets remarkable predominance in the forward direction than in the backward with the increase of incident photon energy.

§ 6. Discussions

As is expected from beginning, it has been found that the resonance behavior occurs at the energy of 280 Mev but not at meson production threshold.

As for the method by which Sachs and Foldy have calculated, a typical diagram can be divided into two ones, for example by the line *a-a* of Fig. 7, first at the energy of threshold where a real process begins to arise. There the denominator of the matrix element becomes small. A virtual meson with enough energy to turn into a real one plays the most important role by being able to be emitted and reabsorbed.

As the nucleon isobar adopted by us has the mass $M=(930+280)$ Mev, our resonance energy accidently corresponds to the threshold energy of two meson production and this might be accounted for in the framework of Sachs-Foldy theory as due to the production at this energy of two real mesons, which appear, e.g., on drawing a line *b-b* in the diagram of Fig. 8.

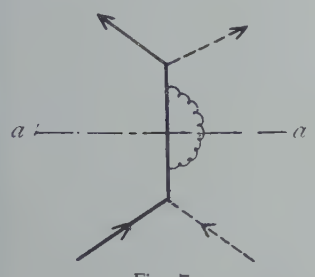

Fig. 7.

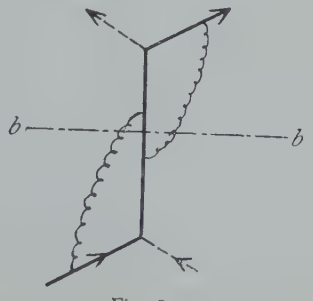

Fig. 8.

As long as Sachs and Foldy take into account the correction up to the order g^2, it is a matter of course to have no peak at the energy of 300 Mev. If an experimental data should show the resonance phenomena both at 160 Mev and at 300 Mev, neither their method nor ours will be sufficient to describe the behavior of virtual meson cloud. Resonance-like phenomena will also be expected at the threshold energies of three, four, ... meson production, although the peaks at higher energies may be so broad as to be scarcely recognizable as resonances.

It may be said that our isobar model is a good approximation so long as we confine ourselves to the energy region of about 250 ~ 350 Mev of γ-ray.

The total cross section for our process at the resonance is estimated at the order of $1/2\times10^{-29}\text{cm}^2$, i.e., about 1/20 times as large as that for meson production. In spite of large back-ground of annihilation γ-rays, the possibility of measuring the effect of this amount cannot be excluded.

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Variational Principle in Hydrodynamics

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A canonical formalism in hydrodynamics is developed by aid of Clebsch's transformation. If we consider the adiabatic process where the entropy is conserved, vortex motion is found to be closely connected with the entropy. Euler's hydrodynamical equation emerges factorised into four fundamental equations which are discussed from the thermodynamical point of view. Many important relations in hydrodynamics are naturally introduced by these equations. Landau's theory of quantum hydrodynamics is criticized upon our theory.

§ 1. Introduction

The variational method or canonical formalism has not so much been developed compared with the study of the motion of liquid itself since it seems to bear only academic interests. The reason why the author pays his attention to this problem is to criticize the possibility of quantum hydrodynamics developed by Landau.[1] According to the current theory of quantum electrodynamics, the canonical formalism of the classical part of hydrodynamics as a field theory is necessary for the purpose of quantization. The most important field quantity in hydrodynamics is the velocity field which is generally composed of two parts, namely gradient field and rotational field. From this point of view, Clebsch[2] has developed skilful representation of the velocity field of which an account can be found in Lamb's "*Hydrodynamics*". Although the Hamilton form of equations established in this book, it is presented in the hydrodynamics of substantial form and it seems inconvenient to use it for our field point of view. But it is not difficult to rewrite it using the Clebsch's transformation as will be discussed in § 2 and § 3. There emerge many interesting points in this formalism if we take into account the thermodynamical consideration. Of course, we must limit our considerations in the framework of usual variational principles to the reversible processes, in particular to adiabatic process, where the entropy is conserved. One will find in § 4 that the formalism of § 3 does not change when the geometrical parameter λ which determines the vortex lines is interpreted as the entropy S. In both cases, the canonical equations will be combined into one equation, that is, Euler's equation of motion where the geometrical parameters or thermodynamical quantities all disappear. The canonical equations themselves have their own physical meanings and many wellknown relations in hydrodynamis are easily derived from them in § 5. In addition to this, the isothermal reversible process is discussed and we have the similar equations where some thermodynamical functions are correspondingly transformed. In § 6,

the equations which govern the small deviations of density and entropy from the uniform distribution will be given. Since in accoustical approximation, the vortex or entropy is irrelevant for the equation of sound, rotational motion must be discussed in the second approximation which is difficult to handle owing to the nonlinearity of the equation. In § 7, Landau's quantum hydrodynamics is discussed in giving some commutation relations by straightforward application of the usual quantization process[3] to our canonical formalism.

§ 2. Clebsch's transformation

Let us assume that velocity field is represented by

$$\boldsymbol{v}=-\nabla\phi-\lambda\nabla\mu\,, \tag{2·1}$$

where ϕ is the velocity potential, and λ and μ are the geometrical parameters, the cross lines of the surfaces $\lambda=$constant $\mu=$const. being the vortex rings. Note that there is no rotational motion when λ is constant. The vorticity is

$$\boldsymbol{w}=\mathrm{rot}\,\boldsymbol{v}=-\nabla\lambda\times\nabla\mu\,. \tag{2·2}$$

ϕ, λ and μ are not uniquely determined by the velocity field and there are possibly a number of equivalent representation where the transformation from one representation $\phi\lambda\mu$ to another $\phi'\lambda'\mu'$ is a sort of gauge transformation. Now we study the gauge invariancy of this transformation. Multiplying the both sides of (2·1) by $d\boldsymbol{r}$, we have

$$\boldsymbol{v}\cdot d\boldsymbol{r}=-d\phi-\lambda d\mu\,, \tag{2·3}$$

where $d\phi$ and $d\mu$ are the total differentials. By the transformation from ϕ, λ and μ to ϕ', λ' and μ', the condition where the left hand side of (2·3) does not change is

$$-d(\phi-\phi'-\lambda'\mu')=\lambda d\mu+\mu' d\lambda'.$$

If we put

$$\phi-\phi'-\lambda'\mu'=w(\mu,\ \lambda'),$$

one has the following relations:

$$\lambda=-\frac{\partial w}{\partial\mu}\,,\quad \mu'=-\frac{\partial w}{\partial\lambda'}\,.$$

It will be clear that the generating function w which gives rise to the identical transformation

$$\lambda=\lambda',\quad \mu=\mu'$$

is $-\mu\lambda'$. If we denote the infinitesimal deviation from this value by the same symbol w, we have

$$\phi-\phi'-\lambda'\mu'=-\mu\lambda'+w(\mu,\ \lambda').$$

Then

$$\lambda=\lambda'-\frac{\partial w}{\partial\mu}$$

$$\mu'=\mu-\frac{\partial w}{\partial\lambda'}$$

and

$$\phi-\phi'=-\lambda'\frac{\partial\varepsilon v}{\partial\lambda'}+\varepsilon v\,.$$

This is the infinitesimal gauge transformation of Clebsch's representation of velocity field.

§ 3. Variational principle and canonical equations

It should be pointed out, however, that the fluid which will be treated in the following is of course the perfect fluid and not viscous one. Phenomena in viscous fluid must be treated as irreversible processes of which variational treatment has not yet been clarified.

The variational method in Clebsch's transformation was already established by Bateman[4]; he used the following expression as Lagrangian.

$$\mathfrak{L}=\rho\left(\frac{1}{2}\boldsymbol{v}^2-U(\rho)+\frac{D\phi}{Dt}+\lambda\frac{D\mu}{Dt}\right), \tag{3·1}$$

where ρ is the fluid density and U is the internal energy per mass and D/Dt is the substantial derivative :

$$\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla.$$

Let us consider the variation of the integral

$$I=\iint\mathfrak{L}\,d\tau\,dt\,,$$

where the quantities to be varied independently are ρ, $\boldsymbol{v}$, ϕ, λ and μ. Varying these variables in such a manner that the variations vanish on a boundary of the region of integration, we have

$$\boldsymbol{v}=-\nabla\phi-\lambda\nabla\mu \tag{3·2}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0 \tag{3·3}$$

$$\frac{D\lambda}{Dt}=0\,,\quad \frac{D\mu}{Dt}=0 \tag{3·4}$$

$$\frac{1}{2}\boldsymbol{v}^2-U-\rho\frac{\partial U}{\partial\rho}+\frac{D\phi}{Dt}+\lambda\frac{D\mu}{Dt}=0\,. \tag{3·5}$$

The equation of motion is

$$\frac{D\boldsymbol{v}}{Dt}=\frac{\partial\boldsymbol{v}}{\partial t}+\frac{1}{2}\nabla\boldsymbol{v}^2-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}$$

$$=-\nabla\left(\frac{\partial\phi}{\partial t}+\lambda\frac{\partial\mu}{\partial t}-\frac{1}{2}\boldsymbol{v}^2\right)+(\nabla\lambda)\frac{D\mu}{Dt}-(\nabla\mu)\frac{D\lambda}{Dt}\,.$$

Then upon remembering (3·4) and (3·5), this becomes

$$\frac{D\boldsymbol{v}}{Dt}=-\nabla\left(U+\rho\frac{\partial U}{\partial\rho}\right).$$

If one puts

$$\frac{\partial U}{\partial \rho}=\frac{P}{\rho^2}\,, \tag{3·6}$$

then one has

$$-dU-d\Big(\rho\frac{\partial U}{\partial \rho}\Big)=-\frac{dP}{\rho}$$

and we arrive at Euler's equation of motion in the usual form.

$$\frac{D\boldsymbol{v}}{Dt}=-\frac{1}{\rho}\nabla P\,. \tag{3·7}$$

This point is obscured in Bateman's preference of $f(\rho)=\rho U$, energy per volume, for our U, f being in relation with P in the form

$$P=\rho f'(\rho)-f(\rho)\,,$$

which allows one to interpret P as pressure. Then the relation (3·6) turns out to be the familiar thermodynamical one between internal energy U and pressure P.

As the Lagrangian has been defined, canonical momenta are derived in the usual way

$$P_\phi=\frac{\partial \mathfrak{L}}{\partial \dot{\phi}}=\rho,\quad P_\mu=\frac{\partial \mathfrak{L}}{\partial \dot{\mu}}=\rho\lambda,$$

$$P_\rho=0,\qquad P_{\rho\lambda}=0\,.$$

These relations teach us that the number of the pairs of canonical variables is two, namely $(\phi,\ \rho)$ and $(\mu,\ \rho\lambda)$. Total Lagrangian may be written as

$$\int\Big\{\rho\Big(\frac{1}{2}\boldsymbol{v}^2-U(\rho)\Big)-\phi\Big(\frac{\partial \rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}\Big)$$

$$-\mu\Big(\frac{\partial \rho\lambda}{\partial t}+\operatorname{div}\rho\lambda\boldsymbol{v}\Big)\Big\}\,d\tau\,, \tag{3·8}$$

which can be obtained by the partial integration from (3·1).

According to this Lagrangian density, we have

$$P_\rho=\phi\,,\quad P_{\rho\lambda}=\mu\,,$$

where the relations in the canonical variables are interchanged. The alternative choice of the two forms of the Lagrangian depends upon the convenience for the problem in question. Hamiltonian density is

$$\mathfrak{H}=\sum_i\rho_i\dot{q}_i-\mathfrak{L}=\frac{1}{2}\rho\boldsymbol{v}^2+\rho U\,,$$

and denotes the energy density as expected. $\mathfrak{H}$ becomes

$$\mathfrak{H}=\frac{1}{2}P_\phi(\nabla\phi)^2+P_\mu(\nabla\phi)\cdot(\nabla\mu)+\frac{1}{2}\frac{P_\mu^2}{P_\phi}(\nabla\mu)^2+P_\phi U(P_\phi) \tag{3·9}$$

in terms of the canonical variables. If the total Hamiltonian $\bar{\mathfrak{H}}$ is used, the canonical equations are

$$\dot{\phi}=\frac{\delta\bar{\mathfrak{H}}}{\delta P_\phi}, \quad \dot{P}_\phi=-\frac{\delta\bar{\mathfrak{H}}}{\delta\phi}, \tag{3·10}$$

$$\dot{\mu}=\frac{\delta\bar{\mathfrak{H}}}{\delta P_\mu}, \quad \dot{P}_\mu=-\frac{\delta\bar{\mathfrak{H}}}{\delta\mu}. \tag{3·11}$$

Note that:

$$\frac{\partial\bar{\mathfrak{H}}}{\partial P_\phi}=\frac{\partial\mathfrak{H}}{\partial P_\phi}, \quad \frac{\partial\bar{\mathfrak{H}}}{\partial\phi}=\frac{\partial\mathfrak{H}}{\partial\phi}-\sum_i\frac{\partial}{\partial x_i}\left(\frac{\partial\mathfrak{H}}{\partial(\partial\phi/\partial x_i)}\right).$$

It will be easily seen that (3·10) and (3·11) are equivalent to the equations (3·4) and (3·5) using the relations

$$P_\phi=\rho, \quad P_\mu=\rho\lambda. \tag{3·12}$$

Remembering (3·2), (3·11) and (3·12),

$$\dot{\phi}=\frac{1}{2}\boldsymbol{v}^2+\lambda\boldsymbol{v}\cdot\nabla\mu+U+\rho\frac{\partial U}{\partial\rho},$$

and

$$\dot{\mu}=\frac{\partial\bar{\mathfrak{H}}}{\partial P_\mu}=-\boldsymbol{v}\cdot\nabla\mu$$

or equivalently, $D\mu/Dt=0$. Then we get the same result as (3·5).

$$\dot{\phi}+\lambda\dot{\mu}-\frac{1}{2}\boldsymbol{v}^2-U-\rho\frac{\partial U}{\partial\rho}=0.$$

In the similar manner, we obtain

$$\dot{P}_\phi=-\frac{\delta\bar{\mathfrak{H}}}{\delta\phi}=\operatorname{div}\left\{P_\phi\left(\nabla\phi+\frac{P_\mu}{P_\phi}\nabla\mu\right)\right\}$$
$$=-\operatorname{div}\rho\boldsymbol{v},$$

that is, the equation of continuity, and

$$\dot{P}_\mu=-\frac{\delta\bar{\mathfrak{H}}}{\delta\mu}$$

becomes

$$\frac{\partial(\rho\lambda)}{\partial t}+\operatorname{div}\rho\lambda\boldsymbol{v}=0$$

which leads to $D\lambda/Dt=0$ on account of the equation of continuity. Thus we may rely exclusively upon the system of the canonical equations of this type instead of the only one equation of Euler's type.

§ 4. Thermodynamical consideration

In the preceding section, we have met with the relation between pressure and internal energy of the same type as the thermodynamical one. Thus we are led to expand our

discussions to thermodynamics and to explore whether Euler's equation should be changed or not. The internal energy U is now not only the function of ρ but also of the entropy S. The parameter λ and μ is conserved along the stream line and we know that the entropy is conserved in the reversible adiabatic process. In (3·8), let us replace λ by S, then we have the variational problem with two additive conditions, namely conservation of mass and of the entropy. The problem of this type was already treated by Zilsel[5] in his two fluid theory of liquid He II. One will see that the modified theory runs quite similarly as that of the preceding section and the thermodynamical consideration does not affect the equation of hydrodynamics.

The variational principle states that

$$\delta\int_{t_0}^{t}\int_{V}\left\{\rho\left(\frac{1}{2}\boldsymbol{v}^2-U(\rho,\ S)\right)-\phi\left(\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}\right)\right.$$
$$\left.-\mu\left(\frac{\partial\rho S}{\partial t}+\operatorname{div}\rho S\boldsymbol{v}\right)\right\}d\tau\,dt=0\,,$$

where ϕ and μ play the rôles of Lagrange's undetermined multipliers. Varying ρ, S and $\boldsymbol{v}$, we have

$$\frac{1}{2}\boldsymbol{v}^2-\rho\frac{\partial U}{\partial\rho}-U+\boldsymbol{v}\cdot\nabla\phi+\frac{\partial\phi}{\partial t}+S\frac{\partial\mu}{\partial t}+S\boldsymbol{v}\cdot\nabla\mu=0\,,$$
$$-\frac{\partial U}{\partial S}+\frac{\partial\mu}{\partial t}+\boldsymbol{v}\cdot\nabla\mu=0\,,$$

and

$$\boldsymbol{v}=-\nabla\phi-\lambda\nabla\mu\,.$$

Of course, these results and the two additive conditions will be obtained from the Lagrangian

$$\mathfrak{L}=\rho\left(\frac{1}{2}\boldsymbol{v}^2-U(\rho,\ S)+\frac{D\phi}{Dt}+S\frac{D\mu}{Dt}\right)$$

by varying ρ, S, ϕ, μ and $\boldsymbol{v}$. Then

$$\frac{D\boldsymbol{v}}{Dt}=-\nabla\left(U+\rho\frac{\partial U}{\partial\rho}\right)+\frac{\partial U}{\partial S}\nabla S\,.$$

The total differential relation in § 3 now becomes

$$-dU-d\left(\rho\frac{\partial U}{\partial\rho}\right)+TdS\qquad\left(\because\ \frac{\partial U}{\partial S}=T\right)$$
$$=-2\frac{\partial U}{\partial\rho}d\rho-\rho d\left(\frac{\partial U}{\partial\rho}\right)$$
$$=-\frac{dP}{\rho}\,.$$

Consequently we have obtained the Euler equation again. Canonical equations are easily obtained by replacing λ in Clebsch's transformation by S and taking account of the fact

that U is also the function of S, namely

$$\dot{\phi}=\frac{1}{2}\boldsymbol{v}^2+S\boldsymbol{v}\cdot\nabla\mu+U+\frac{P}{\rho}-ST, \tag{4·1}$$

$$\frac{D\mu}{Dt}=T, \tag{4·2}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0, \tag{4·3}$$

$$\frac{\partial\rho S}{\partial t}+\operatorname{div}\rho S\boldsymbol{v}=0. \tag{4·4}$$

The last equation for the conservation of the entropy may also be written as follows:

$$\frac{DS}{Dt}=0, \tag{4·5}$$

by means of (4·3). The entropy is conserved along the stream line, and we are dealing with adiabatic changes.

A reversible process may not only be adiabatic but also be isothermal. In an isothermal reversible process the necessary condition is

$$\delta T=0$$

corresponding to $\delta S=0$ in adiabatic process. This amounts to hydrodynamically

$$\frac{\partial T}{\partial t}+\boldsymbol{v}\cdot\nabla T=0$$

or by virtue of the equation of continuity,

$$\frac{\partial\rho T}{\partial t}+\operatorname{div}\rho T\boldsymbol{v}=0.$$

If the canonical formalism in this case is worked out, there appear temperature T and free energy F in places of S and U respectively. Using the total Hamiltonian

$$\tilde{\mathfrak{H}}=\int\rho\left(\frac{1}{2}\boldsymbol{v}^2+F\right)d\tau,$$

we have

$$\dot{\phi}=\frac{1}{2}\boldsymbol{v}^2+T\boldsymbol{v}\cdot\nabla\mu+F+\frac{P}{\rho}+ST,$$

$$\dot{\mu}+\boldsymbol{v}\cdot\nabla\mu=-S,$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0,$$

$$\frac{\partial\rho T}{\partial t}+\rho T\boldsymbol{v}=0.$$

§ 5. Hydrodynamical relations

From the fundamental equations in the preceding section, we can obtain various relations in hydrodynamics. I think, these calculations will be necessary to support the adequacy of our standpoint that the system of canonical equations is fully equivalent to the usual formulation of hydrodynamics.

Vorticity $\boldsymbol{w}$ is given by

$$\boldsymbol{w}=\mathrm{rot}\,\boldsymbol{v}=\nabla\mu\times\nabla S\,, \qquad (adiabatic)$$

or

$$\boldsymbol{w}=\mathrm{rot}\,\boldsymbol{v}'=\nabla\mu'\times\nabla T\,. \qquad (isothermal)$$

Since

$$\frac{D\mu}{Dt}=T \quad \text{and} \quad \frac{DS}{Dt}=0\,,$$

the rate of change of vorticity is

$$\frac{\partial \boldsymbol{w}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$$
$$=(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}-\boldsymbol{w}\,\mathrm{div}\boldsymbol{v}+(\nabla T\times\nabla S)\,.$$

Integrating over the volume V, we have

$$\frac{\partial \boldsymbol{W}}{\partial t}+\int \boldsymbol{w}\boldsymbol{v}\cdot d\boldsymbol{f}=\int \boldsymbol{v}\boldsymbol{w}\cdot d\boldsymbol{f}+\int (\nabla T)\times(\nabla S)\,d\tau\,, \qquad (5\cdot1)$$

where $\boldsymbol{W}$ is the total vorticity

$$\boldsymbol{W}=\int_V \boldsymbol{w}\,d\tau$$

and $d\boldsymbol{f}$ is the elementary surface vector of the volume. If we employ the kinematic transport theorem:

$$\frac{D}{Dt}\int_V \boldsymbol{w}\,d\tau=\left(\frac{\partial}{\partial t}\right)\int_{V_0} \boldsymbol{w}\,d\tau+\int_{\mathfrak{S}} \boldsymbol{w}\boldsymbol{v}\cdot d\boldsymbol{f}\,.$$

Then (5·1) becomes

$$\frac{D}{Dt}\int_V \boldsymbol{w}\,d\tau=\int_{\mathfrak{S}} \boldsymbol{v}\boldsymbol{w}\cdot d\boldsymbol{f}+\int_V (\nabla T\times\nabla S)\,d\tau\,.$$

The first term of the right hand side represents the rate of convection of vorticity through the surface, and the second term may be written by a partial integration as

$$\int_V (\nabla T\times\nabla S)\,d\tau=\int_{\mathfrak{S}} T\nabla S\times d\boldsymbol{f}$$
$$=-\int_{\mathfrak{S}} S\nabla T\times d\boldsymbol{f}\,,$$

which is the rate of diffusion of vorticity. The same result is obtained when the equations in isothermal process are used.

By (4·1) and (4·2)

$$\frac{\partial\phi}{\partial t}+S\frac{\partial\mu}{\partial t}=\frac{1}{2}\boldsymbol{v}^2+U+\frac{P}{\rho}$$
$$=\frac{1}{2}\boldsymbol{v}^2+H,$$

where H is Helmholtz's "*enthalpy*". If the left side is independent of time, we have

$$C=\frac{1}{2}\boldsymbol{v}^2+H$$

where C is a constant and this is Bernoulli's equation. More generally, the rate of the change of the energy is

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{1}{2}\boldsymbol{v}^2+U\right)\right\}=-\operatorname{div}\left\{\rho\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2+H\right)\right\}.$$

In isothermal case, we have

$$C'=\frac{1}{2}\boldsymbol{v}^2+G$$

and

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{1}{2}\boldsymbol{v}^2+F\right)\right\}=-\operatorname{div}\left\{\rho\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2+G\right)\right\}$$

where G is Gibbs' "*thermodynamical potential*. Of course, these results have been obtained by many authors[6,7] using Euler's equation directly, thermodynamical conditions being taken into account afterwards. In this paper, the thermodynamical condition is first assumed and hereafter the field quantities are related with this.

§ 6. The equation of sound

Let us now consider the small deviations of the density and of the entropy from their uniform distributions. These quantities are written as follows:

$$\begin{aligned}\rho&=\rho_0+\rho',\\ S&=S_0+S',\end{aligned}\qquad(6\cdot1)$$

where ρ_0 and S_0 are constant over the whole region, but ρ' and S' are small quantities compared with ρ_0 and S_0. Substituting (6·1) into the equation of continuity and neglecting the last term which contains ρ', we have

$$\frac{\partial\rho}{\partial t}-\rho_0\operatorname{div}(\nabla\phi+S\nabla\mu)=0.$$

Upon differentiation with respect to t, it becomes

$$\frac{\partial^2\rho}{\partial t^2}-\rho_0\operatorname{div}\{\nabla(\dot{\phi}+S\dot{\mu})+\dot{S}\nabla\mu-\dot{\mu}\nabla S\}=0.\qquad(6\cdot2)$$

Using the relations (4·1), (4·2) and (4·5), then we get

$$\frac{\partial^2\rho}{\partial t^2}-\rho_0\,\mathrm{div}\left\{\nabla\left(\frac{1}{2}v^2+U+\frac{P}{\rho}\right)-(\boldsymbol{v}\cdot\nabla S)\nabla\mu\right.$$
$$\left.-T\nabla S+(\boldsymbol{v}\cdot\nabla\mu)\nabla S\right\}=0\,,$$

where the quantity

$$(\boldsymbol{v}\cdot\nabla\mu)\nabla S-(\boldsymbol{v}\cdot\nabla S)\nabla\mu$$

becomes

$$-\boldsymbol{v}\times(\mathrm{rot}\,\boldsymbol{v})\,.$$

Thus the terms in the divergence of (6·2) will be calculated as follows;

$$\left(\frac{\partial U}{\partial\rho}\right)_S\nabla\rho'+\left(\frac{\partial U}{\partial S}\right)_\rho\nabla S'+\frac{1}{\rho_0}\left(\frac{\partial P}{\partial\rho}\right)_S\nabla\rho'+\frac{1}{\rho_0}\left(\frac{\partial P}{\partial S}\right)_\rho\nabla S'$$
$$-\frac{P}{\rho_0^2}\nabla\rho'-T\nabla S'+\frac{1}{2}\nabla(\boldsymbol{v})^2-\boldsymbol{v}\times(\mathrm{rot}\,\boldsymbol{v})$$
$$=\frac{1}{\rho_0}\left(\frac{\partial P}{\partial\rho}\right)_S\nabla\rho'+\frac{1}{\rho_0}\left(\frac{\partial P}{\partial S}\right)_\rho\nabla S'+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\,.$$

The last term can be neglected when we assume the velocity to be small of first order. Now we have

$$\frac{\partial^2\rho}{\partial t^2}=\left(\frac{\partial P}{\partial\rho}\right)_S\Delta\rho+\left(\frac{\partial P}{\partial S}\right)_\rho\Delta S\,.$$

Only under the isentropic condition where the entropy is constant everywhere and hence, also constant in course of time, the usual wave equation will issue:

$$\frac{\partial^2\rho}{\partial t^2}=\left(\frac{\partial P}{\partial\rho}\right)_S\Delta\rho\,.$$

In isothermal process, where the temperature is held constant as in the foregoing case, we have

$$\frac{\partial^2\rho}{\partial t^2}=\left(\frac{\partial P}{\partial\rho}\right)_T\Delta\rho\,.$$

§ 7. Remarks on quantum hydrodynamics

Landau has given the idea of quantum hydrodynamics in his paper on the theory of liquid He II where he asserted that "*phonon*" might be associated with the quantised gradient field and "*roton*" with the rotational field. Of course, the positive support for this idea has not yet been obtained till nowadays, especially for the existence of "*roton*". He laid down the commutation relations among ρ, $\boldsymbol{v}$ and $\mathrm{rot}\,\boldsymbol{v}$, which were calculated by London[8] in somewhat different manner. London concludes that the velocity field cannot be represented by whatever linear operator. By the preceding results, we can construct the

commutation relations among the field quantities and discuss these problems from the view-point of the field theory. Comparing with the ordinary theory, there arise some difficulties in this case, one of which is that the operator ρ appears in Lagrangian or Hamiltonian as denominator, as we have seen in (3·9). However, if the vortex field does not exist, there is no difficulty in this respect, because the kinetic energy can be written as

$$\frac{1}{2}\rho(\nabla\phi)^2$$

and no field quantity appears as denominator. Difficulty appears when the vortex field is considered.

Although the current does not play so much essential a rôle as in the theory of London or Landau, it may be written as follows using the operators

$$\boldsymbol{j}=-\rho\nabla\phi-\eta\nabla\mu \qquad (\eta=\rho S)$$

or in the symmetrical form

$$\boldsymbol{j}=-\frac{\rho(\nabla\phi)+(\nabla\phi)\rho}{2}-\frac{\eta(\nabla\mu)+(\nabla\mu)\eta}{2}$$

which is still manageable. The velocity is defined by

$$\boldsymbol{v}=-\nabla\phi-\frac{\eta}{\rho}\nabla\mu$$

or

$$\boldsymbol{v}=-\nabla\phi-\frac{1}{\rho}\,\frac{\{\eta(\nabla\mu)+(\nabla\mu)\eta\}}{2}\,.$$

This has a significance only when ρ has not eigenvalue zero, and has the reciprocal eigenvalue of that of ρ. In this respect, we favour London's assertion.

While the detailed study will be left to the future, commutation relations are easily written down by assuming all the canonical variables as the operators.

$$[\rho(\boldsymbol{R}),\ \phi(\boldsymbol{S})]=\frac{\hbar}{i}\delta(\boldsymbol{R},\ \boldsymbol{S}),\quad [\eta(\boldsymbol{R}),\ \mu(\boldsymbol{S})]=\frac{\hbar}{i}\delta(\boldsymbol{R},\ \boldsymbol{S}),$$

$$[\phi(\boldsymbol{R}),\ \phi(\boldsymbol{S})]=0,\qquad [\rho(\boldsymbol{R}),\ \eta(\boldsymbol{S})]=0,$$

$$[\rho(\boldsymbol{R}),\ \rho(\boldsymbol{S})]=0,\qquad [\phi(\boldsymbol{R}),\ \mu(\boldsymbol{S})]=0,$$

$$[\mu(\boldsymbol{R}),\ \mu(\boldsymbol{S})]=0,\qquad [\rho(\boldsymbol{R}),\ \mu(\boldsymbol{S})]=0,$$

$$[\eta(\boldsymbol{R}),\ \eta(\boldsymbol{S})]=0,\qquad [\eta(\boldsymbol{R}),\ \phi(\boldsymbol{S})]=0\,.$$

It seems unreasonable to quantize the entropy as it stands. However, we may regard it merely as a geometrical parameter which plays a rôle to produce a vortex line in Clebsch's transformation. Accordingly we now understand it as a dynamical variable to describe the motion of fluid. In any case, the "entropy" is equally qualified to be treated quantum-mechanically as the density which is also a macroscopic quantity.

Now we must examine the gauge invariancy of these commutation relations. The invariancy first of all holds for the Lagrangian density. A gauge transformation for the Clebsch's transformation is of the form

$$\left.\begin{aligned} S' &= S + \frac{\partial w}{\partial \mu}, \\ \mu' &= \mu - \frac{\partial w}{\partial S'}, \\ \phi' &= \phi + S' \frac{\partial w}{\partial S'} - w \end{aligned}\right\} \tag{7·1}$$

as obtained in § 2. Remembering that $\rho = \rho'$, $v = v'$ and (7·1), the Lagrangian density becomes

$$\begin{aligned} &\rho' \left\{ \frac{\partial \phi}{\partial t} + S' \frac{\partial \mu'}{\partial t} - \frac{1}{2} v^2 - U(\rho', S') \right\} \\ &= \rho \left\{ \frac{\partial \phi}{\partial t} + \frac{\partial S'}{\partial t} \frac{\partial w}{\partial S'} + S' \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial S'} \right) - \frac{\partial w}{\partial \mu} \frac{\partial \mu}{\partial t} - \frac{\partial w}{\partial S'} \frac{\partial S'}{\partial t} \right. \\ &\quad \left. + \left(S + \frac{\partial w}{\partial \mu} \right) \left(\frac{\partial \mu}{\partial t} - \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial S'} \right) \right) \right\} - \frac{1}{2} \rho v^2 - \rho U(\rho, S') \\ &= \rho \left(\frac{\partial \phi}{\partial t} + S \frac{\partial \mu}{\partial t} \right) + \rho \left(S' - S - \frac{\partial w}{\partial \mu} \right) \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial S'} \right) \\ &\quad - \frac{1}{2} \rho v^2 - \rho U(\rho, S'). \end{aligned}$$

In order to secure the invariancy of the Lagrangian density, S should not be varied. Thus we have to restrict ourselves to the following transformation formula, instead of (7·1).

$$\left.\begin{aligned} S' &= S, \\ \mu' &= \mu - \frac{\partial w}{\partial S}, \\ \phi' &= \phi + S \frac{\partial w}{\partial S} - w \end{aligned}\right\} \tag{7·2}$$

where w is the function of S only. Expressing (7·2) in canonical variables, they run as follows :

$$\left.\begin{aligned} \eta' &= \eta, \\ \mu' &= \mu - \rho \frac{\partial w}{\partial \eta}, \\ \phi' &= \phi + \eta \frac{\partial w}{\partial \eta} - w. \end{aligned}\right\}$$

$[\rho, \rho]$, $[\eta, \eta]$ and $[\rho, \eta]$ remain untouched because of the invariancy of ρ and η. The relation between ρ and ϕ is

$$[\rho(\boldsymbol{R}),\ \phi'(\boldsymbol{S})]=\left[\rho(\boldsymbol{R}),\ \phi(\boldsymbol{S})+\eta(\boldsymbol{S})\ \frac{\partial w}{\partial \eta}\ (\boldsymbol{S})-w(\boldsymbol{S})\right]$$

$$=[\rho(\boldsymbol{R}),\ \phi(\boldsymbol{S})]+\left[\rho(\boldsymbol{R}),\ \eta(\boldsymbol{S})\frac{\partial w}{\partial \eta}(\boldsymbol{S})-w(\boldsymbol{S})\right].$$

The second term of the last formula will vanish if we assume that ρ has the nonzero eigenvalue and $1/\rho$ is commutative with ρ and η, because w is a function of η/ρ. Similarly, the invariancy of the other commutation relations is certified under the same condition. Now, the commutation relation between ρ and $\boldsymbol{j}$ or η and $\boldsymbol{j}$ is calculated respectively as follows :

$$[\rho(\boldsymbol{R}),\ \boldsymbol{j}(\boldsymbol{S})]=\rho(\boldsymbol{R})\frac{\hbar}{i}\nabla_{\boldsymbol{R}}\delta(\boldsymbol{R},\ \boldsymbol{S}),$$

$$[\eta(\boldsymbol{R}),\ \boldsymbol{j}(\boldsymbol{S})]=\eta(\boldsymbol{R})\frac{\hbar}{i}\nabla_{\boldsymbol{R}}\delta(\boldsymbol{R},\ \boldsymbol{S}),$$

where $$\nabla_{\boldsymbol{R}}=\boldsymbol{i}\frac{\partial}{\partial x_{\boldsymbol{R}}}+\boldsymbol{j}\frac{\partial}{\partial y_{\boldsymbol{R}}}+\boldsymbol{k}\frac{\partial}{\partial z_{\boldsymbol{R}}}.$$

The vector product relation between the current components which was given by London is also obtained. As for the z–component of the vector product of $\boldsymbol{j}(\boldsymbol{R})$ and $\boldsymbol{j}(\boldsymbol{S})$, we have

$$\begin{aligned}
&\boldsymbol{j}_x(\boldsymbol{R})\boldsymbol{j}_y(\boldsymbol{S})-\boldsymbol{j}_y(\boldsymbol{S})\boldsymbol{j}_x(\boldsymbol{R})\\
&=\rho(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R})\rho(\boldsymbol{S})\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})+\rho(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R})\eta(\boldsymbol{S})\frac{\partial}{\partial y_{\boldsymbol{S}}}\mu(\boldsymbol{S})\\
&+\eta(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R})\rho(\boldsymbol{S})\frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})+\eta(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R})\eta(\boldsymbol{S})\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\mu(\boldsymbol{S})\\
&-\rho(\boldsymbol{S})\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})\rho(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R})-\rho(\boldsymbol{S})\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})\eta(\boldsymbol{R})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R})\\
&-\eta(\boldsymbol{S})\frac{\partial}{\partial y_{\boldsymbol{S}}}\mu(\boldsymbol{S})\rho(\boldsymbol{R})\frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R})-\eta(\boldsymbol{S})\ \frac{\partial}{\partial x_{\boldsymbol{S}}}\mu(\boldsymbol{S})\eta(\boldsymbol{R})\frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R})
\end{aligned}$$

which will be written as

$$\begin{aligned}
&\rho(\boldsymbol{R})\left[\frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R}),\ \rho(\boldsymbol{S})\right]\frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})+\eta(\boldsymbol{R})\left[\frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R}),\ \eta(\boldsymbol{S})\right]\frac{\partial}{\partial y_{\boldsymbol{S}}}\mu(\boldsymbol{S})\\
&+\rho(\boldsymbol{R})\eta(\boldsymbol{S})\left[\frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R}),\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\mu(\boldsymbol{S})\right]+\eta(\boldsymbol{R})\rho(\boldsymbol{S})\left[\frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R}),\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\phi(\boldsymbol{S})\right]\\
&+\frac{\hbar}{i}\rho(\boldsymbol{S})\frac{\partial}{\partial y_{\boldsymbol{S}}}\delta(\boldsymbol{R},\ \boldsymbol{S})\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\phi(\boldsymbol{R})+\frac{\hbar}{i}\ \eta(\boldsymbol{S})\ \frac{\partial}{\partial y_{\boldsymbol{S}}}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial x_{\boldsymbol{R}}}\mu(\boldsymbol{R}).
\end{aligned}$$

By the relations

$$\left[\frac{\partial\phi(\boldsymbol{R})}{\partial x_{\boldsymbol{R}}},\ \rho(\boldsymbol{S})\right]=-\frac{\hbar}{i}\ \frac{\partial}{\partial x_{\boldsymbol{R}}}\delta(\boldsymbol{R},\ \boldsymbol{S}),$$

$$\left[\frac{\partial\mu(\boldsymbol{R})}{\partial x_R},\ \eta(\boldsymbol{S})\right]=-\frac{\hbar}{i}\frac{\partial}{\partial x_R}\delta(\boldsymbol{R},\ \boldsymbol{S}),$$

$$\left[\frac{\partial\phi(\boldsymbol{R})}{\partial x_R},\ \frac{\partial\mu(\boldsymbol{S})}{\partial y_S}\right]=0,$$

$$\left[\frac{\partial\mu(\boldsymbol{R})}{\partial x_R},\ \frac{\partial\phi(\boldsymbol{S})}{\partial y_S}\right]=0\ .$$

We have

$$\begin{aligned}&j_x(\boldsymbol{R})j_y(\boldsymbol{S})-j_y(\boldsymbol{S})j_x(\boldsymbol{R})\\&\quad=-\frac{\hbar}{i}\left\{\rho(\boldsymbol{R})\frac{\partial}{\partial x_R}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial y_S}\rho(\boldsymbol{S})+\eta(\boldsymbol{R})\frac{\partial}{\partial x_R}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial y_S}\mu(\boldsymbol{S})\right.\\&\quad\left.-\rho(\boldsymbol{S})\frac{\partial}{\partial y_S}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial x_R}\phi(\boldsymbol{R})-\eta(\boldsymbol{S})\frac{\partial}{\partial y_S}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial x_R}\mu(\boldsymbol{R})\right\}.\end{aligned}$$

Considering the relations

$$\begin{aligned}&\rho(\boldsymbol{R})\frac{\partial}{\partial x_R}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial y_S}\phi(\boldsymbol{S})\\&\quad=\frac{\partial}{\partial x_R}\left\{\rho(\boldsymbol{R})\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial y_S}\phi(\boldsymbol{S})\right\}-\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial\rho(\boldsymbol{R})}{\partial x_R}\frac{\partial\phi(\boldsymbol{S})}{\partial y_S}\\&\quad=\frac{\partial}{\partial x_R}\left\{\delta(\boldsymbol{R},\ \boldsymbol{S})\rho(\boldsymbol{R})\frac{\partial}{\partial y_R}\phi(\boldsymbol{R})\right\}-\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial x_R}\rho(\boldsymbol{R})\frac{\partial}{\partial y_R}\rho(\boldsymbol{R}),\\&\rho(\boldsymbol{S})\frac{\partial}{\partial y_S}\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial x_R}\phi(\boldsymbol{R})\\&\quad=-\frac{\partial}{\partial y_S}\left\{\delta(\boldsymbol{R},\ \boldsymbol{S})\rho(\boldsymbol{S})\frac{\partial}{\partial y_S}\phi(\boldsymbol{S})\right\}-\delta(\boldsymbol{R},\ \boldsymbol{S})\frac{\partial}{\partial y_S}\rho(\boldsymbol{S})\frac{\partial}{\partial x_S}\rho(\boldsymbol{S}),\end{aligned}$$

and the similar relations on η and μ, we have

$$\begin{aligned}&j_x(\boldsymbol{R})j_y(\boldsymbol{S})-j_y(\boldsymbol{S})j_x(\boldsymbol{R})\\&\quad=\frac{\hbar}{i}\left\{\frac{\partial}{\partial x_R}(\delta(\boldsymbol{R},\ \boldsymbol{S})j_y(\boldsymbol{R}))-\frac{\partial}{\partial y_S}(\delta(\boldsymbol{R},\ \boldsymbol{S})j_x(\boldsymbol{S}))\right.\\&\quad\left.-\delta(\boldsymbol{R},\ \boldsymbol{S})\left(\frac{\partial}{\partial x_R}j_y(\boldsymbol{R})-\frac{\partial}{\partial y_R}j_x(\boldsymbol{R})\right)\right\}\\&=\frac{\hbar}{i}\left\{\frac{\partial}{\partial x_R}(\delta(\boldsymbol{R},\ \boldsymbol{S})j_y(\boldsymbol{R}))-\frac{\partial}{\partial y_S}(\delta(\boldsymbol{R},\ \boldsymbol{S})j_x(\boldsymbol{S}))-\delta(\boldsymbol{R},\ \boldsymbol{S})(\mathrm{rot}\,\boldsymbol{j}(\boldsymbol{R}))_z\right\}.\end{aligned}$$

Symmetrizing with respect to $\boldsymbol{R}$ and $\boldsymbol{S}$, it becomes

$$\begin{aligned}&j_x(\boldsymbol{R})j_y(\boldsymbol{S})-j_y(\boldsymbol{S})j_x(\boldsymbol{R})+j_x(\boldsymbol{S})j_y(\boldsymbol{R})-j_y(\boldsymbol{R})j_x(\boldsymbol{S})\\&\quad=\frac{\hbar}{i}\left\{\left(\frac{\partial}{\partial x_R}+\frac{\partial}{\partial x_S}\right)(\delta(\boldsymbol{R},\ \boldsymbol{S})j_y(\boldsymbol{R}))-\left(\frac{\partial}{\partial y_R}+\frac{\partial}{\partial y_S}\right)(\delta(\boldsymbol{R},\ \boldsymbol{S})j_x(\boldsymbol{S}))\right.\end{aligned}$$

$$-2\delta(\boldsymbol{R},\ \boldsymbol{S})\,(\mathrm{rot}_{\boldsymbol{R}}\boldsymbol{j}(\boldsymbol{R}))_z\Big\} = -\frac{\hbar}{i}\delta(\boldsymbol{R},\ \boldsymbol{S})\,(\mathrm{rot}_{\boldsymbol{R}}\boldsymbol{j}(\boldsymbol{R}))_z\,.$$

Generally, one has

$$\boldsymbol{j}(\boldsymbol{R})\times\boldsymbol{j}(\boldsymbol{S})+\boldsymbol{j}(\boldsymbol{S})\times\boldsymbol{j}(\boldsymbol{R}) = -\frac{\hbar}{i}\ \delta(\boldsymbol{R},\ \boldsymbol{S})\,\mathrm{rot}_{\boldsymbol{R}}\boldsymbol{j}(\boldsymbol{R})$$

which is the same result as London's.

§ 8. Conclusions*

Although we have succeeded in constructing a Lagrangian formalism in hydrodynamics, the idea of "*roton*" has not yet been clarified in this paper. The commutation relations do not assume the validity of the standpoint of quantum hydrodynamics if our results come out to be the same as London's or Landau's.

Fluid has of course atomic structure. The proper way of treating the behavior of fluid will be: first to treat the constituent particles quantum-mechanically and then to average the physical quantities over domains of semi-microscopic scale. Whether the property of liquid will be obtained or not depends upon modes of averaging or proper choice of wave functions. But since this way of treatment is very difficult it has a significance to proceed a short-cut way as the theory expounded here. The problems left to the future are that the somewhat mysterious relation between entropy and rotational motion has not been sufficiently clarified and that the ordinary wave equation cannot be obtained without the isentropic condition. The solution of the former point will require a proof based upon the statistical or gas kinetic theory, and the latter leaves open the possibility of accounting for the existence of the "second sound" in He II from the view-point of one fluid theory.

The author is indebted to Professor Kodi Husimi for the suggestion and encouragement during this work.

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* The results which are discussed in 6 have been obtained from the theory of second quantization by T. Nishiyama.

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Some Remarks on the Photodisintegration of Deuteron, II

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Electric interactions in high energies are investigated on the photodisintegration of deuteron. The contributions of the proton current and the exchange effect are separately studied. It is found that the single particle approximation is not necessarily valid in the photodisintegration process of deuteron. The angular distribution of this process in which exchange magnetic interaction is also taken into account, is presented. It shows a fair agreement with the experimental data.

§ 1. Introduction

In the preceding paper of the same title[1]*, we have pointed out that the large cross-section obtained by recent experiments[2] in high energies may be explained by taking into account the virtual meson effect, especially by its exchange magnetic interaction. In (I) the dipole type interaction has been used for the electric interaction of deuteron with γ-ray; and, as is well known, the interaction of this type consists of the proton current part and meson exchange current part[3]. Considering that the exchange magnetic interaction has an important role in high energies, it is of interest for us to investigate the contribution of meson exchange effects in the case of electric interaction.

Moreover, as is pointed out in (I), the expansion of interaction term to multipoles fails away in high energies, and therefore, a more reasonable procedure seems to be necessary. One of this may be a treatment in which no expansion in multipoles is made or a treatment of the interaction as a whole. In this note, we shall investigate these problems, and the angular distribution of ejected protons which has not yet been presented in (I), is also discussed.

§ 2. Treatment in high energies (electric process)

At first, the electromagnetic interaction of deuteron with γ-ray is separated into two parts, the proton current part and meson exchange current part. As for the latter, the general treatment is rather difficult, and we have restricted the latter interaction by the following two conditions: i) the requirement of the gauge invariance, and ii) consistency with dipole type interaction due to meson exchange potential in low energy limits. The restriction of ii) may be implied from the viewpoint of the correspondence theory. But, it must be noticed that there remains yet some arbitrariness by this restriction.[4]

* We quote this paper as (I), hereafter.

Thus, in the first approximation of e, the gauge invariant Hamiltonian is obtained as follows :

$$H=ie\boldsymbol{A}(1/M)\,\mathrm{grad}_1+ie(\Pi_1^*\Pi_2-\Pi_1\Pi_2^*)\int_{r_1}^{r_2}\boldsymbol{A}^s\,ds\,J(\boldsymbol{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) \qquad (1)$$

where $\int ds$ means a line integral and $J(\boldsymbol{r},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$ is nuclear potential. The first and the second term of (1) represent the interaction due to proton current and longitudinal current of meson exchange respectively, and actually agree with the results obtained by Ma and others[5] if one takes the low energy (dipole) approximation.

As is mentioned above, we treat the interaction as a whole and do not expand in multipole terms ; in other words, the contribution from all the high multipoles are taken into accounts. This seems to be a rather consistent way of high energy treatment, but of course, only in nonrelativistic regions. Taking the Hulthén type potential with a finite range for nuclear potential, the calculation has been performed, where the final nucleon states are tentatively approximated by plane waves. As is well known, Born approximation is not quite valid. However, we take this approximation on the following two reasons ; that is, (i) there may be no other practical way to take into account all the multipole terms at the same time, and (ii) the contributions of the proton current and exchange effect are essentially unchanged, because the nuclear potential affects only the restricted final states ($l\sim 1$). For the ground state of deuteron, only the ^3S-state is considered and ^3D-state is neglected. However, the contribution of ^3D-state is not essential for the excitation function in high energies, and no considerable error would be brought into the results by this treatment.

The details of calculation are rather trivial, so only the results obtained are presented here.

a) Contribution of exchange effects.

In Fig. 1, the contribution of exchange effects is plotted comparing with that of proton current, where the exchange effect is significant in high energies and not so small even in low energies. The energy at which their contribution becomes comparable is about 40 Mev. or less. This non-zero contribution of exchange may be due to the fact that virtual mesons have a role even in low energies, and is consistent with the meson theoretical calculation in the lowest order.[7]

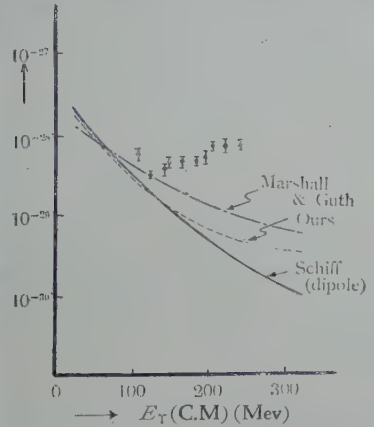

Fig. 1. Total cross-sections due to electric interactions

It may be of interest to note that the following conclusion appears from this analysis : the single particle approximation is not valid in the photodisintegration process of deuteron. At a glance, this seems to contradict with the well-known results of good validity of impulse approximation in the case of π° meson production process by $\gamma-D$ interaction. However, there is a certain difference. In $\gamma-D-\pi^\circ$ process, this interaction can occur even if there is only one nucleon ($\gamma-P-\pi^\circ, \gamma-N-\pi^\circ$), while in the disintegration

case the correlation of two particles plays an essential role. In fact, in the disintegration case, the definition of so-called "collision time" is very unclear. Furthermore, some restrictions may be given by the requirement of gauge invariance. Considering this point, if one remarks only the significant contribution of exchange magnetic moment, the validity of the impulse approximation may be quite doubtful, and thus the break-down of impulse approximation in this case seems not to be a surprising matter.**

b) Effects of high multipoles.

Contrary to our expectation led from the invalidity of expansion in high energies, the contribution of high multipoles is not so large. (Fig. 2) However, this result does not mean the validity of expansion, but indicates that the effect of high multipoles is cancelled away each other only for some accidental reason. This may be understood if one considers the case of magnetic process where the high multipoles have a rather cosiderable role.[6] The situation of cancellation of high multipole terms depends on the feature of deuteron potential, and therefore some remarks would be necessary for this point, especially in the treatment of "quasi-deuteron" in nucleus (deuteron model), where the binding energy of "quasi-deuteron" is different from the real deuteron, because of the positive energy of particles in nucleus.

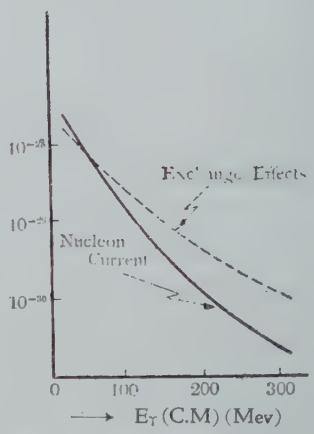

Fig. 2. Cross sections due to electric interaction

Anyhow, as far as one considers the energy region where our treatment is valid (non-relativistic region), the expansion is not so bad. And then, one may expect that the result obtained here may be in agreement with one calculated by taking into account the higher multipole terms of expansion series with an interaction form of dipole type. Marshall and Guth[6] have calculated this by taking the square-well type potential with zero range and obtained a rather large cross section. But their treatment may overestimate the cross section in high energies on account of the square-well potential. In fact, correcting the potential to Hulthén type with a finite range, which seems rather consistent with high energy treatment, a rather sufficient agreement with our results is obtained.

§ 3. Angular distribution

The results of our analysis in (I) have indicated that the main contribution of the excitation curve obtained by experiments can be explained by the exchange magnetic interaction; and if this explanation is true, the angular distribution of ejected protons should be nearly isotropic. Experimental data on the angular distribution have recently been proposed by Littauer and Keck,[2] and is represented here with our calculated results.

** Strictly speaking, the impulse approximation is not yet correctly defined in the case containing the capture process. In usual, however, the word "impulse approximation" seems to be used even in these cases, and here this word is tentatively used in the meaning of usual sense, though not correct.

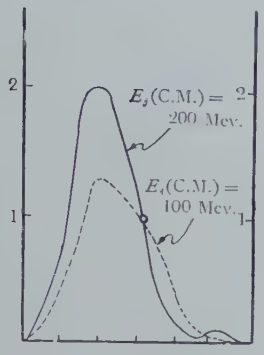

Fig. 3. Angular distribution of ejected protons by electric interactions. (Normarized at 90°)

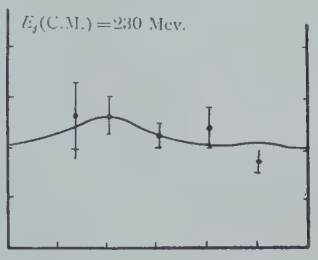

Fig 4. Angular distribution of ejected protons. Experimental results is obtained by Littauer & Keck.

Angular distribution due to electric interaction is calculated by the treatment described in § 2 ; the result is almost the same as that obtained by Schiff and others in the case of dipole and quadrupole interaction. This is because of the cancellation of high multipole effects, as is already mentioned. In addition to it, slight differences appear in high energies, which comes from exchange effects. In Fig. 3. the angular distribution due to electric part is plotted. Thus the final result of angular distribution is obtained by simply summing up the effects of electric and magnetic processes because there is no interference between them, and is shown in Fig. 4, which seems to be in a fair agreement with experiments.

In the above description, the contribution of ^3D-state has been neglected, but it may play some role in the angular distribution. We have roughly estimated the contribution of ^3D-state, which affects as improve the tendency better (give rise the forward scattering).

§ 4. Concluding remarks

In summarizing, our treatment seems to be rather reasonable one explaining the photodisintegration process of deuteron in phenomenological way. And, therefore, to investigate the validity of this treatment by analyzing the other reactions may be of interest. One of the example is the photonuclear reaction process in high energies. In this case, the high frequency part of momentum has an essential role and exchange effects may be important. Thus, except the "real meson effect", the single particle approximation may be not valid, as is mentioned above. A first step to reconcile this defect in nuclear reaction, we should investigate this process regarding it as an ensemble of two nucleon system interacting with γ-ray, or in other words, the quasi-deuteron model, which has been first developed by Levinger[9] in his elaborate work. The result will be reported in the forth-coming paper.

In the calculation of this note, however, we have restricted the treatment in non relativistic regions, and then might contain some error in high energies. To obtain more definite conclusions, the relativistic treatment would be necessary. An attempt has been

made recently by Villars[10] in the calculation of exchange current, but a further development on this point may be necessary.

I should like to express my cordial thanks to Prof. S. Sakata for his kind interest in this work, and to Messrs. S. Ogawa and Y. Nagahara for their valuable discussions. The author is also indebted to Yukawa Yomiuri Fellowship for the financial aid.

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Note added in proof : A new experimental data is presented after this paper is set in press, (W. S. Gilbert and J. W. Rosen, Phys. Rev. **88** (1952), 901), where the angular distribution is rather different from isotropic one as indicated by Keck et al.[2] This cannot be explained by our treatment, and it seems that one of the way is to adopt the isobar model, where the P-state may have a rather significant role.

However, in view of the discrepancy of the two experimental data, it is supposed that a definite conclusion should be postponed until the precise experiment is performed.

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On Yukawa's Theory of Non-local Field, II

— *The Interaction of Non-local Fields* —

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The interaction of non-local fields is investigated using the S-matrix formalism given by Yukawa. In § 2 the self-energy of a local spinor particle interacting with a non-local scalar field is calculated in the second order approximation of the coupling constant. The result agrees with that obtained from the local theory except for a meaningless constant factor. The result is compared with those of Yennie and of Rayski, and it is pointed out that such an agreement is a direct consequence of the quantization rule adopted by Yukawa. In § 3 the general feature of the Yukawa's S-matrix is discussed, and it is shown that it reduces to the ordinary Dyson's S-matrix in the limit when the particle radius tends to zero. The appendix contains a calculation of the vacuum energy of a system composed of two non-local scalar fields mutually interacting.

§ 1. Introduction and summary of the results

In previous papers,[1) 2)] the relation of Yukawa's non-local field theory[3)] to the ordinary local field theory was investigated by the method of canonical transformation. It was shown that the two theories were equivalent in the sense that an irreducible non-local field under Lorentz transformations just corresponds to an irreducible local field. The appearance of the internal freedom was ascribed to the fact that the non-local field was a mixed field with various spins.[1) 4)]

The present paper deals with the interaction problem between non-local fields, to see whether the equivalence is removed in the case of the interaction by introducing an interaction scheme which depends explicitly upon the internal freedom. An example of the method of treating the interaction between non-local fields was introduced by Yukawa in the form of a S-matrix.[5)] No other methods have been proposed until now, and the discussion of this paper is based on this formalism. In § 2 the self-energy of a local spinor particle interacting with a non-local scalar field is calculated, the result of which agrees with that obtained from the ordinary local field theory except for a meaningless constant factor. The situation is analysed comparing this result with those recently obtained by Yennie[6)] or by Rayski,[7)] and it is pointed out that such an agreement is a direct consequence of the quantization rule adopted by Yukawa which restricts the interactions to take place only between the parts of the non-local particles for which the internal coordinates are the same. In § 3, the general feature of the Yukawa's S-matrix is investigated, and

it is proved that it reduces to the ordinary Dyson's S-matrix in the limit when the particle radius tends to zero. Yukawa has calculated the vacuum self-energy of a system composed of two non-local fields mutually interacting, and has obtained the result that it vanishes[5] even in the limit when the particle radius tends to zero, and he concluded from this fact that his S-matrix is of a different structure from that of the local theory. The discrepancy arises from the fact that the integral representation of the time ordering operator used in his calculation does not express correctly the required property. In § 4 a brief discussion of the further outlook of the non-local field theory is given.

§ 2. The self-energy problem

In this section, the self-energy of a local spinor particle interacting with a non-local scalar field is calculated as the simplest example of the interaction problem in the non-local field theory. Although the spinor field is chosen as local in our calculation, the essential part of our discussion is not altered by this simplification. In fact, if the spinor field is also considered as non-local, the result is just the same except for a constant factor which is due to that the non-local field is not correctly normalized. (As an illustration the vacuum self-energy of a system composed of two non-local scalar fields mutually interacting is calculated in the appendix II.)

The interaction Lagrangian density is given in this case by

$$L=f\bar{\psi}U\psi\,, \tag{2·1}$$

where ψ and $\bar{\psi}$ denote a complex local spinor field and U a real non-local scalar field respectively. (2·1) is not charge symmetrical. But as remarked by Dyson in the case of the local theory, a correct result can be obtained using this unsymmetrized form if a suitable care is taken in the actual calculation. Expanding these fields into Fourier series,

$$\begin{aligned}
\psi_\rho(X_\mu) &= \frac{1}{(2\pi)^{3/2}}\int \psi_\rho(p_\mu)e^{ipX}(dp)^*,\\
\bar{\psi}_\rho(X_\mu) &= \frac{1}{(2\pi)^{3/2}}\int \bar{\psi}_\rho(p_\mu)e^{-ipX}(dp),\\
U(X_\mu,P_\mu) &= \int u(k_\mu,l_\mu)e^{-ilP}\,e^{ikX}(dl)\,(dk),
\end{aligned} \tag{2·2}$$

where X_μ and P_μ should also be considered as operators. The commutation relations are

$$\{\bar{\psi}_\beta(p_\mu),\ \psi_\alpha(p_\mu')\} = -\varepsilon(p_\mu)\delta(p^2+\mu^2)\delta(p_\mu-p_\mu')\,(i\gamma p-\mu)_{\alpha\beta} \tag{2·3}$$

and

$$[u(k_\mu,l_\mu),\ u(k_\mu',l_\mu')] = -\varepsilon(k_\mu)\delta(k^2+\varkappa^2)\delta(l^2-\lambda^2)\delta(kl)\delta(l_\mu+l_\mu')\delta(k_\mu+k_\mu'),$$

where μ denotes the mass of the ψ field, $\varkappa$ and λ the mass and the radius of the U field respectively.

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* We use the abbreviations

$$(dp)=(dp_1)\,(dp_2)\,(dp_3)\,(dp_0),$$

and $$AB=A_\mu B^\mu.$$

The lowest order self-energy arises from the third term of the S–matrix given by Yukawa ;[5]

$$S=1+i\{L\}+i^2\{LD_+L\}+\cdots\cdots, \tag{2·4}$$

where

$$\{L\}=\int\cdots\int(x_\mu|L|x_\mu')\,(dx)\,(dx'), \tag{2·5}$$

and

$$(x_\mu|D_+|x_\mu')=\frac{1}{2}(1+\varepsilon^*(x_\mu-x_\mu')), \tag{2·6}$$

with

$$\varepsilon^*(x)=\begin{cases} 1 & \text{if } x_\mu \text{ is time like, and } x_0 \text{ is positive,}\\ 0 & \text{if } x_\mu \text{ is space like,}\\ -1 & \text{if } x_\mu \text{ is time like, and } x_0 \text{ is negative.}\end{cases} \tag{2·7}$$

The (x_μ', x_μ'') element of the interaction Lagrangian density is given by

$$(x_\mu'|L|x_\mu'')=\bar\psi(x_\mu')\,(x_\mu'|U|x_\mu'')\psi(x_\mu''). \tag{2·8}$$

Thus the required term takes the form

$$\begin{aligned}S^{(2)}=&-f^2\int\prod_{i=1}^{4}(dx_i)\bar\psi(x_{1\mu})\,(x_{1\mu}|U|x_{2\mu})\psi(x_{2\mu})\\&\times(x_{2\mu}|D_+|x_{3\mu})\bar\psi(x_{3\mu})\,(x_{3\mu}|U|x_{4\mu})\psi(x_{4\mu}),\end{aligned} \tag{2·9}$$

which can be written, by using (2·2), as

$$\begin{aligned}S^{(2)}=&-f^2/(2\pi)^6\int\prod_{i=1}^{4}(dx_i)\prod_{i=1}^{4}(dp_i)\prod_{i=1}^{2}(dk_i)\prod_{i=1}^{2}(dl_i)\\&\times e^{i(-p_1x_1+k_1x_1+p_2x_2-p_3x_3+k_2x_3+p_4x_4)}\,\delta(x_{1\mu}-x_{2\mu}-l_{1\mu})\,\delta(x_{3\mu}-x_{4\mu}-l_{2\mu})\\&\times\bar\psi(p_{1\mu})u(k_{1\mu},l_{1\mu})\psi(p_{2\mu})\,(x_{2\mu}|D_+|x_{3\mu})\bar\psi(p_{3\mu})u(k_{2\mu},l_{2\mu})\psi(p_{4\mu}).\end{aligned} \tag{2·10}$$

Introducing a new variable ξ_μ by $\xi_\mu=x_{2\mu}-x_{3\mu}$, and performing integrations over $x_{1\mu}$, $x_{3\mu}$ and $x_{4\mu}$,

$$\begin{aligned}S^{(2)}=&-f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{4}(dp_i)\prod_{i=1}^{2}(dk_i)\prod_{i=1}^{2}(dl_i)\\&\times e^{i(-p_1+k_1+p_2)\xi}\times\delta(-p_{1\mu}+k_{1\mu}+p_{2\mu}-p_{3\mu}+k_{2\mu}+p_{4\mu})D_+(\xi_\mu)\\&\times\bar\psi(p_{1\mu})\psi(p_{2\mu})\bar\psi(p_{3\mu})\psi(p_{4\mu})u(k_{1\mu},l_{1\mu})u(k_{2\mu},l_{2\mu})\,e^{-i(p_1l_1+p_4l_2)}.\end{aligned} \tag{2·11}$$

The self-energy of the spinor particle is obtained by calculating the one spinor and zero scalar particle part of $S^{(2)}$. For the one particle part of the spinor field we have, omitting irrelevant terms which vanish upon symmetrization,

$$\begin{aligned}&(\bar\psi_\alpha(p_{1\mu})\psi_\alpha(p_{2\mu})\bar\psi_\beta(p_{3\mu})\psi_\beta(p_{4\mu}))_1\\&\quad=-i\bar\psi(p_{1\mu})S^+(p_{2\mu})\psi(p_{4\mu})\delta(p_{2\mu}-p_{3\mu})-i\bar\psi(p_{3\mu})S^-(p_{1\mu})\psi(p_{2\mu})\delta(p_{1\mu}-p_{4\mu}),\end{aligned} \tag{2·12}$$

where

$$S^{\pm}(p_\mu)=\frac{-i}{2}(\varepsilon(p_\mu)\pm 1)\delta(p^2+\varkappa^2)(i\gamma p-\mu). \tag{2·13}$$

As for the U field, it has already been pointed out by many authors[3)-4)] that it is a superposition of local fields with various spins. Therefore, in order to make clear the correspondence of our calculation to the self-energy problem of a local charged spinor particle interacting with a local scalar field, we should pick up from U the irreducible part which corresponds to the spinless field. Then each $u(k_\mu, l_\mu)$ represents creation and annihilation operator of non-local particles characterized by k_μ according to whether k_0 is negative or is positive, and we have as a zero particle part of the U field,

$$(u(k_{1\mu}, l_{1\mu}),\ u(k_{2\mu}, l_{2\mu}))_0=i\Delta^+(k_{1\mu})\delta(l_1^2-\lambda^2)\delta(k_1l_1)\delta(k_1+k_2)\delta(l_1+l_2), \tag{2·14}$$

where

$$\Delta^{\pm}(k_\mu)=\frac{-i}{2}(\varepsilon(k_\mu)\pm 1)\delta(k^2+\varkappa^2). \tag{2·15}$$

Using (2·12) and (2·14), $S_{10}^{(2)}$ is reduced to

$$\begin{aligned}S_{10}^{(2)}=&-f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)e^{i(-p_1+k_1+p_2)\xi}D_+(\xi_\mu)\Delta^+(k_\mu)\\&\times\{\bar\psi(p_{1\mu})S^+(p_{2\mu})\psi(p_{1\mu})+\bar\psi(p_{2\mu})S^-(p_{1\mu})\psi(p_{2\mu})\}\\&\times\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)\delta(l_1+l_2)e^{-ip(l_1+l_2)}(dl_1)(dl_2).\end{aligned} \tag{2·16}$$

The first integral is independent of l_1 or of l_2, and the effect of the non-localizability of the U field is condensed into the second integral. After integrations over l_1 and l_2,

$$\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)\delta(l_1+l_2)e^{-ip(l_1+l_2)}(dl_1)(dl_2)=2\pi\lambda/\varkappa. \tag{2·17}$$

Therefore,

$$\begin{aligned}S_{10}^{(2)}=&-(2\pi\lambda/\varkappa)f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)e^{i(-p_1+k_1+p_2)\xi}\\&\times D_+(\xi_\mu)\Delta^+(k_\mu)\{\bar\psi(p_{1\mu})S^+(p_{2\mu})\psi(p_{1\mu})+\bar\psi(p_{2\mu})S^-(p_{1\mu})\psi(p_{2\mu})\}.\end{aligned} \tag{2·18}$$

This is identical with that obtained from the local field theory except for a multiplicative factor, which is ascribed, as pointed out before, to the fact that the non-local field in the form used above is not correctly normalized. To make the identity clear, we construct an equivalent expression of (2·18) by changing the variables at the beginning as

$$x_{2\mu}\rightleftarrows x_{3\mu},\quad p_{1\mu}\rightleftarrows p_{3\mu},\quad p_{2\mu}\rightleftarrows p_{4\mu},\quad l_{1\mu}\rightleftarrows l_{2\mu}.$$

This yields

$$\begin{aligned}S_{10}^{(2)}=&-(2\pi\lambda/\varkappa)f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)e^{i(-p_1+k_1+p_2)\xi}\\&\times D_+(-\xi_\mu)\Delta^-(k_\mu)\{\bar\psi(p_{1\mu})S^-(p_{2\mu})\psi(p_{1\mu})+\bar\psi(p_{2\mu})S^+(p_{1\mu})\psi(p_{2\mu})\}.\end{aligned} \tag{2·19}$$

Averaging (2·18) and (2·19),

$$S_{10}^{(2)}=(2\pi\lambda/\varkappa)f^2/4(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)e^{i(-p_1+k_1+p_2)\xi}$$
$$\times\bar{\psi}(p_{1\mu})S_F^*(p_{2\mu})\psi(p_{1\mu})\Delta_F^*(k_{1\mu}), \qquad (2\cdot20)$$

where

$$S_F^*(p_\mu)=\Delta_F^*(p_\mu)(i\gamma p-\mu),$$

and

$$\Delta_F^*(p_\mu)=(1+\varepsilon^*(\xi_\mu)\varepsilon(p_\mu))\delta(p^2+\mu^2). \qquad (2\cdot21)$$

Here it is obvious that we can replace $\varepsilon^*(\xi_\mu)$ by $\varepsilon(\xi_\mu)$, since $\varepsilon^*(\xi_\mu)$ appears always as a product with $\Delta(k_\mu)$. Then (2·20) reduces, except for a constant factor, to the familiar form

$$f^2\int\bar{\psi}(x_{1\mu})S_F(x_{1\mu}-x_{2\mu})D_F(x_{1\mu}-x_{2\mu})\psi(x_{2\mu})(dx_1)(dx_2). \qquad (2\cdot22)$$

Similarly the self-energy of the U particle agrees with that obtained from the local field theory. Moreover we can show easily that these results are not altered even if we take the interaction Lagrangian density as

$$\bar{\psi}\psi U \quad \text{or} \quad U\bar{\psi}\psi$$

instead of (2·1). (Appendix I).

Recently Yennie[6] and Rayski[7] have succeeded in eliminating the divergence of the self-energy problem by introducing somewhat different type of non-local fields from that of Yukawa. The essential difference lies in the quantization rule. They used

$$[u(k_\mu, l_\mu), u(k_\mu', l_\mu')]=-\varepsilon(k_\mu)\delta(k^2+\varkappa^2)\delta(k_\mu+k_\mu')\delta(l^2-\lambda^2)\delta(kl)$$
$$\times\delta(l'^2-\lambda^2)\delta(k'l') \qquad (2\cdot23)$$

instead of (2·3). Then, after a similar calculation as before we have, instead of (2·16),

$$S_{10}^{(2)}=-f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)e^{i(-p_1+k_1+p_2)\xi}D_+(\xi_\mu)\Delta^+(k_\mu)$$
$$\times\{\bar{\psi}(p_{1\mu})S^+(p_{2\mu})\psi(p_{1\mu})+\bar{\psi}(p_{2\mu})S^-(p_{1\mu})\psi(p_{2\mu})\}$$
$$\times\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)\delta(l_2^2-\lambda^2)\delta(k_1l_2)e^{-ip_1(l_1+l_2)}(dl_1)(dl_2). \qquad (2\cdot24)$$

Contrary to the previous case, the second integral is calculated to give

$$\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)\delta(l_2^2-\lambda^2)\delta(k_1l_2)e^{-ip_1(l_1+l_2)}(dl_1)(dl_2)$$
$$=(2\pi\lambda/\varkappa)^2\left(\frac{\sin\lambda\sqrt{(kp_1)^2/\varkappa^2-\mu^2}}{\lambda\sqrt{(kp_1)^2/\varkappa^2-\mu^2}}\right)^2. \qquad (2\cdot25)$$

It is clear that this term is relativistically invariant and serves as a cut off factor. The result is interpreted as follows: In Yukawa's theory the interaction takes place only between the parts of the non-local particles for which the internal coordinates are the same (see figure 1), since the commutation relation (2·3) involves a δ-function of the internal

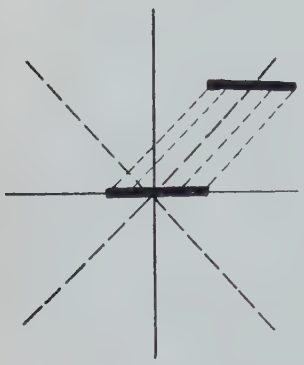

Fig. 1

coordinates. Therefore, we can not expect the averaging out of the high frequency parts of the intermediating waves notwithstanding the introduction of a finite extension for the particle. On the other hand, in Yennie or in Rayski's theory the interaction is averaged over the surface of a three dimensional sphere. This averaging yields a cut off factor (2·25).

§ 3. The S-matrix in non-local field theory

Yukawa calculated the vacuum self-energy of a system composed of two non-local fields mutually interacting using (2·1). Obtaining a vanishing result even in the limit of $\lambda \to 0$ in contrast to ordinary cases, he concluded that his S-matrix is of a different structure from the ordinary Dyson's one. But it has been shown by Koba[3] that Dyson's S-matrix can be written in the form (2·1). Koba's proof is based on the fact that the interaction Lagrangian density at two world points $L(x_\mu)$ and $L(x_\mu')$ commute if x_μ and x_μ' are space like to each other.* Now it is a general demand imposed when generalizing a local field $V(x_\mu)$ into a non-local field $V(x_\mu, r_\mu)$ that it should satisfy

$$V(x_\mu, r_\mu) \to V(x_\mu)\delta(r_\mu) \tag{3·1}$$

in the limit when the particle radius tends to zero. Therefore it should be

$$[V^*(x_\mu, r_\mu),\ V(x_\mu', r_\mu')] \to iD(x_\mu - x_\mu')\delta(r_\mu)\delta(r_\mu') \tag{3·2}$$

as $\lambda \to 0$, where $D(x_\mu - x_\mu')$ is the usual Jordan-Pauli's function. Yukawa's non-local field of course satisfies (3·1), and (3·2), and therefore we may expect that (2·4) is reduced to the ordinary one in the limit of $\lambda \to 0$. The cause of this discrepancy lies in an inappropriate nature of $\varepsilon^*(x_\mu)$ function which was used in Yukawa's calculation. In Koba's theory, $\varepsilon^*(x_\mu)$ is defined by

$$\varepsilon^*(x_\mu) = \begin{cases} 1 & \text{if } x_\mu \text{ is time like or on the light cone and } x_0 \text{ is positive,} \\ 0 & \text{if } x_\mu \text{ is space like,} \\ -1 & \text{if } x_\mu \text{ is time like or on the light cone and } x_0 \text{ is negative.} \end{cases} \tag{3·3}$$

Yukawa used as $\varepsilon^*(x_\mu)$ an integral representation given by Yennie[6];

$$\varepsilon^*(x_\mu) = -\frac{i}{\pi^2} P\int \frac{p_0}{|p_0|}\, \delta'(p^2) e^{ipx}(dp) \tag{3·4}$$

which is transformed, after a simple calculation, into

$$\frac{1}{\pi}\int_0^\infty \frac{1}{p_0}\{\sin p_0(x_0+x) + \sin p_0(x_0-x)\}\, dp_0$$

$$(x = \sqrt{x_1^2 + x_2^2 + x_3^2})$$

* Although Koba's proof is a general one involving the derivative coupling case, we consider here only a simpler case where such couplings do not appear.

$$= \begin{cases} +1 & \text{if } x_\mu \text{ is time like and } x_0 \text{ is positive,} \\ +\frac{1}{2} & \text{if } x_\mu \text{ is on the light cone and } x_0 \text{ is positive,} \\ 0 & \text{if } x_\mu \text{ is space like,} \\ -\frac{1}{2} & \text{if } x_\mu \text{ is on the light cone and } x_0 \text{ is negative,} \\ -1 & \text{if } x_\mu \text{ is time like and } x_0 \text{ is negative.} \end{cases} \tag{3·5}$$

Thus, Yennie's $\varepsilon^*(x_\mu)$ does not agree with Koba's on the light cone. As expressed most clearly in the infinity value of the Jordan-Pauli's function on the light cone, the most parts of the interactions propagate along the light cone with light velocity. Therefore, the interaction along the light cone gives the effect of the same order of magnitude as the whole one which propagates inside the light cone, and therefore the difference between Yennie's and Koba's $\varepsilon^*(x_\mu)$ can not be ignored although the difference occurs only in the zero measured region. If Yennie's $\varepsilon^*(x_\mu)$ is replaced by Koba's, all results agree with those of the local theory in the limit of $\lambda \to 0$. (see Appendix II.)

§ 4. Concluding remarks

In § 2 we saw that Yukawa's theory does not improve the situation in the divergence difficulty. The cause was the appearance of the δ function of the internal coordinates in the commutation relation. This term also played a decisive role in deriving the commutation relation from that of the local theory by a canonical transformation. Thus, we may conclude that the non-local field which is characterized by a commutation relation of the form (2·3) is an equivalent dynamical system to the ordinary local field, and fails to express the essential feature of the non-local field. To remove this equivalence would be essentially important for the further development of the non-local field theory. Although Yennie or Rayski's theories succeed partly in avoiding this difficulty, their theories lack necessity in the way of the modification, and a more radical consideration seems to be indispensable for the ultimate solution of the problem.

With all these unsatisfactory features in the present form, however, the concept of the non-local field seems very interesting to us. First it is in connection with the non-local interaction. Recently, Sakata et al.[9] and Stückelberg[10] classified the interactions into the first kind for which the renormalization can be performed in a closed form, and the second for which this is not the case. The interactions of the second kind necessarily lead to the non-local interactions. To dynamical systems with non-local interactions, however, Hamiltonian formalism is not applicable, and it seems quite dubious whether the traditional local theory can be adapted to include them. The second view-point is related to the mass spectrum of elementary particles. Nambu[11] pointed out that the mass of elementary particles measured in the unit of $137\times$ (electron mass) take integer or half integer values. The existence of such a simple regularity between the mass strongly suggests the notion of "*Urmaterie*". That is, we may suppose that the present elementary particles correspond to various energy levels of the internal motion of this Urmaterie, and it seems to us likely that the field that describes the Urmaterie is nothing but the non-local field.

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Appendix I. The use of another form of the interaction Lagrangian density

We shall show in this appendix that the result obtained in § 2 is not altered if we change the ordering of the field variables in the interaction Lagrangian density. Taking for example the interaction Lagrangian density as

$$L=f\bar{\psi}\psi U\,, \tag{A·1}$$

we get

$$(x_\mu'|L|x_\mu'')=\psi(x_\mu')\bar{\psi}(x_\mu')\,(x_\mu'|U|x_\mu''). \tag{A·2}$$

Thus corresponding to (2·11),

$$S^{(2)}=-f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{4}(dp_i)\prod_{i=1}^{2}(dk_i)\prod_{i=1}^{2}(dl_i)\,e^{i(-p_1+k_1+p_2)(\xi+l_1)}D_+(\xi_\mu)$$
$$\times\delta(-p_1+k_1+p_2-p_3+k_2+p_4)\bar{\psi}(p_{1\mu})\psi(p_{2\mu})\psi(p_{3\mu})\bar{\psi}(p_{4\mu})u(k_{1\mu},l_{1\mu})u(k_{2\mu},l_{2\mu}). \tag{A·3}$$

The one spinor and zero scalar particle part of this expression is calculated to be

$$S_{10}^{(2)}=-f^2/(2\pi)^2\int(d\xi)\prod_{i=1}^{2}(dp_i)\cdot(dk_1)\,e^{i(-p_1+k_1+p_2)\xi}D_+(\xi_\mu)$$
$$\times\{\bar{\psi}(p_{1\mu})S^+(p_{2\mu})\psi(p_{1\mu})+\psi(p_{2\mu})S^-(p_{1\mu})\psi(p_{2\mu})\}\Delta^+(k_\mu)$$
$$\times\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)e^{i(-p_1+k_1+p_2)l_1}(dl_1), \tag{A·4}$$

which is further transformed, after a similar symmetrization as before, into

$$S_{10}^{2}=f^2/4\,(2\pi)^2\int(d\xi)\prod_{i=0}^{2}(dp_i)\cdot(dk_1)\,e^{i(-p_1+k_1+p_2)\xi}$$
$$\times\bar{\psi}(p_{1\mu})S_F^*(p_{2\mu})\psi(p_{1\mu})\Delta_F^*(k_{1\mu})$$
$$\times\int\delta(l_1^2-\lambda^2)\delta(k_1l_1)e^{i(-p_1+k_1+p_2)l_1}(dl_1), \tag{A·5}$$

In (A·5), $\varepsilon^*(\xi_\mu)$ appears always as a product with $\varepsilon(p_\mu)$ or with $\varepsilon(k_\mu)$. But for any function of p_μ or of k_μ which does not diverge more strongly than quadratically in p_0 or in k_0,

$$\int F(p_\mu)\delta({p_\mu}^2+\mu^2)\,(1+\varepsilon^*(\xi_\mu)\varepsilon(p_\mu))\,e^{ipx}(dp)=\frac{-2i}{2\pi}\int\frac{F(p_\mu)e^{ipx}}{p^2+\mu^2-i\varepsilon}\,(dp),$$
$$\int F(k_\mu)\delta({k_\mu}^2+\varkappa^2)\,(1+\varepsilon^*(\xi_\mu)\varepsilon(k_\mu))\,e^{ikx}(dk)=\frac{-2i}{2\pi}\int\frac{F(k_\mu)}{k^2+\varkappa^2-i\varepsilon}\,(dk).$$

(ε is a small real positive number.)

Thus, terms in the bracket in (A·5) become effectively independent of ξ_μ, and an integration over ξ_μ yields a factor

$$\delta(-p_{1\mu}+k_{1\mu}+p_{2\mu}).$$

Therefore, the characteristic factor of the non-local theory

$$\int \delta(l_1^2-\lambda^2)\,\delta(k_1 l_1)\,e^{i(-p_1+k_1+p_2)l_1}(dl_1)$$

is again reduced to a mere multiplicative factor $2\pi\lambda/\varkappa$, and (A·5) agrees with (2·18).

Appendix II. The vacuum self-energy of a system composed of two non-local scalar fields mutually interacting

Denoting by U, V, V^* the neutral non-local, charged non-local, and its conjugate fields respectively, we expand them into a Fourier series as before;

$$\begin{aligned} U(X_\mu, P_\mu) &= \int u(k_\mu, l_\mu)\,e^{ikX}e^{-iuP}(dk)\,(dl), \\ V(X_\mu, P_\mu) &= \int v(q_\mu, l_\mu)\,e^{iqX}e^{-iuP}(dq)\,(dl), \\ V^*(X_\mu, P_\mu) &= \int v^*(q_\mu, l_\mu)\,e^{-iqX}e^{iuP}(dq)\,(dl). \end{aligned} \tag{A·6}$$

Each field variables satisfies the commutation relations*

$$\begin{aligned} &[u(k_\mu, l_\mu),\ u(k_\mu' l_\mu')] = -\varepsilon(k_\mu)\,\delta(k^2+\varkappa_U^2)\,\delta(k_\mu+k_\mu')\,\delta(l_\mu+l_\mu')\,\rho_U(k_\mu, l_\mu), \\ &[v^*(q_\mu, l_\mu),\ v(q_\mu, l_\mu)] = -\varepsilon(q_\mu)\,\delta(q_\mu{}^2+\varkappa_V{}^2)\,\delta(q_\mu-q_\mu')\,\delta(l_\mu-l_\mu')\,\rho_V(q_\mu, l_\mu), \end{aligned} \tag{A·7}$$

where we have used abbreviations

$$\rho_U(k_\mu, l_\mu)=\delta(l^2-\varkappa_U{}^2)\,\delta(kl),\quad \rho_V(q_\mu, l_\mu)=\delta(q^2-\varkappa_V{}^2)\,\delta(ql).$$

As the interaction Lagrangian density we assume

$$L=fV^*UV. \tag{A·8}$$

Thus, corresponding to (2·9),

$$\begin{aligned} S^{(2)} = -f^2\int \prod_{i=1}^{8}(dx_i)\,(x_{1\mu}|V^*|x_{2\mu})\,(x_{2\mu}|U|x_{3\mu})\,(x_{3\mu}|V|x_{4\mu})\,(x_{4\mu}|D_+|x_{5\mu}) \\ \times (x_{5\mu}|V^*|x_{6\mu})\,(x_{6\mu}|U|x_{7\mu})\,(x_{7\mu}|V|x_{8\mu}). \end{aligned} \tag{A·9}$$

In order to make the comparison with Yennie's result easy, we perform the following substitutions;

$$\begin{aligned} x_{1\mu}&=x_\mu'+r_{1\mu}+r_{2\mu}, & x_{5\mu}&=x_\mu''+r_{4\mu}, \\ x_{2\mu}&=x_\mu'+r_{2\mu}, & x_{6\mu}&=x_\mu'', \\ x_{3\mu}&=x_\mu', & x_{7\mu}&=x_\mu''-r_{5\mu}, \\ x_{4\mu}&=x_\mu'-r_{3\mu}, & x_{8\mu}&=x_\mu''-r_{5\mu}-r_{6\mu}. \end{aligned} \tag{A·10}$$

It should be noted that the $(x_{1\mu}, x_{2\mu})$ element of V^* is written as

$$(x_{1\mu}'|V^*|x_{2\mu}')=\int v^*(q_\mu, -x_{1\mu}'+x_{2\mu}')\,e^{iq(x_{1\mu}'+x_{2\mu}')/2}(dq). \tag{A·11}$$

Thus, substituting (A·6) into (A·9),

---

* $\varkappa_U$, λ_U, $\varkappa_V$, and λ_V denote the mass and the radius of the U and V fields respectively.

$$S^{(2)}=-f^2\int (dx')\,(dx'')\prod_{i=1}^{6}(dr_i)\prod_{i=1}^{4}(dq_i)\prod_{i=1}^{2}(dk_i)D_+(x'-x''-r_3-r_4)$$
$$\times\exp i\{(-q_1+k_1+q_2)x'+(-q_3+k_2+q_4)x''+(-q_1r_1+k_1r_2-q_2r_3-q_3r_4$$
$$-k_2r_5-q_4r_6)/2-(q_1r_2+q_4r_5)\}$$
$$\times v^*(q_{1\mu},\,-r_{1\mu})u(k_{1\mu},\,r_{2\mu})v(q_{2\mu},\,r_{3\mu})v^*(q_{3\mu},\,-r_{4\mu})u(k_{2\mu},\,r_{5\mu})v(q_{4\mu},\,r_{6\mu}). \tag{A·12}$$

As a simplest example, we calculate the vacuum self-energy. First we obtain

$$(v^*(q_{1\mu},\,-r_{1\mu})v(q_{2\mu},\,r_{3\mu})v^*(q_{3\mu},\,-r_{4\mu})v(q_{4\mu},\,r_{6\mu}))_0$$
$$=-\Delta^-(q_{1\mu})\Delta^+(q_{2\mu})\rho_V(q_{1\mu},\,r_{1\mu})\rho_V(q_{2\mu},\,r_{3\mu})\delta(q_{1\mu}-q_{4\mu})\delta(q_{2\mu}-q_{3\mu})$$
$$\times\delta(r_{1\mu}+r_{6\mu})\delta(r_{3\mu}+r_{4\mu}), \tag{A·13}$$

and

$$(u(k_{1\mu},\,r_{2\mu}),\;u(k_{2\mu},\,r_{5\mu}))_0=i\Delta^+(k_{1\mu})\rho_U(k_{1\mu},\,r_{2\mu})\delta(k_{1\mu}+k_{2\mu})\delta(r_{2\mu}+r_{5\mu}). \tag{A·14}$$

Substituting (A·13) and (A·14) into (A·12), and performing required integrations, we find

$$S_{00}^{(2)}=i(2\pi)^4f^2\int(d\xi)\prod_{i=1}^{3}(dr_i)\prod_{i=1}^{2}(dq_i)\cdot(dk_1)e^{i(-q_1+k_1+q_2)\xi}D_+(\xi_\mu)$$
$$\times\Delta^-(q_{1\mu})\Delta^+(q_{2\mu})\Delta^+(k_{1\mu})\rho_V(q_{1\mu},\,r_{1\mu})\rho_V(q_{2\mu},\,r_{3\mu})\rho_U(k_{1\mu},\,r_{2\mu}),$$
$$\xi_\mu=x_\mu'-x_\mu''. \tag{A·15}$$

Therefore the integrations over $r_{1\mu}$, $r_{2\mu}$ and $r_{3\mu}$ can be performed independently, yielding

$$\int\rho_V(q_\mu,\,r_{1\mu})\,(dr_1)=2\pi\lambda_V/\varkappa_V\,, \tag{A·16}$$

$$\int\rho_V(q_\mu,\,r_{3\mu})\,(dr_3)=2\pi\lambda_V/\varkappa_V\,,$$

$$\int\rho_U(k_\mu,\,r_{2\mu})\,(dr_2)=2\pi\lambda_U/\varkappa_U\,. \tag{A·17}$$

Thus the non-localizability of the fields becomes meaningless, and we obtain a usual result again except for a multiplicative factor.

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On the Group of Transformations in Six-dimensional Space

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The possible irreducible representations of Lie algebra of the transformation group which leaves the fundamental quadratic form,

$$(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2+(x^5)^2-(x^6)^2,$$

invariant, are investigated. Three invariants which commute with all the elements of the algebra are given and every irreducible representation is characterised by the values of these invariants.

§ 1. Introduction

Several years ago Snyder[1] has considered the quantization of space-time on the basis of the infinitesimal transformation group in five-dimensional space and given one direction to the theory of non-local field. A little later Yang[2] has proposed the similar operators related to the transformation group in six-dimensional space. In this case the operators constitute a Lie algebra, while Snyder's were not. Then Tokuoka[3] has examined the equations given by Snyder from the view point of reciprocity, and also taken the infinitesimal transformations in six-dimensional space, of which the fundamental quadratic form was different from Yang's in the sign of one term. This type of operators was, then, taken up by Segal[4], in his work on "A class of operator algebras which are determined by groups". Here he emphasizes the following four points: 1) An analogue to the square of rest-mass is automatically obtained. 2) The analogue to the angular momentum automatically includes the spin angular momentum. 3) The space associated with the particle is discrete. 4) The Lie algebra of the group is not a limiting case of any non-isomorphic Lie algebra. We shall add a little about the last statement, as the others are shown in the course of constructing representations. The fact that one Lie algebra is the limiting case of another is visualized by the following example, which he gave in his paper. We take $X_1=ip$, $X_2=iq$ and $X_3=iI$ ($[p, q]=\frac{\hbar}{i}I$), then the Lie algebra generated by X_i has the commutation relations $[X_1, X_2]=\hbar X_3$, $[X_1, X_3]=[X_2, X_3]=0$. Now the Lie algebra of the group of transformations which leave invariant $kt_1^2+t_2^2-t_3^2$ has as bases (Y_1, Y_2, Y_3), among which hold the commutation relations $[Y_1, Y_2]=\hbar Y_3$, $[Y_1, Y_3]=\hbar kY_2$, $[Y_2, Y_3]=\hbar^{-1}kY_1$. These Lie algebras are not isomorphic and when $k\to 0$ the latter tends to the former. In this sense the Lie algebra under consideration is not a limiting case of any non-isomorphic Lie algebra. (About the proof, see the last paragraph of his paper.)

Therefore, if we can construct the theory of elementary particle on the basis of this Lie algebra, we may expect this to be the most fundamental theory, provided that the relativistic wave equations are intimately related to some group such as the inhomogeneous Lorentz group.

On the other hand Schouten[5] has called attention of physicists to the conformal geometry and thereby stated at the outset of his argument that the Maxwell equations are invariant not only under the Lorentz transformation but also under the conformal transformation. Now, the group under consideration, if the value of the fundamental quadratic form is kept equal to zero, has the conformal group in Minkowski space as a covering group[6]. Therefore, they have the same Lie algebra. The conformal group in Minkowski space contains as a special case the inhomogeneous Lorentz transformation, of which Wigner[7] clssified the irreducible reprsentations and then Wigner and Bargmann[8] have applied them to the theory of relativistic wave equations.

Our intension is to form a theory of elementary particle on the basis of the conformal group in Minkowski space or the orthogonal group in six-dimensional pseudo-Euclidean space. Of course, the wave equations will appear in non-localizable form. Here, for the time being, we publish the result of classification of the irreducible representations of the above mentioned Lie algebra, since it may have some merit by itself. And in the next paper we shall interprete the result in a rather simple way and obtain a mass spectrum tentatively.

§ 2. Structure of the Lie algebra[9]

We take as the fundamental quadratic form,

$$F\equiv(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2+(x^5)^2-(x^6)^2. \tag{2·1}$$

The Lie algebra of the group of transformations which leave (2·1) invariant is of order 15. Let χ denote an element of the Lie algebra $\mathfrak{G}$.

$$\chi=(\chi^1(a),\chi^2(a),\ldots\ldots,\chi^{15}(a)), \tag{2·2}$$

where a denotes parameters contained in the group element, i.e., in

$$y^i=w^i_j(a)\,x^j. \tag{2·3}$$

Corresponding to each χ, we shall introduce the differential operator

$$\chi\equiv\chi^i(a)\frac{\partial}{\partial a^i}. \tag{2·4}$$

Then,

$$\chi y=L_\chi y, \text{ or } \chi y^i=l^i_j\chi^j. \tag{2·5}$$

As $\chi(F)=0$, we have

$$l_{ij}+l_{ji}=0.$$

We define L_{kl} by

$$(L_{kl}\,x)^i=\delta^i_l\,x_k-\delta^i_k\,x_l. \tag{2·6}$$

Fifteen χ_{kl}, which correspond to L_{kl}, can be taken as the bases of $\mathfrak{G}$.
The commutation relations for L_{kl} are,

$$[L_{ij}, L_{kl}] = -g_{jk} L_{il} + g_{ik} L_{jl} - g_{il} L_{jk} + g_{jl} L_{ik}. \tag{2·7}$$

Next we shall consider the related spinor group. Following Cartan's procedure we first represent six-dimensional vector (x^i) by the matrix X.

$$X = \begin{pmatrix} 0 & \Xi \\ \Xi & 0 \end{pmatrix}; \quad \Xi = \begin{pmatrix} x^1+ix^2 & x^3+x^0 & x^5+x^6 & 0 \\ x^3-x^0 & -x^1+ix^2 & 0 & x^5+x^6 \\ x^5-x^6 & 0 & -x^1+ix^2 & -x^3-x^0 \\ 0 & x^5-x^6 & -x^3+x^0 & x^1+ix^2 \end{pmatrix}. \tag{2·8}$$

The following relations hold,

$$\begin{aligned} XX &= \{(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2+(x^5)^2-(x^6)^2\}E, \\ \text{Det. } \Xi &= \{(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2+(x^5)^2-(x^6)^2\}. \end{aligned} \tag{2·9}$$

The rotation in the six-dimensional space can be expressed by the equation,

$$\begin{pmatrix} 0 & \Xi' \\ \Xi' & 0 \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix} \begin{pmatrix} 0 & \Xi \\ \Xi & 0 \end{pmatrix} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^{-1*} \end{pmatrix}, \tag{2·10}$$

or

$$\Xi' = W \Xi W^{*-1}, \tag{2,11}$$

where W is a certain unimodular matrix of 15 parameters. We shall decompose W in the following way :

$$W = R_4' R^{06'} R^{10} R^{06} R_4. \tag{2·12}$$

Here R_4, R^{06} and R^{10} are rotations in (x^1, x^2, x^3, x^5)-space, x^0-x^6 plane and x^1-x^0 plane respectively. The desired forms of matrices can be easily obtained if we consider a rotation, for example R^{06}, as a result of double reflections by two planes, whose normals lie in the x^0-x^6 plane. The result is

$$R_4 = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta^* & \alpha^* & \delta^* & -\gamma^* \\ -\gamma^* & -\delta^* & \alpha^* & \beta^* \\ -\delta & \gamma & -\beta & \alpha \end{pmatrix}, \quad R^{06} = \begin{pmatrix} \cos\xi & & & \sin\xi \\ & \cos\xi & -\sin\xi & \\ & \sin\xi & \cos\xi & \\ -\sin\xi & & & \cos\xi \end{pmatrix}, \quad R^{10} = \begin{pmatrix} \cosh\eta & \sinh\eta & & \\ \sinh\eta & \cosh\eta & & \\ & & \cosh\eta & \sinh\eta \\ & & \sinh\eta & \cosh\eta \end{pmatrix}$$

$$(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^* = 1, \quad \alpha\delta^* - \alpha^*\delta = \beta\gamma^* - \beta^*\gamma). \tag{2·13}$$

Further the matrix R_4 can be decomposed as follows :

$$R_4 = \begin{pmatrix} \cos\vartheta & & & \sin\vartheta \\ & \cos\vartheta & \sin\vartheta & \\ & -\sin\vartheta & \cos\vartheta & \\ -\sin\vartheta & & & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\varphi & i\sin\varphi & & \\ i\sin\varphi & \cos\varphi & & \\ & & \cos\varphi & -i\sin\varphi \\ & & -i\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} e^{i\omega} & & & \\ & e^{-i\omega} & & \\ & & e^{-i\omega} & \\ & & & e^{i\omega} \end{pmatrix}$$

$$\cdot\begin{pmatrix} \cos\psi & & \sin\psi & \\ & \cos\psi & & -\sin\psi \\ -\sin\psi & & \cos\psi & \\ & \sin\psi & & \cos\psi \end{pmatrix}\begin{pmatrix} \cos\delta & i\sin\delta & & \\ i\sin\delta & \cos\delta & & \\ & & \cos\delta & -i\sin\delta \\ & & -i\sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} \cos\chi & & & \sin\chi \\ & \cos\chi & \sin\chi & \\ & -\sin\chi & \cos\chi & \\ -\sin\chi & & & \cos\chi \end{pmatrix}. \tag{2·14}$$

The group composed of above W and our original rotation group are locally isomorph, and so they have the same Lie algebra. If we apply an infinitesimal transformation χ to both forms of transformations

$$y=w(a)x \text{ and } H=W\Xi W^{-1*},$$

we have

$$\chi y=L_\chi y \text{ and } \chi W=M_\chi W, \tag{2·15}$$

where M_χ is a matrix to be determined:

$$\chi H=(\chi W)\Xi W^{-1*}+W\Xi(\chi W^{-1*}).$$

To compute χW^{-1*}, we use the decomposition (2.12), i.e.,

$$W=U_1 R U_2,$$

then from (2.13),

$$U_i^{-1}=U_i^{*\prime}, \quad R=R^*=R'.$$

Therefore,

$$\chi W^{-1*}=\chi(U_2'R^{-\prime}U_1')=\{\chi(U_1R'U_2)\}'=(U_1R^{-1}U_2)'M_\chi^{0\prime}=U_2^{-1*}R^{-1}U_1^{-1*}M_\chi^{0\prime}$$
$$=W^{-1*}M_\chi^{0\prime},$$

where $\chi(U_1R^{-1}U_2)=M_\chi^0(U_1RU_2)$. And we get

$$\chi H=M_\chi H+HM_\chi^{0\prime}, \tag{2·16}$$

of which M_χ can be obtained and found to be

$$M_\chi=\frac{1}{2}$$
$$\cdot\begin{pmatrix} x^{03}-x^{65}-ix^{12} & -x^{01}-ix^{02}-x^{31}+ix^{23} & x^{15}+ix^{25}+x^{61}+ix^{62} & x^{05}-x^{60}+x^{35}+x^{63} \\ -x^{01}+ix^{02}+x^{31}+ix^{23} & -x^{03}-x^{65}+ix^{12} & -x^{05}+x^{60}+x^{35}+x^{63} & -x^{15}+ix^{25}-x^{61}+ix^{62} \\ -x^{15}+ix^{25}+x^{61}-ix^{62} & -x^{05}-x^{60}-x^{35}+x^{63} & x^{03}-x^{65}+ix^{12} & -x^{01}+ix^{02}-x^{31}-ix^{23} \\ x^{05}+x^{60}-x^{35}+x^{63} & x^{15}+ix^{25}-x^{61}-ix^{62} & -x^{01}-ix^{02}+x^{31}-ix^{23} & -x^{03}-x^{65}-ix^{12} \end{pmatrix}$$

where $\chi=x^{12}\chi_{12}+x^{23}\chi_{23}+\cdots$. (2·17)

Corresponding to χ_{kl} we have M_{kl} and with the aid of these matrices we can write as follows (cf. (2.13), (2.14) and (2.17)):

$$R_4=\exp(2\vartheta M_{35})\exp(2\varphi M_{23})\exp(2\omega M_{12})\exp(2\psi M_{15})\exp(2\delta M_{23})\exp(2\rho M_{35}),$$
$$R^{06}=\exp(2\xi M_{60}), \quad R^{10}=\exp(2\eta M_{01}), \tag{2·18}$$

and the corresponding elements of the original algebra are the same forms, M_A being replaced by χ_A.

One parameter subgroup is written as $a(t)=\exp(t\chi)$, and its representative matrix $U_t=U(a(t))$ is, according to Stone's theorem, $\exp(-itH_\chi)$, with a selfadjoint operator H_χ. As is easily seen the commutation relations among H are

$$[H_A, H_B]=i c_{AB}^{C} H_C, \tag{2·19}$$

if $[\chi_A, \chi_B]=c_{BA}^{C}\chi_C$. Therefore the structure constants are determined from $[L_A, L_B]=c_{AB}^{C} \cdot L_C$(2.7). Writing $X_i, L_i, M_i, p_i, X_0, p_0, N$(i=1,2,3) for $H_{i5}, H_{jk}, H_{0i}, H_{i6}, H_{05}, H_{06}$, H_{56}, the commutation relations are as follows:

$$\begin{aligned}
&[X_i, L_j]=iX_k, \quad [X_i, L_k]=-iX_j, \quad [X_i, L_i]=0, \quad [X_i, p_0]=0, \quad [X_i, N]=-ip_i,\\
&[p_i, L_j]=ip_k, \quad [p_i, L_k]=-ip_j, \quad [p_i, L_i]=0, \quad [p_i, p_0]=iM_i, \quad [p_i, N]=-iX_i,\\
&[M_i, L_j]=iM_k, \quad [M_i, L_k]=-iM_j, \quad [M_i, L_i]=0, \quad [M_i, p_0]=-ip_i, \quad [M_i, N]=0,\\
&[X_i, X_0]=-iM_i, \qquad [L_i, p_0]=[L_i, N]=[L_i, X_0]=0, \qquad (2·20)\\
&[p_i, X_0]=0, \qquad [X_i, X_j]=[L_i, L_j]=iL_k.\\
&[M_i, X_0]=-iX_i, \qquad [p_i, p_j]=[M_i, M_j]=-iL_k,\\
&\qquad [X_i, p_j]=i\delta_{ij}N, \quad [p_i, M_j]=i\delta_{ij}p_0, \qquad [X_i, M_j]=i\delta_{ij}X_0,\\
&\qquad [X_0, p_0]=-iN, \quad [p_0, N]=-iX_0, \qquad [X_0, N]=-ip_0.
\end{aligned}$$

This algebra is of rank 3. We take L_3, X_3 and p_0 as the elements of Cartan subalgebra and in accordance with notation used by Weyl, we form the following bases besides the above ones (this is because the following bases are convenient in constructing representations, as we shall see later).

$$\begin{aligned}
&E_{\begin{pmatrix}1\\1\\0\end{pmatrix}}=\frac{1}{4\sqrt{2}}(L_1+iL_2+X_1+iX_2), \quad E_{\begin{pmatrix}-1\\-1\\0\end{pmatrix}}=\frac{1}{4\sqrt{2}}(L_1-iL_2+X_1-iX_2),\\
&E_{\begin{pmatrix}1\\-1\\0\end{pmatrix}}=\frac{1}{4\sqrt{2}}(L_1+iL_2-X_1-iX_2), \quad E_{\begin{pmatrix}-1\\1\\0\end{pmatrix}}=\frac{1}{4\sqrt{2}}(L_1-iL_2-X_1+iX_2),\\
&E_{\begin{pmatrix}1\\0\\1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(M_1+iM_2+ip_1-p_2), \quad E_{\begin{pmatrix}-1\\0\\-1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(M_1-iM_2-ip_1-p_2), \qquad (2·21)\\
&E_{\begin{pmatrix}1\\0\\-1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(M_1+iM_2-ip_1+p_2), \quad E_{\begin{pmatrix}-1\\0\\1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(M_1-iM_2+ip_1+p_2),\\
&E_{\begin{pmatrix}0\\1\\1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(X_0-iM_3+p_3+iN), \quad E_{\begin{pmatrix}0\\-1\\-1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(X_0+iM_3+p_3-iN),\\
&E_{\begin{pmatrix}0\\1\\-1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(X_0-iM_3-p_3-iN), \quad E_{\begin{pmatrix}0\\-1\\1\end{pmatrix}}=\frac{i}{4\sqrt{2}}(X_0+iM_3-p_3+iN).
\end{aligned}$$

Further we put

$$H_1=L_3, \quad H_2=X_3, \quad H_3=p_0. \tag{2·22}$$

Then the commutation relations become

$$[H_i, H_j]=0, \quad [H_i, E_\alpha]=\alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}]=\frac{1}{8}\sum \alpha_i H_i, \quad [E_\alpha, E_\beta]=N_{\alpha\beta} E_{\alpha+\beta}. \tag{2·23}$$

Here the subscript α is a root and means $\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$. The last commutation relations are, explicitly written,

$$[E_{\binom{1}{1}_0}, E_{\binom{-1}{0}_{-1}}]=\frac{i}{2\sqrt{2}}E_{\binom{0}{1}_{-1}}, \quad [E_{\binom{1}{1}_0}, E_{\binom{-1}{0}_1}]=\frac{i}{2\sqrt{2}}E_{\binom{0}{1}_1},$$

$$[E_{\binom{1}{1}_0}, E_{\binom{0}{-1}_{-1}}]=\frac{-i}{2\sqrt{2}}E_{\binom{1}{0}_{-1}}, \quad [E_{\binom{1}{1}_0}, E_{\binom{0}{-1}_1}]=\frac{-i}{2\sqrt{2}}E_{\binom{1}{0}_1},$$

$$[E_{\binom{1}{-1}_0}, E_{\binom{0}{1}_1}]=\frac{i}{2\sqrt{2}}E_{\binom{1}{0}_1}, \quad [E_{\binom{0}{-1}_{-1}}, E_{\binom{0}{1}_{-1}}]=\frac{i}{2\sqrt{2}}E_{\binom{1}{0}_1},$$

$$[E_{\binom{1}{-1}_0}, E_{\binom{-1}{0}_1}]=\frac{-i}{2\sqrt{2}}E_{\binom{0}{-1}_0}, \quad [E_{\binom{1}{-1}_0}, E_{\binom{-1}{0}_{-1}}]=\frac{-i}{2\sqrt{2}}E_{\binom{0}{-1}_{-1}},$$

$$[E_{\binom{0}{1}_1}, E_{\binom{-1}{0}_{-1}}]=\frac{i}{2\sqrt{2}}E_{\binom{-1}{1}_0}, \quad [E_{\binom{0}{1}_1}, E_{\binom{1}{0}_{-1}}]=\frac{-i}{2\sqrt{2}}E_{\binom{1}{1}_0},$$

$$[E_{\binom{1}{0}_1}, E_{\binom{0}{1}_{-1}}]=\frac{i}{2\sqrt{2}}E_{\binom{1}{1}_0}, \quad [E_{\binom{1}{0}_{-1}}, E_{\binom{0}{-1}_1}]=\frac{-i}{2\sqrt{2}}E_{\binom{-1}{1}_0}. \tag{2·24}$$

$N_{\alpha\beta}=N_{-\alpha-\beta}$ and $N_{\alpha,\beta}$ is zero if $\alpha+\beta$ is not a root.

This algebra has three invariants, of which one is the so-called Casimir operator:

$$Q=\boldsymbol{L}^2+\boldsymbol{X}^2+p_0{}^2-\boldsymbol{p}^2-\boldsymbol{M}^2-N_0^2-N^2, \tag{2·25}$$

and the others are

$$R=(\boldsymbol{L}\cdot\boldsymbol{p})X_0-(\boldsymbol{L}\cdot\boldsymbol{M})N-X_i(L_i p_0+M_j p_k-M_k p_j), \tag{2·26}$$

$$W=(\boldsymbol{L}\cdot\boldsymbol{X})^2-(\boldsymbol{L}\cdot\boldsymbol{p})^2-(\boldsymbol{L}\cdot\boldsymbol{M})^2+\sum(L_i p_0+M_j p_k-M_k p_j)^2 + \sum(M_i N+p_0 X_i-p_i X_0)^2 - \sum(L_i N+p_j X_k-p_k X_j)^2-\sum(L_i X_0+M_j X_k-M_k X_j)^2. \tag{2·27}$$

§ 3. General forms of the representative matrices

In the following argument, the first procedure is almost the same as what Thomas[10] has done in his "On the Unitary Representations of the Group of De Sitter Space". So we shall state briefly.

Two subalgebras composed of H_1+H_2, $E_{\binom{1}{1}_0}, E_{\binom{-1}{-1}_0}$ and H_1-H_2, $E_{\binom{1}{-1}_0}, E_{\binom{-1}{1}_0}$ respectively are commutative with each other. Among them hold the very familiar commutation relations, for example,

$$[H_1+H_2,\ E_{\binom{1}{1}_0}]=2E_{\binom{1}{1}_0},\quad [H_1+H_2,\ E_{\binom{-1}{-1}_0}]=-2E_{\binom{-1}{-1}_0},\quad [E_{\binom{1}{1}_0},\ E_{\binom{-1}{-1}_0}]=\frac{1}{8}(H_1+H_2).$$

Therefore the following representation matrices are obtained:

$$(m_1|H_1+H_2|m_1)=m_1,\qquad (m_2|H_1-H_2|m_2)=2m_2,$$

$$(m_1|E_{\binom{1}{1}_0}|m_1-1)=\frac{1}{2\sqrt{2}}\sqrt{(j_1+m_1)(j_1-m_1+1)},$$

$$(m_2|E_{\binom{1}{-1}_0}|m_2-1)=\frac{1}{2\sqrt{2}}\sqrt{(j_2+m_2)(j_2-m_2+1)},$$

$$(m_1|E_{\binom{-1}{-1}_0}|m_1+1)=\frac{1}{2\sqrt{2}}\sqrt{(j_1-m_1)(j_1+m_1+1)}, \tag{3·1}$$

$$(m_2|E_{\binom{-1}{1}_0}|m_2+1)=\frac{1}{2\sqrt{2}}\sqrt{(j_2-m_2)(j_2+m_2+1)}.$$

$$j_i=0,\ \frac{1}{2},\ 1,\ldots\qquad m_i=j_i,\ j_i-1,\ldots\ldots,\ -j_i+1,-j_i.$$

Then from the commutation relations

$$[H_1+H_2,E_{\binom{1}{0}_1}]=E_{\binom{1}{0}_1},\qquad [H_1+H_2,\ E_{\binom{0}{-1}_1}]=-E_{\binom{0}{-1}_1},$$

$$[E_{\binom{1}{1}_0},\ E_{\binom{1}{0}_1}]=0,\qquad [E_{\binom{1}{1}_0},\ E_{\binom{0}{-1}_1}]=\frac{-i}{2\sqrt{2}}E_{\binom{1}{0}_1},$$

$$[E_{\binom{-1}{-1}_0},\ E_{\binom{1}{0}_1}]=\frac{i}{2\sqrt{2}}E_{\binom{0}{-1}_1},\qquad [E_{\binom{-1}{-1}_0},\ E_{\binom{0}{-1}_1}]=0,$$

we get the following relations:

$$(j_1,\ m_1|E_{\binom{1}{0}_1}|j_1+\frac{1}{2},\ m_1-\frac{1}{2})=\sqrt{j_1-m_1+1}\ A,$$

$$(j_1,\ m_1|E_{\binom{1}{0}_1}|j_1-\frac{1}{2},\ m_1-\frac{1}{2})=\sqrt{j_1+m_1}\ B,$$

$$(j_1,m_1|E_{\binom{0}{-1}_1}|j_1+\frac{1}{2},\ m_1+\frac{1}{2})=i\sqrt{j_1+m_1+1}\ A,$$

$$(j_1,m_1|E_{\binom{0}{-1}_1}|j_1-\frac{1}{2},\ m_1+\frac{1}{2})=-i\sqrt{j_1-m_1}\ B,$$

where A and B are certain matrices. In this way using all the commutation relations except such ones as $[H_i,\ E_\alpha]=*$ and $[E_\alpha,\ E_\beta]=*$ ($\alpha_i\neq 0,\ \beta_i\neq 0$), we find the similar relations as above, and combining these results we have finally:

$$(j_1, j_2, m_1, m_2|E_{\binom{0}{1}}|j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1-\frac{1}{2}, m_2+\frac{1}{2})$$
$$=\sqrt{(j_1-m_1+1)(j_2+m_2+1)}f_1(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1+\frac{1}{2}, j_2-\frac{1}{2} \quad ,, \quad)$$
$$=\sqrt{(j_1-m_1+)(j_2-m_2)}f_2(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1-\frac{1}{2}, j_2+\frac{1}{2} \quad ,, \quad)$$
$$=\sqrt{(j_1+m_1)(j_2+m_2+1)}f_3(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1-\frac{1}{2}, j_2-\frac{1}{2} \quad ,, \quad)$$
$$=\sqrt{(j_1+m_1)(j_2-m_2)}f_4(j_1, j_2),$$

$$(\quad ,, \quad |E_{\binom{-1}{0}{1}}|j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2})$$
$$=-i\sqrt{(j_1+m_1+1)(j_2+m_2+1)}f_1(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad , \quad)$$
$$=-i\sqrt{(j_1+m_1+1)(j_2-m_2)}f_2(f, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1-\frac{1}{2}, j_2+\frac{1}{2} \quad ,, \quad)$$
$$=i\sqrt{(j_1-m_1)(j_2+m_2+1)}f_3(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad)$$
$$=i\sqrt{(j_1-m_1)(j_2-m_2)}f_4(j_1, j_2),$$

$$(\quad ,, \quad |E_{\binom{1}{0}{1}}|j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1-\frac{1}{2}, m_2-\frac{1}{2})$$
$$=i\sqrt{(j_1-m_1+1)(j_2-m_2+1)}f_1(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad)$$
$$=-i\sqrt{(j_1-m_1+1)(j_2+m_2)}f_2(j_1, j_2),$$

$$(\quad ,, \quad |\ ,, \ |j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad ,, \quad)$$
$$=i\sqrt{(j_1+m_1)(j_2-m_2+1)}f_3(j_1, j_2);$$

$$(\quad „ \quad | \quad „ \quad | j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= -i\sqrt{(j_1+m_1)(j_2+m_2)}f_4(j_1, j_2),$$

$$(\quad „ \quad | E_{\binom{0}{-1}{1}} | j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2-\frac{1}{2})$$
$$= -\sqrt{(j_1+m_1+1)(j_2-m_2+1)}f_1(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= \sqrt{(j_1+m_1+1)(j_2+m_2)}f_2(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad „ \quad)$$
$$= \sqrt{(j_1-m_1)(j_2-m_2+1)}f_3(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= -\sqrt{(j_1-m_1)(j_2+m_2)}f_4(j_1, j_2),$$

$$(\quad „ \quad | E_{\binom{0}{1}{-1}} | j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1-\frac{1}{2}, m_2+\frac{1}{2})$$
$$= \sqrt{(j_1-m_1+1)(j_2-m_2+1)}g_4(j_1, j_2), \qquad (3\cdot2)$$

$$(\quad „ \quad | \quad „ \quad | j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= \sqrt{(j_1-m_1+1)(j_2-m_2)}g_3(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad „ \quad)$$
$$= \sqrt{(j_1+m_1)(j_2+m_2+1)}g_2(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= \sqrt{(j_1+m_1)(j_2-m_2)}g_1(j_1, j_2),$$

$$(\quad „ \quad | E_{\binom{1}{0}{-1}} | j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1-\frac{1}{2}, m_2-\frac{1}{2})$$
$$= i\sqrt{(j_1-m_1+1)(j_2-m_2+1)}g_4(j_1, j_2),$$

$$(\quad „ \quad | \quad „ \quad | j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad „ \quad)$$
$$= -i\sqrt{(j_1-m_1+1)(j_2+m_2)}g_3(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad ,, \quad\right)$$
$$= i\sqrt{(j_1+m_1)(j_2-m_2+1)}\, g_2(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad\right)$$
$$= -i\sqrt{(j_1+m_1)(j_2+m_2)}\, g_1(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| E_{\binom{-1}{-1}} \middle| j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2}\right)$$
$$= -i\sqrt{(j_1+m_1+1)(j_2+m_2+1)}\, g_4(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad\right)$$
$$= -i\sqrt{(j_1+m_1+1)(j_2-m_2)}\, g_3(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad ,, \quad\right)$$
$$= i\sqrt{(j_1-m_1)(j_2+m_2+1)}\, g_2(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad\right)$$
$$= i\sqrt{(j_1-m_1)(j_2-m_2)}\, g_1(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| E_{\binom{0}{-1}} \middle| j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2-\frac{1}{2}\right)$$
$$= -\sqrt{(j_1+m_1+1)(j_2-m_2+1)}\, g_4(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1+\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad\right)$$
$$= \sqrt{(j_1+m_1+1)(j_2+m_2)}\, g_3(j_1, j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2+\frac{1}{2}, \quad ,, \quad\right)$$
$$= \sqrt{(j_1-m_1)(j_2-m_2+1)}\, g_2(j_1 j_2),$$

$$\left(\quad ,, \quad \middle| \quad ,, \quad \middle| j_1-\frac{1}{2}, j_2-\frac{1}{2}, \quad ,, \quad\right)$$
$$= -\sqrt{(j_1-m_1)(j_2+m_2)}\, g_1(j_1 j_2),$$

where f_i, g_i are certain matrices.

Next we shall take up H_3. Being commutative with H_1 and H_2, it may be diagonalized. We put

$$(\lambda|H_3|\lambda)=\lambda, \tag{3·3}$$

then the non-vanishing elements are $(\lambda|f_i(j_1,j_2)|\lambda-1)$ and $(\lambda|g_i(j_1,j_2)|\lambda+1)$, which we shall write as $f_i(j_1,j_2,\lambda)$ and $g_i(j_1,j_2,\lambda)$ hereafter.

Of the remaining commutation relations, we have to consider only

$$[E_{\begin{pmatrix}1\\0\\1\end{pmatrix}},\ E_{\begin{pmatrix}0\\-1\\-1\end{pmatrix}}]=\frac{i}{2\sqrt{2}}E_{\begin{pmatrix}1\\-1\\0\end{pmatrix}},\quad [E_{\begin{pmatrix}0\\1\\1\end{pmatrix}},\ E_{\begin{pmatrix}0\\-1\\-1\end{pmatrix}}]=\frac{1}{8}(H_2-H_3)$$

and $$[E_{\begin{pmatrix}1\\0\\1\end{pmatrix}},\ E_{\begin{pmatrix}-1\\0\\1\end{pmatrix}}]=[E_{\begin{pmatrix}-1\\0\\-1\end{pmatrix}},\ E_{\begin{pmatrix}1\\0\\-1\end{pmatrix}}]=0,$$

the others being deduced from them. These commutation relations give the following relations among f_i and g_i:

$$\begin{aligned}
&j_1(f_4g_3+g_1f_2)=(j_1+1)(f_2g_1+g_3f_4), && f_4g_3-g_1f_2=-f_2g_1+g_3f_4,\\
&j_2(f_4g_2+g_1f_3)=(j_2+1)(f_3g_1+g_2f_4), && f_4g_2-g_1f_3=-f_3g_1+g_2f_4,\\
&j_1(f_3g_4+g_2f_1)=(j_1+1)(f_1g_2+g_4f_3), && f_3g_4-g_2f_1=-f_1g_2+g_4f_3,\\
&j_2(f_2g_4+g_3f_1)=(j_2+1)(f_1g_3+g_4f_2), && f_2g_4-g_3f_1=-f_1g_3+g_4f_2,
\end{aligned}$$

$$f_1g_4=g_4f_1,\quad f_2g_3=g_3f_2,\quad f_3g_2=g_2f_3,\quad f_4g_1=g_1f_4,$$

$$\begin{aligned}
&j_1f_3f_1=(j_1+1)f_1f_3, && (j_1+1)g_3g_1=j_1g_1g_3,\\
&j_1f_4f_2=(j_1+1)f_2f_4, && (j_1+1)g_4g_2=j_1g_2g_4,\\
&j_2f_2f_1=(j_2+1)f_1f_2, && (j_2+1)g_2g_1=j_2g_1g_2,\\
&j_2f_4f_3=(j_2+1)f_3f_4, && (j_2+1)g_4g_3=j_2g_3g_4,
\end{aligned} \tag{3·4}$$

$$\begin{aligned}
&\{m_1(j_2+1)+m_2(j_1+1)\}f_1f_4+\{m_2(j_1+1)-m_1j_2\}f_2f_3\\
&\qquad+\{m_1(j_2+1)-m_2j_1\}f_3f_2-\{m_1j_2+m_2j_1\}f_4f_1=0,
\end{aligned}$$

$$\begin{aligned}
&\{m_1(j_2+1)+m_2(j_1+1)\}g_4g_1+\{m_2(j_1+1)-m_1j_2\}g_3g_2\\
&\qquad+\{m_1(j_2+1)-m_2j_1\}g_2g_3-\{m_1j_2+m_2j_1\}g_1g_4=0,
\end{aligned}$$

$$\begin{aligned}
&(j_1+m_1+1)(j_2-m_2+1)f_1g_1-(j_1+m_1+1)(j_2+m_2)f_2g_2-(j_1-m_1)(j_2-m_2+1)f_3g_3\\
&\qquad+(j_1-m_1)(j_2+m_2)f_4g_4\\
&-(j_1+m_1)(j_2-m_2)g_1f_1+(j_1+m_1)(j_2+m_2+1)g_2f_2+(j_1-m_1+1)(j_2-m_2)g_3f_3\\
&\qquad-(j_1-m_1+1)(j_2+m_2+1)g_4f_4=\frac{1}{8}(m_1-m_2-\lambda)I.
\end{aligned}$$

In these relations f_i or g_i, which stands at the head of a term, has always the arguments (j_1,j_2,λ), and then those of the following matrix are automatically determined, if we consider (3·2): for example, f_4g_3 means $f_4(j_1,j_2,\lambda)g_3(j_1-\frac{1}{2},j_2-\frac{1}{2},\lambda-1)$.

By (3·1), (3·2), (3·3) and (3·4), three invariants Q, R, and IV can be expressed as follows;

$$Q=2j_1(j_1+1)+2j_2(j_2+1)+\lambda^2+4\lambda-2^6[j_1j_2g_1f_1-j_1(j_2+1)g_2f_2-(j_1+1)j_2g_3f_3 + (j_1+1)(j_2+1)g_4f_4], \quad (3\cdot5)$$

$$=2j_1(j_1+1)+2j_2(j_2+1)+\lambda^2-4\lambda-2^6[j_1j_2f_1g_1-j_1(j_2+1)f_3g_3-(j_1+1)j_2f_2g_2 + (j_1+1)(j_2+1)f_1g_1]. \quad (3\cdot5')$$

$$R=-(j_1-j_2)(j_1+j_2+1)(\lambda+2)+2^5(j_1-j_2)[(j_1+1)(j_2+1)g_4f_4-j_1j_2g_1f_1] + 2^5(j_1+j_2+1)[j_1(j_2+1)g_2f_2-(j_1+1)j_2g_3f_3], \quad (3\cdot6)$$

$$=(j_1-j_2)(j_1+j_2+1)(-\lambda+2)-2^5(j_1-j_2)[(j_1+1)(j_2+1)f_1g_1-j_1j_2f_4g_4] -2^5(j_1+j_2+1)[j_1(j_2+1)f_3g_3-(j_1+1)j_2f_2g_2]. \quad (3\cdot6')$$

$$W=(j_1+j_2+1)^2(j_1-j_2)^2-2^5(j_1-j_2)^2[(j_1+1)(j_2+1)(f_1g_1+g_4f_4) + j_1j_2(f_4g_4+g_1f_1)]+2^5(j_1+j_2+1)^2[(j_1+1)j_2(f_2g_2+g_3f_3) + j_1(j_2+1)(f_3g_3+g_2f_2)]$$

$$+2^9j_1(j_1+1)[(j_2+1)(g_4f_4+g_2f_2)-j_2(g_1f_1+g_3f_3)-\frac{1}{16}(\lambda+1)]^2$$

$$+2^9j_2(j_2+1)[(j_1+1)(g_4f_4+g_3f_3)-j_1(g_1f_1+g_2f_2)-\frac{1}{16}(\lambda+1)]^2$$

$$-2^9[(j_1+1)(2j_1+3)\{(j_2+1)g_4f_2-j_2g_3f_1\}\{(j_2+1)g_2f_4-j_2g_1f_3\} \quad (3\cdot7)$$
$$+j_1(2j_1-1)\{(j_2+1)g_2f_4-j_2g_1f_3\}\{(j_2+1)g_4f_2-j_2g_3f_1\}$$
$$+(j_2+1)(2j_1+3)\{(j_1+1)g_4f_3-j_1g_2f_1\}\{(j_1+1)g_3f_4-j_1g_1f_2\}$$
$$+j_2(2j_2-1)\{(j_1+1)g_3f_4-j_1g_1f_2\}\{(j_1+1)g_4f_3-j_1g_2f_1\}].$$

4. Classification of representations

Now the preparation is ready and we are going to classify the possible representations using the relations (3·5), (3·6), (3·7) and the last equation in (3·4). Before that, however, we shall discuss about the proper values of H_1, H_2 and H_3. Since the proper values of H_1 and H_2 are m_1+m_2 and m_1-m_2 respectively, they are both integral or both half integral. As to the proper value of H_3, we consider, returning to the expression (2·18), the subgroup of two parameters $\exp(2\vartheta\mathcal{X}_{35})\exp(2\xi\mathcal{X}_{60})$. To this rotation corresponds the matrix

$$\begin{pmatrix} \cos\vartheta & & & \sin\vartheta \\ & \cos\vartheta & \sin\vartheta & \\ & -\sin\vartheta & \cos\vartheta & \\ -\sin\vartheta & & & \cos\vartheta \end{pmatrix}\begin{pmatrix} \cos\xi & & & -\sin\xi \\ & \cos\xi & \sin\xi & \\ & -\sin\xi & \cos\xi & \\ \sin\xi & & & \cos\xi \end{pmatrix}=\begin{pmatrix} \cos(\vartheta-\xi) & & & -\sin(\vartheta+\xi) \\ & \cos(\vartheta+\xi) & \sin(\vartheta+\xi) & \\ & -\sin(\vartheta+\xi) & \cos(\vartheta+\xi) & \\ \sin(\vartheta+\xi) & & & \cos(\vartheta-\xi) \end{pmatrix}.$$

Therefore $a(t, s)=\exp(t\mathcal{X}_{35})\exp(s\mathcal{X}_{60})$ becomes an identity when $t=s=2\pi$. This is represented by $\exp(-itH_2)\exp(-isH_3)$, and so $2\pi(H_2+H_3)$ must be zero, modulo 2π, i.e., the proper values of H_2 and H_3 have to be both integral or both half integral. Therefore the proper values of H_1, H_2, H_3 are all integral or all half integral. (This

property can be found, in the finite dimensional representation, out of consideration of the behavior of weights under Weyl reflection.)

From now on we use $p=j_1+j_2$ and $q=j_1-j_2$ instead of j_1 and j_2. Then

$$Q=p(p+2)+q^2+\lambda^2+4\lambda-2^4[(p^2-q^2)g_1f_1-(p+q)(p-q+2)g_2f_2 - (p-q)(p+q+2)g_3f_3+(p+q+2)(p-q+2)g_4f_4], \quad (4\cdot1)$$

$$R=-(p+1)q(\lambda+2)+2^3q[(p+q+2)(p-q+2)g_4f_4-(p^2-q^2)g_1f_1] +2^3(p+1)[(p+q)(p-q+2)g_2f_2-(p-q)(p+q+2)g_3f_3], \quad (4\cdot2)$$

and from the last equation of (3·4) we get

$$(p+q+1)(p-q+1)f_4g_4=g_1f_1+(p-q+2)g_2f_2+(p+q+2)g_3f_3 +(p+q+2)(p-q+2)g_4f_4-\frac{1}{8}(p+\lambda+2). \quad (4\cdot3)$$

These relations are to be understood as those of operators which are applied to $f(p, q, \lambda)$. Making use of them we can express g_2f_2, g_3f_3, and g_4f_4 by g_1f_1, as follows:

$$-2^5p(p+q+1)(p-q+2)g_2f_2=p\{p^2+(\lambda-q+1)^2-5-Q\}-2R -2^4(p-q)(p+q+1)\{(p+q-1)g_1f_1+(p-q+1)f_4g_4\}, \quad (4\cdot4)$$

$$-2^5p(p-q+1)(p+q+2)g_3f_3=p\{p^2+(\lambda+q+1)^2-5-Q\}+2R -2^4(p+q)(p-q+1)\{(p-q-1)g_1f_1+(p+q+1)f_4g_4\}, \quad (4\cdot5)$$

$$2^4p(p+q+1)(p+q+2)(p-q+1)(p-q+2)g_4f_4=p(p+1)\{(p+2)(3p+2\lambda) +\lambda^2-q^2-Q\}+2qR+2^4(p+q)(p+q+1)(p-q)(p-q+1)\{(p+1)f_4g_4-g_1f_1\}. \quad (4\cdot6)$$

As $p\geqq0$, in every representation there exists the minimum p, say p_0. Then $f_1f(p_0, q, \lambda)$ and $g_4f(p_0, q, \lambda)$ are zero, if q is equal neither to p_0 nor to $-p_0$. Therefore we find from (4·4), (4·5) and (4·6)

$$-g_2f_2f(p_0, q, \lambda)=\frac{p_0\{p_0^2+(\lambda-q+1)^2-5-Q\}-2R}{2^5p_0(p_0+q+1)(p_0-q+2)}f(p_0, q, \lambda), \quad (-p_0<q\leqq p_0) \quad (4\cdot7)$$

$$-g_3f_3f(p_0, q, \lambda)=\frac{p_0\{p_0^2+(\lambda+q+1)^2-5-Q\}+2R}{2^5p_0(p_0-q+1)(p_0+q+2)}f(p_0, q, \lambda), \quad (-p_0\leqq q<p_0) \quad (4\cdot8)$$

$$g_4f_4f(p_0, q, \lambda)=\frac{p_0(p_0+1)\{(p_0+2)(3p_0+2\lambda)+\lambda^2-q^2-Q\}+2qR}{2^4p_0(p_0+q+1)(p_0+q+2)(p_0-q+1)(p_0-q+2)}f(p_0, q, \lambda). \quad (-p_0\leqq q\leqq p_0) \quad (4\cdot9)$$

(We suppose here $p_0\neq0$.) We need not the value of g_2f_2 for $q=-p_0$ and that of g_3f_3 for $q=p_0$; because, if we form vectors in the representation space, which are proper func-

tions of operators H_1, H_2 and H_3 and belong to (p, q), starting from a definite vector, we have to multiply an appropriate factor shown in (3·2) on $f_2 f$ or $f_3 f$, and this factor is zero for $q=-p_0$ in the former case and for $q=p_0$ in the latter case. The corresponding expressions for $f_1 g_1$, $f_2 g_2$ and $f_3 g_3$ can be obtained by changing the signs of λ and R in $g_4 f_4$, $g_3 f_3$ and $g_2 f_2$ respectively, for example,

$$f_1 g_1 f(p_0, q, \lambda) = \frac{p_0(p_0+1)\{(p_0+2)(3p_0-2\lambda)+\lambda^2-q^2-Q\}-2qR}{2^4 p_0(p_0+q+1)(p_0+q+2)(p_0-q+1)(p_0-q+2)} f(p_0, q, \lambda),$$
$$(-p_0 \leqq q \leqq p_0). \quad (4\cdot10)$$

Now we apply $f_4(p_0+1, q, \lambda+1)$ to both sides of (4·9),

$$f_4(p_0+1, \lambda+1) g_4(p_0, q, \lambda) f_4(p_0+1, q, \lambda+1) f(p_0, q, \lambda)$$
$$= \frac{p_0(p_0+1)\{(p_0+2)(3p_0+2\lambda)+\lambda^2-q^2-Q\}+2qR}{2^4 p_0(p_0+q+1)(p+q+2)(p_0-q+1)(p-q+2)} f_4(p_0+1, q, \lambda+1) f(p_0, q, \lambda).$$

We denote $f_4(p_0+1, q, \lambda+1)f$, which, multiplied by an appropriate factor, is a proper vector of H_1, H_2 and H_3 (of proper value $\lambda+1$) and belongs to p_0+1 and q, by $f(p_0+1, q, \lambda+1)$. Thus we have

$$f_4 g_4 f(p_0+1, q, \lambda) = \frac{p_0(p_0+1)\{(p_0+2)(3p_0+2\lambda-2)+(\lambda-1)^2-q^2-Q\}+2qR}{2^4 p_0(p_0+q+1)(p_0+q+2)(p_0-q+1)(p_0-q+2)} f(p_0+1, q, \lambda).$$

Similarly from (4·10),

$$g_1 f_1 f(p_0+1, q, \lambda) = \frac{p_0(p_0+1)\{(p_0+2)(3p_0-2\lambda-2)+(\lambda+1)^2-q^2-Q\}-2qR}{2^4 p_0(p_0+q+1)(p_0+q+2)(p_0-q+1)(p_0-q+2)} f(p_0+1, q, \lambda).$$

Using these results we can find, by means of (4·4), (4·5) and (5·6), the expressions for $g_2 f_2$, $g_3 f_3$ and $g_4 f_4$ as operators applying to $f(p_0+1, q, \lambda)$. Following in this way we can get the similar expressions for $p=p_0+2$, $p_0+3, \cdots\cdots$. And from the first few cases we can perceive the general forms, which are legitimated by inserting into (4·4), (4·5) and (4·6);

$g_1 f_1 f(p_0+s, q, \lambda)$

$$= \frac{s(p_0+1/2(s+1))}{2^4 p_0(p_0+1)(p_0+q+s)(p_0+q+s+1)(p_0-q+s)(p_0-q+s+1)} [p_0(p_0+1)$$
$$\cdot\{3p_0(p_0+2)+(s-1)(2p_0+s+3)-2(\lambda+1)(p_0+s+3)+(\lambda+1)^2-q^2-Q\}-2qR] f,$$

$$-g_2 f_2 f(p_0+s, q, \lambda)$$
$$= \frac{(p_0+q)(p_0-q+1)}{2^5 p_0(p_0+1)(p_0+q+s)(p_0+q+s+1)(p_0-q+s+1)(p_0-q+s+2)}$$
$$\times[p_0(p_0+1)\{p_0^2-2sp_0-s(s+2)-5+(\lambda-q+1)^2-Q\}-2(p_0+s+1)R] f,$$
$$(4\cdot11)$$

$$-g_3 f_3 f(p_0+s, q, \lambda)$$

$$=\frac{(p_0-q)(p_0+q+1)}{2^5 p_0(p_0+1)(p_0-q+s)(p_0-q+s+1)(p_0+q+s+1)(p_0+q+s+2)}$$

$$\times[p_0(p_0+1)\{p_0^2-2sp_0-s(s+2)-5+(\lambda+q+1)^2-Q\}+2(p_0+s+1)R]f,$$

$$g_4 f_4 f(p_0+s, q, \lambda)$$

$$=\frac{(s+1)(p_0+1+s/2)}{2^4 p_0(p_0+1)(p_0+q+s+1)(p_0+q+s+2)(p_0-q+s+1)(p_0-q+s+2)}$$

$$\times[p_0(p_0+1)\{3p_0(p_0+2)+s(2p_0+s+4)+2\lambda(p_0+s+2)+\lambda^2-q^2-Q\}+2qR]f.$$

The corresponding expressions for $f_1 g_1, f_2 g_2, f_3 g_3$ and $f_4 g_4$ are obtained by changing the signs of λ and R in $g_4 f_4$, $g_3 f_3$, $g_2 f_2$ and $g_1 f_1$ respectively.

From (2·21) and (3·2), we see that

$$f_1(p, q, \lambda) \text{ and } g_1(p+1, q, \lambda-1),$$

$$f_2(p, q, \lambda) \text{ and } -g_2(p, q+1, \lambda-1),$$

$$f_3(p, q, \lambda) \text{ and } -g_3(p, q-1, \lambda-1),$$

$$f_4(p, q, \lambda) \text{ and } g_4(p-1, q, \lambda-1)$$

are Hermitian conjugate with each other. Therefore, forming scalar products of both sides of (4·11) with $f(p_0+s, q, \lambda)$, we have,

$$\begin{aligned}
&\|f_1(p_0+s-1, q, \lambda+1)f(p_0+s, q, \lambda)\|^2=\sigma_1(s, q, \lambda)\|f(p_0+s, q, \lambda)\|^2,\\
&\|f_2(p_0+s, q-1, \lambda+1)f(\quad ,, \quad)\|^2=\sigma_2(\quad ,, \quad)\|f(\quad ,, \quad)\|^2,\\
&\|f_3(p_0+s, q+1, \lambda+1)f(\quad ,, \quad)\|^2=\sigma_3(\quad ,, \quad)\|f(\quad ,, \quad)\|^2,\\
&\|f_4(p_0+s+1, q, \lambda+1)f(\quad ,, \quad)\|^2=\sigma_4(\quad ,, \quad)\|f(\quad ,, \quad)\|^2,\\
&\|g_1(p_0+s+1, q, \lambda-1)f(\quad ,, \quad)\|^2=\rho_1(\quad ,, \quad)\|f(\quad ,, \quad)\|^2, \qquad (4\cdot12)\\
&\|g_2(p_0+s, q+1, \lambda-1)f(\quad ,, \quad)\|^2=\rho_2(\quad ,, \quad)\|f(\quad ,, \quad)\|^2,\\
&\|g_3(p_0+s, q-1, \lambda-1)f(\quad ,, \quad)\|^2=\rho_3(\quad ,, \quad)\|f(\quad ,, \quad)\|^2,\\
&\|g_4(p_0+s-1, q, \lambda-1)f(\quad ,, \quad)\|^2=\rho_4(\quad ,, \quad)\|f(\quad ,, \quad)\|^2.
\end{aligned}$$

We simplify the expression for σ_i and ρ_i by introducing two constants A and B related to Q and R by:

$$Q=2(p_0^2+p_0-2)+AB,\ 2R=p_0(p_0+1)(A+B). \qquad (4\cdot13)$$

Then σ_i and ρ_i come to be as follows;

$$\sigma_1(s, q, \lambda)=\frac{s(p_0+1/2(s+1))[(p_0+s-\lambda)^2-(q+A)(q+B)]}{2^4(p_0+q+s)(p_0+q+s+1)(p_0-q+s)(p_0-q+s+1)}, \qquad (4\cdot14\text{–}1)$$

$$\sigma_2(\quad,,\quad)=\frac{(p_0+q)(p_0-q+1)[(q-\lambda-1)^2-(p_0+s+1+A)(p_0+s+1+B)]}{2^5(p_0+q+s)(p_0+q+s+1)(p_0-q+s+1)(p_0-q+s+2)}, \tag{4·14-2}$$

$$\sigma_3(\quad,,\quad)=\frac{(p_0-q)(p_0+q+1)[(q+\lambda+1)^2-(p_0+s+1-A)(p_0+s+1-B)]}{2^5(p_0-q+s)(p_0-q+s+1)(p_0+q+s+1)(p_0+q+s+2)}, \tag{4·14-3}$$

$$\sigma_4(\quad,,\quad)=\frac{(s+1)(p_0+1+s/2)[(p_0+s+\lambda+2)^2-(q-A)(q-B)]}{2^4(p_0+q+s+1)(p_0+q+s+2)(p_0-q+s+1)(p_0-q+s+2)}, \tag{4·14-4}$$

$$\rho_1(\quad,,\quad)=\frac{(s+1)[(p_0+1+s/2)[(p_0+s-\lambda+2)^2-(q+A)(q+B)]}{2^4(p_0+q+s+1)(p_0+q+s+2)(p_0-q+s+1)(p_0-q+s+2)}, \tag{4·15-1}$$

$$\rho_2(\quad,,\quad)=\frac{(p_0-q)(p_0+q+1)[(q-\lambda+1)^2-(p_0+s+1+A)(p_0+s+1+B)]}{2^5(p_0-q+s)(p_0-q+s+1)(p_0+q+s+1)(p_0+q+s+2)}, \tag{4·15-2}$$

$$\rho_3(\quad,,\quad)=\frac{(p_0+q)(p_0-q+1)[(q+\lambda-1)^2-(p_0+s+1-A)(p_0+s+1-B)]}{2^5(p_0+q+s)(p_0+q+s+1)(p_0-q+s+1)(p_0-q+s+2)}, \tag{4·15-3}$$

$$\rho_4(\quad,,\quad)=\frac{s(p_0+1/2(s+1))[(p_0+s+\lambda)^2-(q-A)(q-B)]}{2^4(p_0+q+s)(p_0+q+s+1)(p_0-q+s)(p_0-q+s+1)}. \tag{4·15-4}$$

At first we see that always (i.e., for any value of s) $-p_0\leqq q\leqq p_0$. Then except σ_1 and ρ_4 for $s=0$, it depends upon the expression in the square brackets, whether σ_i or ρ_i is positive, negative or zero.

We first consider (4·14–2, 3) and (4·15–2, 3) for a fixed s. The existence of $f(p_0+s, q, \lambda)$ being assumed for some values q and λ, we may construct other vectors by applying g_2, g_3, f_2 or f_3 on $f(q, \lambda)$. From the condition that σ_i and ρ_i can not be negative, there arise restrictions for the domain on which $f(q, \lambda)$ exists. We see that these domains are necessarily limited by a boundary-line $q+\lambda=$const. or $q-\lambda=$const.. From the consideration of σ_2 and ρ_2 we have the following three different cases.

1) $(p_0+s+1+A)(p_0+s+1+B)<0$ or $(p_0+s+1+A)(p_0+s+1+B)<1$ (when $q\pm\lambda$ is even). In this case σ_2 and ρ_2 are always positive and starting from any vector, we can form other vectors of which q ranges from $-p_0$ to p_0. (Fig. 1)

2. $(p_0+s+1+A)(p_0+s+1+B)=(t-1)^2$, $t>0$.

Let σ_2 be zero for $q-\lambda=t$ and positive for $q-\lambda=t+2$, then $(p_0+s+1+B)(p_0+s+1+B)=(t-1)^2$ and $t>0$. For $q-\lambda\geqq t$, ρ_2 is positive and the allowable domain becomes as such one shown in Fig. 2a. Similarly let ρ_2 be zero for $q-\lambda=t'$ and positive for $q-\lambda=t'-2$, then $(p_0+s+1+A)(p_0+s+1+B)=(t'+1)^2$ and $t'<0$. For $q-\lambda\leqq t$, σ_2 is positive. Putting $t'=-t$ we have the same condition as above. (Fig. 2b)

3. $(p_0+s+1+A)(p_0+s+1+B)=1,\ q-\lambda=0.$

In this case $\sigma_2=\rho_2=0$ and $f(q,\lambda)$ can exist only for $q-\lambda=0$. (Fig. 3)

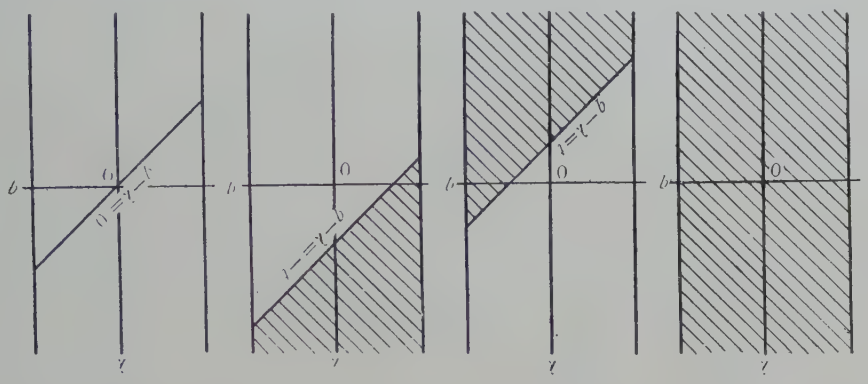

Fig. 1 Fig. 2a Fig. 2b Fig. 3

In the same way from (4·14–3) and (4·15–3) we have three cases as to the behavior of f_3 and g_3.

1′. $(p_0+s+1-A)(p_0+s+1-B)<0$ or $(p_0+s+1-A)(p_0+s+1-B)<1$, (when $q\pm\lambda$ is even).

2′. $(p_0+s+1-A)(p_0+s+1-B)=(r-1)^2,\ r>0.$

3′. $(p_0+s+1-A)(p_0+s+1-B)=1,\ q+\lambda=0.$

Combining these results we have nine cases in all. (1–1′, 1–2′, etc.) Now we shall annul the inappropriate combinations examining the behavior of vectors under the application of f_1, g_1, f_4 and g_4.

1) $(p_0+s+1+A)(p_0+s+1+B)<0,\ (p_0+s+1-A)(p_0+s+1-B)<0,$

or $(p_0+s+1+A)(p_0+s+1+B)<1,\ (p_0+s+1-A)(p_0+s+1-B)<1,$ ($q\pm\lambda$ is even).

In this case, since σ_4 and ρ_4 are always positive the domain $(-p_0\leqq q\leqq p_0,\ -\infty<\lambda<+\infty)$ is extended from $p=p_0$ to $p<+\infty$. Therefore above condition must be satisfied for any value of s. But it is impossible with definite A and B. This case should be excluded. This is the combination $1-1'$.

2) $(p_0+s+1-A)(p_0+s+1-B)=(r-1)^2$ and $(p_0+s+1+A)(p_0+s+1+B)<0,$ (r is positive odd integer),

or $(p_0+s+1+A)(p_0+s+1+B)<1,$ (r is positive even integer).

For $p=p_0+s$, vectors are supported on the domain $q+\lambda\geqq r$ or $q+\lambda\leqq -r$. We shall

consider the former case where the domain is $q+\lambda \geqq r$; the latter case is treated exactly in the same way. In this case σ_4 is always positive and if ρ_1 is not zero on $q+\lambda=r$, the domain comes to be limited by $q+\lambda=r-1$ for $p=p_0+s+1$. But such a situation can not continue without limit, since the domain $q+\lambda \leqq r'$ $(r' \leqq 0)$ is not permissible for any s. Therefore ρ_1 must vanish for $p=p_0+s+k$, on $q+\lambda=r-k$ $(r>k \geqq 0)$. But such an equation can not be satisfied as A and B are compelled to satisfy only one equation and an inequality. This case should be excluded too. This is the combination $1-2'$, and $1'-2$ is treated similarly.

3) $(p_0+s+1+A)(p_0+s+1+B)<1$, $(p_0+s+1-A)(p_0+s+1-B)=1$, $q+\lambda=0$.

If σ_4 or ρ_1 is not zero for $q+\lambda=0$, we have as a domain $q+\lambda=1$ or $q+\lambda=-1$ for $p=p_0+s+1$. Therefore, if possible, the domain has to be confined in the plane $p=p_0$. But, for $s=0$ σ_4 and ρ_1 can not be made equal to zero on $q+\lambda=0$, without destroying above condition. Thus the combination $1-3'$ is also to be expelled. The same about the combination $1'-3$.

4) $(p_0+s+1+A)(p_0+s+1+B)=(t-1)^2$, $(p_0+s+1-A)(p_0+s+1-B)=1$, $q+\lambda=0$ and t is a positive integer.

The same argument as was made in 3) shows that the domain is to be restricted in $p=p_0$. In order that σ_4 and ρ_1 vanish on $q+\lambda=0$, A and B must be both equal to p_0+2. Then from the first condition $t=2(p_0+2)$ and $q \geqq p_0+2$. Therefore the combination $2-3'$, and similarly $2'-3$, are excluded.

5) $(p_0+s+1-A)(p_0+s+1-B)=1$, $(p_0+s+1+A)(p_0+s+1+B)=1$, $q=\lambda=0$.

s must be zero and $A+B=0$ and $AB=1-(p_0+1)^2$. But as in 4) A must be equal to B, and as $p_0 \neq 0$ this can not be satisfied. This is the combination $3-3'$.

Thus we have seen that all the combinations except $2-2'$ are to be excluded. In the case of $2-2'$ we have the condition

$$(p_0+s+1+A)(p_0+s+1+B)=(t-1)^2,$$
$$(p_0+s+1-A)(p_0+s+1-B)=(r-1)^2.$$

t and r are positive integers.

The domain of vectors is $q+\lambda \geqq r$, $q-\lambda \leqq -t$ or $q+\lambda \leqq -r$, $q-\lambda \geqq t$. Since this is the only possibility remained here, the similar condition and the similar shape of the domain must be maintained for any value of s. From the simple consideration of above equations we see that $A=B$ and when s changes by ± 1, t and r change also by ± 1. We take s to be zero, then

$$p_0+1+A=\pm(t-1), \quad p_0+1-A=\pm(r-1).$$

Thus there are four possibilities for the choice of signs.

a) $(+,+)$. In this case $t+r=2(p_0+2)$, $2A=t-r$. As to the values of σ_4 and ρ_1, we have the following result;

$$\begin{aligned}
&\text{on } q+\lambda=-r, \quad \sigma_4=0, \quad \rho_1>0,\\
&\text{on } q+\lambda=r, \quad \sigma_4>0, \quad \rho_1=0,\\
&\text{on } q-\lambda=t, \quad \sigma_4=0, \quad \rho_1>0,\\
&\text{on } q-\lambda=-t, \quad \sigma_4>0, \quad \rho_1=0.
\end{aligned}$$

(cf. Fig. 4). When s increases the similar situation continues as is seen from (4.14–4) and (4.15–4).

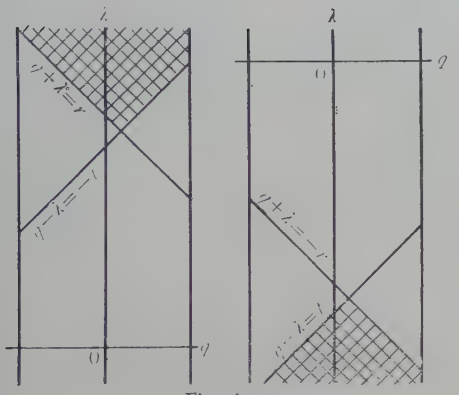

Fig. 4

b) $(-,-)$. This should be excluded since $2(p_0+1)=2-t-r<0$ appears.

c) $(+,-)$. $2(p_0+1)=t-r$, $2A=t+r-2$.

on $q+\lambda=-r$, $\sigma_4\leqq 0$, $(=0$ if $r=1)$, $\rho_1>0$,

on $q+\lambda=r$, $\sigma_4>0$, $\rho_1\leqq 0$, $(=0$ if $r=1)$,

on $q-\lambda=t$, $\sigma_4=0$, $\rho_1>0$,

on $q-\lambda=-t$, $\sigma_4>0$, $\rho_1=0$.

In this case, therefore, $r=1$ and $t=2p_0+3$, i.e., $A=p_0+1$ is the only case to be taken into account, but this is included in a).

d) $(-,+)$. Similarly as above the permissible values are $t=1$ and $r=2p_0+3$, i.e., $A=-(p_0+1)$ and this is included in a).

To find the value of W, e.g., in the case where the domain is $q+\lambda\leqq -r$, $q-\lambda\geqq t$ for $p=p_0$, $q+\lambda\leqq -r-1$, $q-\lambda\geqq t+1$ for $p=p_0+1,\cdots\cdots$, we apply (3.7) to the vector $f(p=p_0=\frac{1}{2}(t+r-4),\ q=\frac{1}{2}(t-r),\ \lambda=-\frac{1}{2}(t+r))$, (when $|t-r|\leq 2p_0$). We get the following result,

$$W=\frac{1}{16}(t+r-2)(t+r-4)[(t+r-2)(t+r-4)+2(t-r+2)(t-r-2)]. \quad (4\cdot 16)$$

When $|t-r|>2p_0$, the values of t and r are forced to be $t=2p_0+3$, $r=1$ or $t=1$, $r=2p_0+3$. And if we apply W on $f(p_0,\ q=p_0,\ \lambda=p_0-t)$ or on $f(p_0,\ q=-p_0,\ \lambda=p_0-r)$, we see that the result is nothing else the special case of (4.16).

If vectors are confined in the domain $q+\lambda\geqq r$, $q-\lambda\leqq -t$ for $p=p_0$, the value of W is the same as above. Because this case is obtained from the former by changing the signs of λ and exchanging the role of t and r (this comes to be $\lambda\to-\lambda$, and $R\to-R$ and so $f_1g_1, f_2g_2, f_3g_3, f_4g_4$ are replaced by $g_1f_4, g_3f_3, g_2f_2, g_1f_1$ respectively and vice versa), and thereby the expression W (3.7) is seen to be left invariant.

Now we shall study the case $p_0=0$. In the same way as before, starting from

$$Q=\lambda^2-4\lambda-2^6 f_1 g_1=\lambda^2+4\lambda-2^6 g_4 f_4,$$

(Q being considered as an operator on $f(0, 0, \lambda)$) we get

$$g_4 f_4 f(s, 0, \lambda)=\frac{\lambda^2+2\lambda(s+2)+s(s+4)-Q}{2^5(s+1)(s+2)}f=\frac{(s+\lambda+2)^2+A}{2^2(s+1)(s+2)}f=\sigma_4 f,$$

$$f_1 g_1 f(s, 0, \lambda)=\frac{\lambda^2-2\lambda(s+2)+s(s+4)-Q}{2^5(s+1)(s+2)}f=\frac{(s-\lambda+2)^2+A^2}{2^5(s+1)(s+2)}f=\rho_1 f,$$

$f_2 g_2=f_3 g_3=g_2 f_2=g_3 f_3=0$, $R=0$ and q is always zero. (We put $Q=-4-A^2$ in the last expression.) There are three cases as to the value of A.

1) $A^2>0$.

In this case σ_4 and ρ_1 is positive for any s and λ.

2) $0\geqq A^2>-1$.

Only when λ is an odd integer, σ_4 and ρ_1 is positive for any s.

3) $A^2=-t^2$, $t=0, 1, 2, \cdots\cdots$

In this case the support of vectors, to which both f_1 and g_1 are applicable, is confined to $s-\lambda\leqq-(t+2)$ or $s-\lambda\geqq t$, and similarly for f_4 and g_4 the support is $s+\lambda\geqq t$ or $s+\lambda\geqq-(t+2)$. Combining them we have for the domain of vectors, to which any operator can be applied, the following:

a) $s-\lambda\leqq-(t+2)$, b) $s+\lambda\leqq-(t+2)$, c) $s+\lambda\geqq 0$, $s-\lambda\leqq 0$, (only for $t=0$).

In these cases W is obviously zero.

§ 5. Summary

We write down all the permissible values of three invariants and the range of the proper values of H_1, H_2, and H_3 in each case.

I. $Q<-4$, $\quad R=W=0$.

$j_1+j_2=0, 1, 2, \cdots\cdots, j_1-j_2=0$, λ integer $(-\infty, +\infty)$.

II. $-4\leqq Q<-3$, $\quad R=W=0$.

$j_1+j_2=0, 1, 2, \cdots\cdots, j_1-j_2=0$, λ integer $(-\infty, +\infty)$,
$\lambda+j_1+j_2$; odd integer.

III. $Q=t^2-4$, $t=0, 1, 2, 3, \cdots\cdots$, $R=W=0$.

$$\left.\begin{matrix} j_1+j_2=0 \\ \lambda\geqq t+2 \end{matrix}\right\}\ \left.\begin{matrix} 1 \\ t+3 \end{matrix}\right\}\ \left.\begin{matrix} 2 \\ t+4 \end{matrix}\right\}\cdots\cdots, j_1-j_2=0.$$

III′ $Q=t^2-4$, $t=0,\ 1,\ 2,\ 3,\cdots\cdots$, $R=W=0$.

$$\left.\begin{array}{l} j_1+j_2=0 \\ \lambda \leqq -(t+2) \end{array}\right\} \begin{array}{c} 1 \\ -(t+3) \end{array} \left.\begin{array}{c} 2 \\ -(t+4) \end{array}\right\} \cdots\cdots,\ j_1-j_2=0.$$

IV. $Q=-4$, $R=W=0$.

$$\left.\begin{array}{l} j_1+j_2=0 \\ \lambda=0 \end{array}\right\} \begin{array}{c} 1 \\ -1,\ 1 \end{array} \left.\begin{array}{c} 2 \\ -2,\ 0,\ 2 \end{array}\right\} \cdots\cdots,\ j_1-j_2=0.$$

V. $$Q=\frac{1}{4}\left[2(t+r-2)(t+r-4)+(t-r+4)(t-r-4)\right],$$

$$R=\frac{1}{8}(t-r)(t+r-2)(t+r-4),$$

$$W=\frac{1}{16}(t+r-2)(t+r-4)\left[(t+r-2)(t+r-4)+2(t-r-2)(t-r-2)\right],$$

$t=1,\ 2,\ 3,\ 4,\cdots\cdots$, $r=1,\ 2,\ 3,\ 4,\cdots\cdots,(t+r\geqq 5)$.

$$\left.\begin{array}{l} j_1+j_2=\frac{1}{2}(t+r-4) \\ j_1-j_2+\lambda \geqq r \\ j_1-j_2-\lambda \leqq -t \end{array}\right\} \begin{array}{c} \frac{1}{2}(t+r-2) \\ r+1 \\ -(t+1) \end{array} \left.\begin{array}{c} \frac{1}{2}(t+r) \\ r+2 \\ -(t+2) \end{array}\right\} \cdots$$

$$\cdots\cdots,\ |j_1-j_2| \leqq \frac{1}{2}(t+r-4).$$

V′. Q, R and W are the same as above,

$$\left.\begin{array}{l} j_1+j_2=\frac{1}{2}(t+r-4) \\ j_1-j_2+\lambda \leqq -r \\ j_1-j_2-\lambda \geqq t \end{array}\right\} \begin{array}{c} \frac{1}{2}(t+r-2) \\ -(r+1) \\ t+1 \end{array} \left.\begin{array}{c} \frac{1}{2}(t+r) \\ -(r+2) \\ t+2 \end{array}\right\} \cdots, |j_1-j_2| \leqq \frac{1}{2}(t+r-4).$$

Though we have said little about the domain of vectors to which E_α applies, all the necessary arguments are stated in Bargmann's paper. In fact using (3.2) and (4.11) we can write

$$E_\alpha f(j_1, j_2, m_1, m_2, \lambda) = \Sigma (j_1, j_2, m_1, m_2, \lambda \mid h^\alpha \mid j_1', j_2', m_1', m'_2 \lambda') \cdot f(j_1', j_2', m_1', m_2', \lambda'),$$

$((A|h|B)=(B|h|A)^*)$, and f span a Hilbert space, which is invariant with respect to $\mathfrak{G}$. And the classification of them furnishes that of the irreducible representation of $\mathfrak{G}$. We have not tried to extend these representations to those of the original group. We shall do this in the course of application to physics, if it demands.

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On the Natural Decay of the Free Neutron

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The natural decay of the free neutron is investigated where the beta-transition of the nucleon is considered to be caused by the intermediation of a boson field. The results are compared with those of Fermi's theory, and the problem of whether the β-coupling belongs to the interaction of the first kind or not is discussed.

§ 1. Introduction

As is well known, two different theories of the beta-disintegration have been suggested, that is, the theory of Fermi's type and that of Yukawa's type. The former theory consider the β-transition to be caused by a direct interaction between nucleons and leptons, while the latter theory by the intermediation of a boson field. However, in the theory of Yukawa's type there exists a familiar and serious difficulty concerning the competing process between (π, μ) and (π, e) decays, when this boson is considered to be identical with the π-meson. On the other hand, as the direct interaction belongs to the interaction of the second kind, the theory of Fermi's type seems to lie beyond the limit of the applicability of the renormalization.

In order to get rid of these difficulties, we assume here the existence of a new kind of boson which intermediates the beta-transition. By treating chiefly the natural decay of the free neutron, we shall analyse the connection between the theory of Yukawa's type and that of the Fermi's type. In the last section we also discuss the problem whether or not the β-coupling can be explained only in terms of the interaction of the first kind.

As the elementary process of the β-transition we take the following one which is the same process as treated previously by S. Sakata[1] :

$$\begin{aligned} N &\longrightarrow P + B^- \\ B^- &\longrightarrow e^- + (\nu), \end{aligned} \tag{1.1}$$

where N, P, e^- and (ν) denote neutron, proton, electron and antineutrino respectively and $B^{\pm}$ express the positive and negative boson.

The Hamiltonian density of this interaction is given by

$$H(x) = \sum_{i=1}^{4} H_i(x) \,. \tag{1.2}$$

with

$$H_1 = R\,U^* + \text{comp. conj.}\,,$$
$$H_2 = R'U^* + \text{comp. conj.}\,,$$

$$H_3 = S_\mu \partial_\mu U^* + \text{comp. conj.},$$
$$H_4 = S_\mu' \partial_\mu U^* + \text{comp. conj.},$$

where

$$R = f \bar{\psi} \gamma_5' \tau_+ \psi,$$
$$R' = f' \bar{\varphi} \gamma_5' \phi,$$
$$S_\mu = g/\varkappa \bar{\psi} \gamma_\mu \gamma_5' \tau_+ \psi,$$
$$S' = g'/\varkappa \bar{\varphi} \gamma_\mu \gamma_5' \phi,$$

(scalar or pseudoscalar theory)

or

$$H = \sum_{i=1}^{6} H_i \text{ *)}$$

with

$$H_1 = M_\mu U_\mu^* + \text{comp. conj.},$$
$$H_2 = M'_\mu U_\mu^* + \text{comp. conj.},$$
$$H_3 = T^{\mu\nu}(\partial_\mu U_\nu^* - \partial_\nu U_\mu^*) + \text{comp. conj.},$$
$$H_4 = T'_{\mu\nu}(\partial_\mu U_\nu^* - \partial_\nu U_\mu^*) + \text{comp. conj.},$$
$$H_5 = 1/\varkappa^2 \, n_\mu n_\nu (M_\mu + M'_\mu)(M_\mu^* + M'^*_\mu),$$
$$H_6 = 4 \, n_\mu n_\nu (T_{\sigma\mu} + T'_{\sigma\mu})(T_{\sigma\nu}^* + T'^*_{\sigma\nu}),$$

where

$$M_\mu = f \bar{\psi} \gamma_5' \gamma_\mu \tau_+ \psi,$$
$$M'_\mu = f' \bar{\phi} \gamma_5' \gamma_\mu \varphi,$$
$$T_{\mu\nu} = g/\varkappa \bar{\psi} \gamma_5' \gamma^{[\mu\nu]} \tau_- \psi,$$
$$T'_{\mu\nu} = g'/\varkappa \bar{\phi} \gamma_5' \gamma^{[\mu\nu]} \varphi,$$

(vector or pseudovector theory).

Among these expressions, ψ, φ, ϕ and U describe the nucleon, neutrino, electron and boson field respectively and γ_5' denotes 1 or γ_5 for scalar and vector or pseudoscalar and pseudovector boson theory respectively and further n_μ is a time-like unit vector.

§ 2. Qualitative discussion on the effect caused by the boson field

In terms of mathematics the deviation of the theory of Yukawa's type from this of Fermi's type may be expressed by the non-local interaction between the Dirac particles caused by the propagation function F of the boson field, and the effects of the derivative on the F-function, which arise from the derivative-coupling or the commutation relation of the boson field. At first, we obtain a general equation by which we can estimate the deviation effects caused by the F-function from the case of Fermi's type interaction in such a

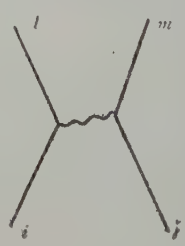

Fig. 1.

* This interaction differs from the one treated by S. Sakata by $2(T_{\sigma\mu} + T'_{\sigma\mu})(T^*_{\sigma\mu} + T'^*_{\sigma\mu})$.

general Möller type interaction of the four Dirac particles via the virtual boson. (see Fig. 1.)

In the general case, neglecting normal dependent terms, we have the interaction Hamiltonian density:

$$H(x)=\sum_{i,k=1}^{4} f_{ik}(\bar{\psi}\,\gamma_{\alpha\beta}\,\overset{ik}{\tau}\,\psi)\,\Gamma_{\beta}(\partial)\,U^{\alpha}+\text{comp. conj.} \tag{2.1}$$

where $\gamma_{\alpha\beta}$ denotes a some product of γ-matrices, and suffix α refers to the boson-type and β to the derivative operator, while $\overset{ik}{\tau}$ is τ-spin matrix of these four Dirac particles.

The second order S-matrix with the vacuum value of the boson field for this process is written as follows,

$$\begin{aligned}\langle U(\infty)\rangle_{\text{vac}}&=-1/(\hbar c)^{2}\int_{-\infty}^{\infty}\!\!\int dx_1 dx_2 \sum_{i,j,k,l} f_{ik}f_{jl}{}^{*}(\bar{\psi}(x_1)\gamma_{\alpha\beta}\overset{ik}{\tau}\psi(x_1))\\ &\quad\times(\bar{\psi}(x_2)\gamma_{\alpha'\beta'}\overset{jl}{\tau}\psi(x_2))\,\Gamma^{\beta\beta'}(\partial_1,\partial_2)\,\langle P(U^{\alpha}(x_1),U^{*\alpha'}(x_2))\rangle_{\text{vac}}\\ &=-1/(2\pi\hbar c)^{2}\int_{-\infty}^{\infty}\!\!\int dx_1 dx_2 \sum_{i,j,k,l} f_{ik}f_{jl}{}^{*}(\bar{\psi}(x_1)\gamma_{\alpha\beta}\overset{ik}{\tau}\psi(x_1))\\ &\quad\times(\bar{\psi}(x_2)\gamma_{\alpha'\beta'}\overset{jl}{\tau}\psi(x_2))\,\Gamma^{\beta\beta'}(\partial_1,\partial_2)\,C^{\alpha\alpha'}(\partial^{1}/\varkappa)F(x_1-x_2)\end{aligned} \tag{2.2}$$

where

$$\Gamma^{\beta\beta'}(\partial_1,\partial_2)=\Gamma_{\beta}(\partial_1)\,\Gamma_{\beta'}(\partial_2),$$

$$\langle P(U^{\alpha}(x_1),U^{*\alpha'}(x_2))\rangle_{\text{vac}}=1/(2\pi)^{2}C^{\alpha\alpha'}(\partial_1/\varkappa)F(x_1-x_2).^{*}$$

Further (2.2) can be transformed into an integral of one space-time coordinate x. In fact,

$$\begin{aligned}\langle U(\infty)\rangle_{\text{vac}}&=-(2\pi/\hbar c)^{2}\sum_{i,j,k,l}\int_{-\infty}^{\infty}dx(\bar{\psi}(x)\gamma_{\alpha\beta}\overset{ik}{\tau}\psi(x))\\ &\quad\times\Gamma^{\beta\beta'}(\partial,-\partial)\,G(\Box)\,C^{\alpha\alpha'}(\partial/\varkappa)\,(\bar{\psi}(x)\gamma_{\alpha'\beta'}\overset{jl}{\tau}\psi(x))\end{aligned} \tag{2.3}$$

where $G(-p^2)$ denotes the Fourier transform of $F(x)$ and $G(\Box)$ is defined as follows:

$$\begin{aligned}G(\Box)f(x)&=\int_{|k|<\varkappa}A\sum_{n=0}^{\infty}F_n\times(\Box/\varkappa^2)^{n}g(k)\exp(i\,k_\mu x_\mu)\,d^4k\\ &\quad+\int_{|k|>\varkappa}\sum_{n=1}^{\infty}F_n\times(\Box/\varkappa^2)^{-n}g(k)\exp(ik_\mu x_\mu)\,d^4k,\end{aligned} \tag{2.4}$$

where $f(x_1)$ and $g(k)$ are an arbitrary function of x and its Fourier transform respectively, F_n denotes the coefficient divided by A of the power series expansion of $G(-p^2)$ in $-p^2/\varkappa^2$ and A is a normalization constant in order that the relation $F_0=1$ is satisfied.

---

* $F(x)$ is to be a function of x^2 in order to be scalar.

Using (2.3) the transition probability per unit time for the process, where particles of i's and k's kind with the energy-momentum of $p^{(i)}$ and $p^{(k)}$ change into particles of l's and m's kind with that of $p^{(l)}$ and $p^{(m)}$:

$$P_R(p^{(i)}, p^{(j)} \to p^{(l)}, p^{(m)}) = F_R^{\alpha\beta\alpha'\beta'}(p^{(i)}, p^{(j)} \to p^{(l)}, p^{(m)}) A^{-2} \mid G(-\overline{p^{(m)} - p^{(j)}}^2) \mid^2$$
$$\times \left\{ C^{\alpha\alpha'}\left(i \frac{p^{(m)} - p^{(j)}}{\varkappa}\right) C^{*\beta\beta'}\left(i \frac{p^{(m)} - p^{(j)}}{\varkappa}\right) \right\}, \qquad (2.5)$$

where P_R, whose explicit form is given in the next section and called hereafter the reduced direct interaction part of the transition probability, determines the reduced direct interaction type between the Dirac particles, and the last two factors refer to the retardation effects caused by the virtual boson and the one from the commutation relation.

Applying the above consideration to the elementary process (1.1),

$$A^{-1}F(\square) = [1 + \{\partial/\partial\varkappa^2 \log(1 + \square/\varkappa^2)\} / \partial/\partial\varkappa^2 \log \varkappa^2]$$

and the energy spectrum of the created electron is given by

$$W_R(p_0) = \int_{-1}^{+1} W_R^{\alpha\beta\alpha'\beta'}(p_0, \cos\theta) d\cos\theta$$
$$\times [+ \partial/\partial\varkappa^2 \log \{1 - (2(\bar{p}_0 - p_0)(p^0 + p\cos\theta) + \varkappa_0^2)/\varkappa^2\} / \partial/\partial\varkappa^2 \log \varkappa^2]^2$$
$$\times \{\delta_{\alpha\beta} + (q_\alpha - p_\alpha)(q_\beta - p_\beta)/\varkappa^2\} \{\delta_{\alpha'\beta'} + (q_{\alpha'} - p_{\alpha'})(q_{\beta'} - p_{\beta'})/\varkappa^2\} \qquad (2.6)$$

where the last two factors appear only in the vector or pseudovector theory. In the expression of (2.6) p_μ, q_μ and θ denote the energy-momentum four vector of the electron, neutrino and the angle between their momenta respectively and further $\varkappa_0$ denotes the electron mass.

Considering that the approximate value of $[\ ]^2$ in the above expression can be written as

$$[1 + 2\{2(\bar{p}_0 - p_0)(p_0 - p\cos\theta) + \varkappa_0^2\}/\varkappa^2]$$

provided that we take a legitimate assumption that $\varkappa > \varkappa_0$ and the value $(q_\mu - p_\mu)$ is of the order of $\varkappa_0$,* we can easily conclude that the effects of the β-transition in the theory of Yukawa's type are divided into two parts, that is to say, the reduced direct interaction part W_R and the correction part, whose dimension is of the order of $(\varkappa_0/\varkappa)^2$ compared with the first part.

§ 3. The type of the β-coupling

In this section we investigate what kinds of types of the five β-couplings of Fermi's theory arise from the theory of Yukawa's type. This analysis will be equivalent to deter-

* This is due to the fact that, from the conservation law of energy and momentum, q_0 and the maximum value $\bar{p}_0$ of p_0 in the laboratory system are given as follows:

$$q_0 = \varkappa_N(p_0 - \bar{p}_0)/(p_0 + p\cos\theta - \varkappa_N), \quad \bar{p}_0 = (\varkappa_N^2 - \varkappa_p^2 + \varkappa_0^2)/2\varkappa_N.$$

mine the explicit form of the reduced direct interaction part given in the preceding section. For this purpose it is sufficient to eliminate the derivatives in the interaction term of the boson field with the nucleon and lepton field through a certain unitary transformation in the Schrödinger equation.

This will be done following the analogous method as in Case's paper. Then, neglecting the terms higher than the second order in the coupling constant and those independent of the β-transition, the Schrödinger equation becomes as follows,

$$i\hbar c\frac{\delta\psi(\sigma)}{\delta\sigma(x)}=\sum_{i=1}^{2}H_i(x)\psi(\sigma) \tag{3.1}$$

with

$$H_1=\underline{g}\,R\,U^*+\text{comp. conj.},$$

$$H_2=\underline{g'}\,R'\,U^*+\text{comp. conj.}$$

where

$$\underline{g}=\{f-g/\varkappa(\varkappa_P-\epsilon\,\varkappa_N)\},$$

$$\underline{g'}=\{f'+g'/\varkappa\epsilon\,\varkappa_0\},$$

(scalar or pseudoscalar theory)

and

$$\epsilon=\begin{cases}1 & \text{for scalar theory,}\\ -1 & \text{for pseudoscalar theory,}\end{cases}$$

or

$$i\hbar c\frac{\partial\psi(\sigma)}{\partial\sigma(x)}=\sum_{i=1}^{4}H_i(x)\psi(\sigma) \tag{3.2}$$

with

$$H_1=\underline{f}\,M_\mu U_\mu^*+\text{comp. conj.},$$

$$H_2=f'\,M'_\mu U_\mu^*+\text{comp. conj.},$$

$$H_3=2g/\varkappa\{\partial_\mu\bar{\psi}\,\tau_-\gamma_5'\,\psi-\bar{\psi}\,\gamma_5'\,\tau_-\partial_\mu\,\psi\}U_\mu^*+\text{comp. conj.},$$

$$H_4=2g'/\varkappa\{\partial_\mu\,\phi\,\gamma_5'\,\varphi-\phi\,\gamma_5'\,\partial_\mu\varphi\}U_\mu^*+\text{comp. conj.}$$

where

$$\underline{f}=\{f-2g/\varkappa(\varkappa_p+\epsilon\,\varkappa_N)\},$$

$$f'=\{f'-2g'/\varkappa\,\epsilon\,\varkappa_0\},$$

(vector or pseudovector theory)

and

$$\epsilon=\begin{cases}1 & \text{for vector theory,}\\ -1 & \text{for pseudovector theory.}\end{cases}$$

From (3.1), (3.2) and (2.3), we can easily find that the reduced direct β-couplings are: (1) scalar or pseudoscalar coupling for $S(S)$, $PS(PS)$, $S(V)$ and $PS(PV)$,* (2)

* Hereafter, we denote the scalar boson theory with the scalar coupling as $S(S)$ and so on.

vector or pseudovector coupling for $I(I')$ and $PI'(PV)$, (3) vector or pseudovector coupling and a coupling of the new type containing the derivatives on the spinor field for $V(T)$ and $PV(PT)$.

§ 4. The calculation of the energy distribution

Using the completely covariant formalism of Feynman and Dyson and neglecting the electro-magnetic interaction of the created electron with the proton, we can reach the following expression for the energy distribution of the β-coupling of the free neutron in the assumption $\varkappa \gg \varkappa_0$.**

(I) Scalar or pseudoscalar boson theory.

$$W(p_0)=1/(2\pi)^3 \,|\, \underline{g}\,\underline{g'} \,|^2/(\hbar c)^2(\bar{p}_0-p_0)^2 pp_0/K^2\varkappa_N\varkappa_p$$
$$\times\Big[\{\varkappa_N+\epsilon\varkappa_p)^2-\varkappa_0^2\}\Big\{1+1/2pp_0\sum_{n=1}^{\infty}(-1)^n$$
$$\times\binom{-2}{n}\Big(\frac{2(\bar{p}_0-p_0)}{K^2}\Big)^n\frac{(p_0+p)^{n+2}-(p_0-p)^{n+2}}{n+2}\Big\}$$
$$-(\bar{p}_0-p_0)\Big\{2p^0+\frac{2}{3}p^2/p_0+1/pp_0\sum_{n=1}^{\infty}(-1)^n$$
$$\times\binom{-2}{n}\Big(\frac{2(\bar{p}_0-p_0)}{K^2}\Big)^n\frac{(p_0+p)^{n+3}-(p_0-p)^{n+3}}{n+3}\Big\}\Big]. \qquad (4.1)$$

In this expression $\varkappa_p$, $\varkappa_N$, p, p_0 and $\bar{p}_0=(\varkappa_N^2-\varkappa_p^2+\varkappa_0^2)/2\varkappa_N$ denote respectively proton, neutron mass, the momentum, energy and maximum energy of the emitted electron, and further $K^2=\varkappa^2-\varkappa_0^2$. The reduced direct interaction part of the spectrum becomes

$$W_R(p_0)=1/(2\pi)^3 \,|\, \underline{g}\,\underline{g'} \,|^2/(\hbar c)^2(\bar{p}_0-p_0)^2 pp_0/K^2\varkappa_N\varkappa_p$$
$$\times\Big[(\varkappa+\epsilon\varkappa_p)^2-\varkappa_0^2-(\bar{p}_0-p_0)\Big(2p_0+\frac{2}{3}p^2/p_0\Big)\Big]. \qquad (4.2)$$

From the above expression we find that the first term in (4.2) becomes the main term for the scalar theory, but this term becomes comparable with the rest terms for the pseudoscalar theory.

(II) Vector or pseudovector boson theory.

a). $V(V)$ or $PV(PV')$.

$$W(p_0)=\frac{4}{(2\pi)^3}\,|ff'|^2/(\hbar c)^2(p_0-p_0)^2/K^4\varkappa_N$$

** Here the laboratory frame of reference is taken in which the incident neutron is at rest.

$$\times\Big[-(1+\varkappa_0^2/\varkappa^2)\Big\{2(\epsilon\varkappa_p-\varkappa_N+\bar{p}_0)pp_0+\frac{1}{3}\Big(1/\varkappa_N+\frac{4(\epsilon\varkappa_p-\varkappa_N+\bar{p}_0)}{K^2}\Big)$$
$$\times(\bar{p}_0-p_0)p(3p_0^2+p^2)+4(\bar{p}_0-p_0)p_0p^2/K^2\varkappa_N(p^2+p_0^2)\Big\}$$
$$-\Big\{(\varkappa_N-\bar{p}_0)+\bar{p}_0(\varkappa_p^2-\varkappa_N^2+\bar{p}_0\varkappa_N)/\varkappa^2\Big(2+\frac{2\varkappa_N(\varkappa_N-\bar{p}_0)-(\varkappa_N^2+\varkappa_p^2)}{\varkappa^2}\Big)$$
$$+(\bar{p}_0-p_0)-\frac{\varkappa_p^2-\varkappa_N^2}{\varkappa^2}\bar{p}_0+p_0\bar{p}_0\varkappa_N/\varkappa^2\Big\}$$
$$\times\{pp_0+4(\bar{p}_0-p_0)p/3K^2(3p_0^2+p^2)\}$$
$$+\{2\varkappa_Np_0+(\varkappa_0^2/\varkappa^2)(\varkappa_p^2-\varkappa_N^2)\}p(1+4(\bar{p}_0-p_0)p_0/K^2)\}$$
$$+2\bar{p}_0/\varkappa^2\{p(p_0^2+p^2/3)+8(\bar{p}_0-p_0)pp^0(p^2+p_0^2)/K^2)\}\Big], \qquad (4.4)$$

where we have neglected $(\varkappa_0/\varkappa)^4$ comparing with 1.
The reduced direct interaction part of the spectrum is given by

$$W_R(p_0)=4/(2\pi)^3\,|ff'\,|^2/(\hbar c)^2\varkappa^4(\bar{p}_0-p_0)^2pp_0 \qquad \text{for } V(V),\ (4.4)$$

$$W_R(p_0)=20/(2\pi)^3\,|ff'|^2/(\hbar c)^2\varkappa^4(\bar{p}_0-p_0)^2pp_0 \qquad \text{for } PV(PV).\ (4\text{-}5)$$

β). $V(T)$ or $PV(PT)$.

It is remarkable that the main terms are cancelled for vector theory with tensor coupling. The reduced direct interaction part in unit $\varkappa_0=1$ becomes as follows,

$$W_R(p^0)\cong(2^5\cdot 2.6/(2\pi)^3K^2)\varkappa_N|\,gg'|^2/(\hbar c)^2(\bar{p}_0-p_0)^2$$
$$\times[p(3p_0^2+p^2)-6(\bar{p}_0-2p_0)pp_0-(\bar{p}_0-p_0)p(3p_0^2+p^2)(\bar{p}_0-2p_0)]$$
$$\text{for } V(T),\ (4.7)$$

$$W_R(p^0)\cong(2^8/(2\pi)^3K^2)\,|gg'|^2/(\hbar c)^2(\bar{p}_0-p_0)^2$$
$$\times[1.85pp_0+1.24(\bar{p}_0-p_0)p(3p_0^2+p^2)-0.50(\bar{p}_0-p_0)p-(\bar{p}_0-p_0)^2pp_0$$
$$-0.25(\bar{p}_0-2p_0)^2pp_0-1.27(\bar{p}_0-2p_0)pp_0]$$
$$\text{for } PV(PT).\ (4.7)$$

§ 5. Alternative theory of β-transition

As is well known, the renormalizable theory is restricted only to $S(S)$, $PS(PS)$ and neutral $V(V)$ theory. But as shown in the preceding section $S(S)$, $PS(PS)$ theories give only the scalar and pseudoscalar β-coupling, but not tensor-coupling.

Therefore, it may be desirable to look for the another theory, which is at first renormalizable and further able to give the various Fermi's β-couplings.[2] Such a possibility may be found in the early works of Y. Tanikawa,[3] where the boson field was considered

as the charged scalar or pseudoscalar, and the corresponding Feynman diagram of the β-transition is shown in Fig. 2.* However, in this case Fermi's five types appear simultaneously in the same weight apart from their signs. In order to eliminate this defect, it is sufficient to introduce (I) : the neutral scalar or pseudoscalar boson, or (II) : the neutral vector or pseudovector boson, besides the charged scalar or pseudoscalar boson and to choose the both coupling constants adequately. Then, we can select the various combinations of the Fermi's β-couplings. Unfortunately, however, this theory, due to the neutral boson, necessarily leads to such a process as denoted in Fig. 2 with the same order as the process of the β-transition and would not explain the stability of the nucleus.**

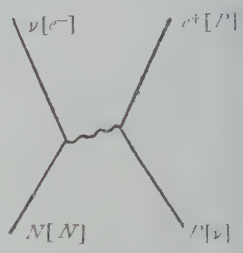

Fig. 2.

However, this difficulty is also saved by the somewhat unfamiliar assumption that the neutral boson field is described by the complex field operator and has a vanishing charge.***

In this case the interaction Hamiltonian density is as follows,

$$H(x)=\sum_{i=1} H_i(x) \tag{5.1}$$

with

$$\begin{aligned} H_1&=f_1\,\overline{\varphi}\,\gamma_5'\,\psi_P\,U+\text{ comp. comj.},\\ H_2&=f_2\,\bar{\phi}\,\gamma_5'\,\psi_N\,U+\text{ comp. conj.},\\ H_3&=f_3\,\overline{\psi}'\,\gamma_5'\,\psi_P U_0+\text{ comp. conj.},\\ H_4&=f_4\,\overline{\psi}'\,\gamma_5'\,\varphi\,U_0+\text{ comp. conj.}, \end{aligned} \qquad \text{for (I), (5.2)}$$

and

$$\begin{aligned} H_1&=f_1\,\overline{\varphi}\,\gamma_5'\,\psi_P\,U+\text{ comp. conj.},\\ H_2&=f_2\,\bar{\phi}\,\gamma_5'\,\psi_N\,U+\text{ comp. conj.},\\ H_3'&=f_3'\bar{\phi}\,\gamma_5\,\gamma_\mu\,\psi_P\,U_0+\text{ comp. conj.},\\ H_4&=f_4'\overline{\psi}'\gamma_5'\gamma_\mu\,\varphi\,U_0^{\mu}+\text{ comp. conj.} \end{aligned} \qquad \text{for (II), (5.3)}$$

where U, U^{μ} and U_0, U_0^{μ} are the complex field operators which describe the charged and neutral boson fields, and suffix μ denotes the four components of the vector or pseudovector.

Using the Fierz's identity of the γ-matrix[4] and neglecting those terms which are independent of the β-transition, the second order S-matrix with the case (I),

$$\langle U(\infty)\rangle_{\text{vac}}=\frac{1}{8\hbar c}\int\!\!\int_{-\infty}^{\infty}dx_1dx_2\sum_{k=1}^{5}\left[\rho_1(k)f_1f_2(\bar{\phi}(x_2)A^k\varphi(x_1))(\overline{\psi}_P(x_1)A^k\psi_N(x_2))\triangle_F(x_1-x_2)\right.$$

$$\left.+\rho_2(k)\zeta(k)f_3f_4(\bar{\phi}(x_1)A^k\varphi(x_2))(\overline{\psi}_P(x_1)A^k\psi_N(x_2))\triangle_F^0(x_1-x_2)\right], \tag{5.4}$$

* Here we mean the process expressed by letters in brackets in Fig. 2.

** We are indebted to Mr. E. Yamada for indication of this point.

*** See note added in proof.

where $A^k(k=1, 2, 3, 4, 5)$ describe $(1, \gamma_\mu, \gamma^{[\mu\nu]}, \gamma^{[\lambda\mu\nu]}, \gamma_5)$ according to the Fierz's notation, $\rho_1(k)$, $\rho_2(k)$, and $\zeta(k)$ are defined by

$$\rho_1(k)A^k=\rho_2(k)A^k=\gamma_5' A^k\gamma_5'=\begin{cases}A^k & \text{for } \gamma_5'=1 \text{ (scalar theory)},\\ (-1)^{k+1}A^k & \text{for } \gamma_5'=\gamma_5 \text{ (pseudoscalar theory)},\end{cases} \tag{5.5}$$

and

$$\zeta(k)A_k=C(A_k)^TC^{-1}*,$$

$$\zeta(k)=\begin{cases}1 & \text{for } k=1, 4, 5,\\ -1 & \text{for } k=2, 3,\end{cases} \tag{5.6}$$

and further

$$\langle P(U(x), U^*(x'))\rangle_{\text{vac}}=\frac{\hbar c}{2}\triangle_F(x-x'),$$

$$\langle P(U_0(x), U_0^*(x'))\rangle_{\text{vac}}=\frac{\hbar c}{2}\triangle^0{}_F(x-x'). \tag{5.7}$$

Analogous expression of $\langle U(\infty)\rangle_{\text{vac}}$ for the case (II) is obtained in the approximation $(x_0/x)^2\cong 0$, when only $\rho_2(k)$ and $\triangle^0{}_F$, defined by (5.5) and (5.7) respectively are replaced by $\rho_2'(k)$ and $\triangle^{0\prime}{}_F$;

$$\rho_2'(k)A^k=\gamma_5'\gamma_\mu A^k\gamma_5'\gamma_\mu=\begin{cases}4\ A^1\\ -2\ A^2\\ 0\\ 2\ A^4\\ -4\ A^5\end{cases}\quad\text{for}\quad k=\begin{cases}1\\2\\3\\4\\5\end{cases}\text{ in the case } \gamma_5'=1 \text{ (vector theory)}, \tag{5.8}$$

$$=\begin{cases}-4\ A^1\\ -2\ A^2\\ 0\\ 2\ A^4\\ 4\ A^5\end{cases}\quad\text{for}\quad k=\begin{cases}1\\2\\3\\4\\5\end{cases}\text{ in the case } \gamma_5'=\gamma_5 \text{ (pseudovector theory)},$$

and

$$\langle P(U_0{}^\mu(x), U_0{}^{\nu*}(x'))\rangle_{\text{vac}}=\frac{\hbar c}{2}\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\varkappa^2}\right)\triangle^{0\prime}{}_F(x-x'). \tag{5.9}$$

From the expression (5.4), we can obtain, for the five coefficients of the spinor products referring to $k=1, 2, 3, 4, 5$, the values expressed in the following Tab. I. and Tab. II., where we have neglected the difference of the space-time coordinates of spinor products.

---

* C is the charge conjugation operator defined by J. Schwinger

	Type of boson		Coefficients of spinor products				
	charged	neutral	S(k=1)	$V(k=2)$	$T(k=3)$	$Pv(k=4)$	$Ps(k=5)$
(A)	S	S	$G+G_0$	$G-G_0$	$G-G_0$	$G+G_0$	$G+G_0$
(B)	Ps	Ps	$G+G_0$	$-G+G_0$	$G-G_0$	$-G-G_0$	$G+G_0$
(C)	Ps	S	$G+G_0$	$-G-G_0$	$G-G_0$	$-G+G_0$	$G+G_0$
(D)	S	Ps	$G+G_0$	$G+G_0$	$G-G_0$	$G-G_0$	$G+G_0$

Tab. I. The coefficients of spinor products for the case (I), where $G=f_1f_2\triangle_F$, $G_0=f_3f_4\triangle^0{}_F$.

	Type of boson		Coefficients of spinor products				
	charged	neutral	$S(k=1)$	$V(k=2)$	$T(k=3)$	$Pv(k=4)$	$Ps(k=5)$
(A)′	S	V	$G+4G'_0$	$G+2G'_0$	G	$G+2G_0'$	$G-4G_0'$
(B)′	Ps	Pv	$G-4G'_0$	$-G+2G'_0$	G	$-G+2G_0'$	$G+4G_0'$
(C)′	Ps	V	$G-4G'_0$	$-G+2G'_0$	G	$-G+2G_0'$	$G-4G_0'$
(D)′	S	Pv	$G-4G'_0$	$G+2G^{0\prime}$	G	$G+2G_0'$	$G+4G_0'$

Tab. II. The coefficients of spinor products for the case (II), where $G=f_1f_2\triangle_F$, $G_0'=f_3'f_4'\triangle^{0\prime}{}_F$.

Considering the above Tab. I. and Tab. II., we can expect that; (i) it is generally possible to obtain the well known five β-couplings in Fermi's theory but with the restricted weights of their five coefficients and (ii) if we assume the relation between G and G_0 or G and G_0', we can obtain the various combinations of the β-couplings from among five types according as we adopt the different types of boson fields which are distinguished by (A), (B), (C), (D) and (A)′, (B)′, (C)′, (D)′, in the Tab. I and Tab. II. Namely, for example, we can select the several combinations under setting a simple relation between G and G_0 or G and G_0':

$$\begin{aligned}
&\text{i).}\quad (\text{S, PV, PS}) \quad \text{for } G=-G_0 \text{ in the cases: (A), (B).}\\
&\text{ii).}\quad (\text{V, T}) \quad \text{,, } G=G_0 \quad \text{,, : (A), (B).}\\
&\text{iii).}\quad (\text{S, V, PS}) \quad \text{,, } G=-G_0 \quad \text{,, : (C), (D).}\\
&\text{iv).}\quad (\text{PV, T}) \quad \text{,, } G=G_0 \quad \text{,, : (C), (D).} \qquad (5.10)\\
&\text{v).}\quad (\text{S, PS, T}) \quad \text{,, } \begin{cases} G=-2G_0' \text{ ,, : (A)′, (D)′,}\\ G=2G_0' \text{ ,, : (B)′, (C)′.}\end{cases}
\end{aligned}$$

§ 6. Summary and concluding remarks

In § 2 and 3, we have obtained simple results for the β-transition of Yukawa's type

that the effects of the β-transition are divided into two parts, that is, the reduced direct interaction part and the correction part, that the dimension of the latter is of the order of $(\varkappa_0/\varkappa)^2$ comparing with first part, and that the reduced direct β-coupling consists of (i) scalar or pseudoscalar coupling for S or PS boson theory, (ii) vector or pseudovector coupling for V(V) or PV(PV) theory and (iii) vector or pseudovector coupling and the coupling of the new type containing the derivatives on the spinor fields for V(T) or PV(PT) theory.

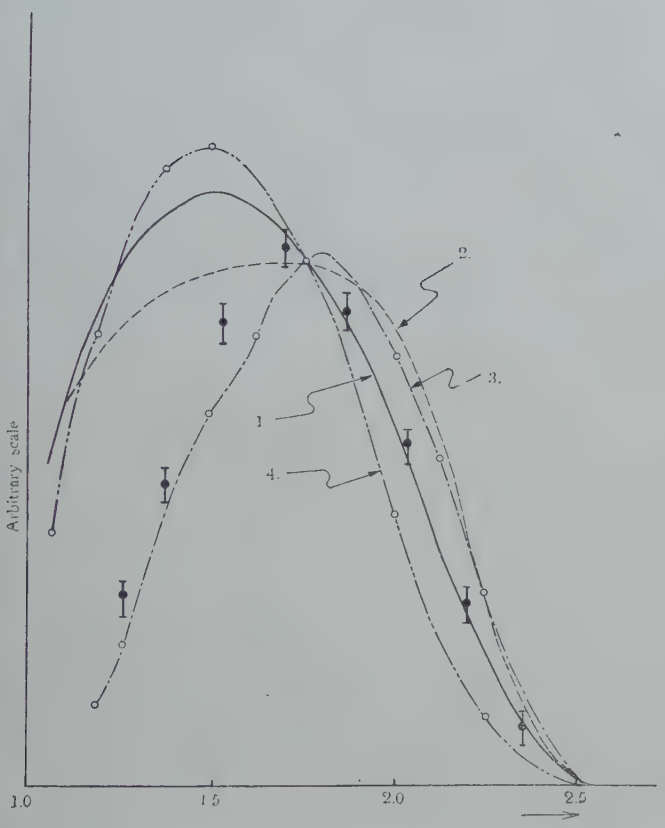

Fig. 3. Energy distribution of the electron in unit of $\varkappa_0=1$. All curves are written to coincide on the portion $p_0/\varkappa_0=1.75$, numbers affixed to curves distinguish the following cases:

1. ..... S(S) equ. (4.2), S(V) „ „, V(V) „ (4.4), PV(PV) „ (4.5),
2. ..... PS(PS) „ (4.2), PS(PV) „ „
3. ..... V(T) equ. (4.6),
4. ..... PV(PT) equ. (4.7),

and further $\overset{\top}{\bullet}_{\perp}$ indicates the experimental results of Robson. For V(T) and PV(PT) we have made numerical calculations only at the points with circle.

In § 5, we have an alternative theory of the β-transition. It is characteristic for this theory that this interaction (except the spin 1 boson theory) belongs to the interaction of the first kind* and gives the five β-couplings in the Fermi's theory and that if we choose the magnitudes of the coupling constants adequately, then we can select the various combinations of the Fermi's β-couplings, as given by (5.10).

Hitherto, the importance of the linear combination of five β-couplings has been discussed by many authors[4), 6)]. It is noticeable that the recent experiments of the nuclear beta-decay process also suggest the existence of such linear combinations.

Recently, Petschek and Marshak[7)], who have analysed the RaE spectrum, have pointed out that only the linear combination of tensor and pseudoscalar interaction can be regarded as giving a satisfactory fit of the RaE spectrum and explain its forbidden shape. However, we can not draw

any decisive conclusion about the magnitude of the ratio of the admixture of two interactions before we have more accurate information about the nuclear matrix element of RaE. On the other hand, due to their thoughts, " the $0 \rightarrow 0$ yes transition which is indicated for RaE cannot decide whether the S or V interaction must be also added to the (T, PS) combination ".

According to the above consideration, (S, T, PS) given in (5.10) seems to be consistent with the results of the analysis of the RaE spectrum given by above authors. Further, in our theory the ratio of the linear combination of scalar, tensor and pseudoscalar interaction is uniquely determined corresponding to the different types of boson fields as follows :

$$(-1 : 1 : 3) \text{ for the case } (A)', (B)',$$

$$(3 : 1 : -1) \text{ „ „ „ } (C)', (D)'.$$

The authors should like to express their sincere thanks to Prof. S. Sakata for indicating the idea of this work and to Dr. H. Umezawa for an extremely helpful advice, continual guidance and encouragement, without which this work would not have been possible. They are also deeply indebted to Mr. E. Yamada and to the members of the research group of the theory of elementary particle in Tokyo for their valuable advice and helpful discussion.

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Note added in proof : In § 5, we have introduced the neutral boson field described by the complex variable. In this case we should notice the fact that the neutral vector theory is not renormalizable since, when using the Glauber's formalism, we cannot eliminate the derivative term of the boson field. Thus, (A)′, (B)′, (C)′ and (D)′ in § 5 can not be obtained from the theory of the first kind interaction but from the theory of the second kind one. While, irrespective of the above circumstances, we have considered the problem from the viewpoint of the general treatment.

* See note added in proof

Letters to the Editor

Difficulty of Divergence of the Perturbation Method in the Quantum Theory of Field

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January 31, 1953

Let us consider a system of a neutral scalar meson field $\varphi(x)$ with a non-linear and non-local interaction. The Lagrangian density of this system is given by

$$L(x)=-\frac{1}{2}[\varphi_\mu(x)\varphi_\mu(x)+m^2\varphi^2(x)]$$
$$+\frac{g}{3}\int U(x,x',x'')\varphi(x)\varphi(x')\varphi(x'')dx'dx'', \quad (1)$$

and the S-matrix can be written as

$$S=\sum_{n=0}^{\infty}\frac{(+i)^n}{n!\,3^n}\int U(x_1y_1z_1)\cdots U(x_ny_nz_n)g^n\times$$
$$T[\varphi_0(x_1)\varphi_0(y_1)\varphi_0(z_1),\cdots\varphi_0(x_n)\varphi_0(y_n)\varphi_0(z_n)]\times$$
$$dx_1\cdots dx_ndy_1\cdots dy_ndz_1\cdots dz_n, \quad (2)$$

where $\varphi_0(x)$ is a wave function of the free system. In deriving (2), some assumptions concerning the property of the source function U and those of the so-called adiabatic procedure have been used. Since each term in the right hand side of (2) is finite by virtue of U, we shall investigate the convergence of the series (2). To do this it is sufficient to make a study of the convergence of the Green-function $G(x,x')$ of the one-meson problem. It is seen, however, that the convergence of G shares the fate of that of the mass operator M.

The mass operator M has the expression

$$M(x_1-y_{2m+2})=ig^2\sum_{m=0}^{\infty}g^{2m}C_m\times$$
$$\sum_{\text{graph(prop.conn)}}'\int\prod_{j=1}^{2m+2}U(x_jy_jz_j)\prod_{j=1}^{3m+2}G_0(\xi_j-\eta_j)\times$$
$$dx_2\cdots dx_{2m+2}dy_1dy_2\cdots dy_{2m+1}dz_1\cdots dz_{2m+2}, \quad (3)$$

where

$$G_0(x)=i\Delta^F(x)=\frac{i}{2}\{\Delta^{(1)}(x)+i\varepsilon(x)\Delta(x)\},$$

and

$$C_m=\frac{(-i)^m}{(2m)!\,3^{2m}}.$$

The variables $\xi_1\cdots\xi_{3m+2}$, $\eta_1\cdots\eta_{3m+2}$ are a set of variables derived from the set $(x_2\cdots x_{2m+2}, y_1\cdots y_{2m+1}, z_1\cdots z_{2m+2})$ by some permutation. In permuting the set $(x_2\cdots y_1\cdots z_1\cdots z_{2m+2})$, however, we must determine $(\xi_1\cdots, \eta_1\cdots)$ in such a way that each term of (3) corresponds to a proper self-energy diagram. The notation $\sum_{\text{graph}}$ in (3) means that we must sum up all the terms corresponding to every possible choice of the set of variables $(\xi_1\cdots, \eta_1\cdots)$.

The eq. (3) can be written in the momentum space as

$$M(k)=ig^2\sum_{m=0}^{\infty}g^{2m}C_mA_m(k), \quad (4)$$

where

$$A_m(k)=\sum_{\text{graph}}i^{m+1}\int_{-\infty}^{\infty}dq_1\cdots dq_{m+1}\times$$
$$\prod_{j=1}^{2m+2}U(k_jl_j)\prod_{i=1}^{3m+2}G_0(p_i), \quad (5)$$

and $U(k,l)$, $G_0(p)$ are the Fourier components of $U(x,x',x'')$ and $G_0(x)$ respectively. The notation $\sum_{\text{graph}}$ has the same meaning as that of (3). The variables $p_1\cdots p_{3m+2}$ in (5) are the energy-momentum of each internal line, k_j, l_j are those being incident on the j-th U, and $q_1\cdots q_{m+1}$ are the linearly independent variables chosen out of $p_1\cdots p_{3m+2}$.

Suppose now that $k^2+m^2>0$, then we can transform the 4-th component of each q_j in (5) into the imaginary variables without any difficulty concerning the displaced poles. Let us assume that U is a definite function. Then we have

$$|A_m(k)|<\sum_{\text{graph(prop.irred)}}\int_{-Q}^{Q}dq_1\cdots dq_{m+1}\prod_j|U(j)|\times$$
$$\prod_i G_0(p_i)<(G_0)^{3m+2}(|U|)^{2m+2}V^{m+1}N_m, \quad (6)$$

where, the domain of integration is restricted within a four dimensional sphere with a large but finite radius Q in a four-dimensional Euclidian space, and $\sum_{\text{graph}}$ means that we must only take those summands which correspond to proper self-energy diagrams not including self-energy parts or any branches inside them. G_0, $|U|$ are the minimums of G_0 and $|U|$ in this do-

main of integration; V is the volume of the sphere stated above; N_m is a number of diagrams just mentioned above.

Now the necessary and sufficient condition for the convergence of (4) is

$$\lim_{m\to\infty} C_m a^m N_m = 0, \tag{7}$$

where $a = G_0^3 |U|^2 V g^2$.

Unfortunately, (7) can not hold for any a, that is to say, the series (4) has no non-vanishing radius of convergence. This can be easily proved by analysing the Eqs.

$$\Gamma(x,y,z) = U(x,y,z) - \frac{1}{2g}\frac{\delta M(x,y)}{\delta\langle\varphi(z)\rangle}$$

and

$$M(x,y) = 2ig^2 \int U(x,x',x'')G(x'y')\times G(x''y'')\Gamma(y'y''y)dx'dx''dy'dy''$$

without using the explicit expression of N_m. Since the result we obtained essentially depends on the character of Boson fields, it is quite doubtful to consider that similar result will be obtained in case of Fermion fields.

The detailed explanation of this letter will be published in this journal in the near future.

A Note on Many Particle Problems

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February 1, 1953

In quantum theory of many particle systems, the system wave functions are represented by determinant or quasideterminant, which is replaced by plus signs what is to be assigned by plus or minus signs when one wishs to construct ordinary determinant, so as to make the system wave functions always antisymmetric or symmetric in particle coordinates according as the system obeys Fermi-statistics or Bose-statistics.

In field theory, this determinant or quasideterminant is automatically derived, as Dyson[1] showed in the case of two electron system. For example, in the case of Dirac electron, Lagrangian is

$$L = -\sum_{i=1}^{n} \hbar c\bar{\psi}\left(\gamma_{\nu i}\frac{\partial}{\partial X_{\nu i}} + \mu_i\right)\psi. \tag{1}$$

Field equations are

$$\sum_i \left(\gamma_{\nu i}\frac{\partial}{\partial X_{\nu i}} + \mu_i\right)\psi = 0, \tag{2}$$

$$\sum_i \left(-\frac{\partial\psi}{\partial X_{\nu i}}\gamma_{\nu i} - \mu_i\psi\right) = 0. \tag{3}$$

The general solution of (2) is

$$\psi(r,t) = \sum_\lambda b_\lambda \psi_\lambda(r) e^{-\frac{i}{\hbar}E_\lambda t}, \tag{4}$$

where b_λ are operators independent of r and t.

Covariant commutation relations are

$$[b_\lambda, b_\mu]_+ = 0, \quad [b_\lambda^*, b_\mu^*]_+ = 0, \quad [b_\lambda, b_\mu^*] = \delta_{\lambda\mu}, \tag{5}$$

b_λ and b_λ^* stand respectively for absorption and creation operators.

Vacuum state Ψ_0 is given by

$$b_\lambda \Psi_0 = 0. \tag{6}$$

Now consider a system with n-electrons in state $\psi_1\cdots\psi_n$, all other systems being empty. Then the state of the system is given by

$$\Psi = b_1^* b_2^* b_3^* \cdots b_n^* \Psi_0. \tag{7}$$

A n-particles interaction operator is assumed to be given by

$$V = \frac{1}{n!}\int\cdots\int d_3r_1 d_3r_2 \cdots d_3r_n (\psi^*(r_1)\psi(r_1))\cdots (\psi^*(r_n)\psi(r_n))W \tag{8}$$

where W is any potential between n particles.

When these n-states make transition to a state

$$\Psi = b_{n+1}^* b_{n+2}^* b_{n+3}^* \cdots b_{2n}^* \Psi_0, \tag{9}$$

matrix element of V becomes

$$M = (\Psi_0^* b_{2n} b_{2n-1}\cdots b_{n+1} V b_1^* b_2^* \cdots b_n^* \Psi_0). \tag{10}$$

Substitute (4) into (8), then contribution to M come from the $n!^2$ terms in V proportional to $b_1 b_2 \cdots b_n b_{n+1}^* b_{n+2}^* \cdots b_{2n}^*$. Using (5) we find the value of

$$(\Psi_f^* b_1 b_2 \cdots b_n b_{n+1}^* b_{n+2}^* \cdots b_{2n}^* \Psi_i) \tag{11}$$

is

1) $n = 4m, +1,$
2) $n = 4m+1, -1,$
3) $n = 4m+2, -1,$
4) $n = 4m+3, +1,$ (12)

m being any integer.

Parities of other $(n!^2-1)$ terms are given by

$$\mathfrak{p}+\mathfrak{p}'+\mathfrak{p}'', \tag{13}$$

where $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{p}''$ stand respectively for the parity of (11) given by the rules of (12), the parity of any permutation $(b_1 b_2 \cdots b_n)$, and that of any permutation $(b_{n+1}{}^* b_{n+2}{}^* \cdots b_{2n}{}^*)$.

Hence we can derive the determinant form

$$M=\frac{1}{n!}\sum_{PP'}(-1)^{P'+P}\prod_{l,m}P_{lm}P'_{lm'}\times$$

$$\int\psi_{lm}{}^*\psi_{lm'}Wdx \tag{14}$$

where P and P' stand respectively for permutation operator redistributing the m's among l's and m''s among l's.

In the case of photons, one can show in the same way that,

$$M=\frac{1}{n!}\sum_{PP'}(+1)^{P'+P}\prod_{l,m}P_{lm}P'_{lm'}\times$$

$$\int\psi_{lm}{}^*\psi_{lm'}Wdx\,. \tag{15}$$

Combining (14) with (15),

$$M=\frac{1}{n!}\sum_{PP'}(\pm 1)^{P'+P}\prod_{l,m}P_{lm}P'_{lm'}\times$$

$$\int\psi_{lm}{}^*\psi_{lm'}Wdx\,, \tag{16}$$

$+$: for Bosons,

$-$: for Fermions.

So far the writer described the field theory gives determinent or quasideterminent for n-electrons and n-photons, only in the same way as Dyson did for two electrons.

Now consider the calculation of (16), with the object of which this letter has originally been written. To do this, we may devide the above expression in the following way as Kirkwood,[2] the sum over P and P' may be arranged so that the first member contains all terms in which P and P' are the same permutation, the second those in which P and P' differ only with respect to a single pair of particles and so on.

Clearly this procedure corresponds to divide $n!$ terms to families of cycle form, though not 'classify' the symmetric group in the strict sense, that is, n unary cycles, one binary cycle plus $(n-2)$ unary cycles, one ternay cycle plus $(n-3)$ unary cycles, $(n-4)$ unary cycles plus other all possible cycles, i.e. one quartenary cycle or two binary cycles inclusive, and so on. To effect this design, the following considerations can conveniently be used, that the number of terms belonging to the family of $(n-s)$ unary cycles plus other cycles total letters being s, which is designated as $(n-s)$ family, is given by

$$\sum_{i=0}^{n}(-1)^{s-i}{}_nC_{s\,s}P_i, \tag{17}$$

for $s=1$, (17) being always zero.

Out of these terms, the number of those terms which have even parity is given by

$$\frac{1}{2}\left\{\sum_{i=0}^{s}(-1)^{s-1}{}_sP_i+(-1)^{s-i_s}+(-1)^s\right\}{}_nC_s\,, \tag{18}$$

and the number of those terms which have the odd parity is given by

$$\frac{1}{2}\left\{\sum_{i=0}^{s}(-1)^{s-i}{}_sP_i-(-1)^{s-1}{}_s-(-1)^s\right\}{}_nC_s\,. \tag{19}$$

In fact, when (17) are added from $s=0$ to $s=n$, one obtains

$$\sum_{s=0}^{n}\sum_{i=0}^{s}(-1)^{s-i}{}_nC_{s\,s}P_i=n!\,. \tag{20}$$

Of course, these mathematical formulae, which were derived recently by the writer, are proved by the use of mathematical induction.[3]

The above method may be more convenient than conventional way using the idea of class of symmetrical group. For when one follows the latter one, one ought first to find the all possible solutions of the equation

$$C_1+2C_2+3C_3+\cdots+nC_n=n, \tag{21}$$

and after that one must calculate the order of the class using the formula:

$$n_i=n!/C_1!\,2^{C_2}C_2!\,3^{C_3}C_3!\cdots n^{C_n}C_n! \tag{22}$$

and consider the parity of each class.[4] This method is thus very troublesom when n is great, and over-necessary for the present purpose.

The above stated is directly applicable, for example, to the calculation of the partition function of almost classical assemblies,[2,5] or many body problems exerted with many body forces in general, on which the writer shall report on some future, and may facilitate the further attack of problems already done to the half.[2,6]

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On the Relativistic Two-body Problem

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February 21, 1953

In the previous paper,[1] we have derived the formula which connects the wave functions in the configuration space at $\sigma=+\infty$ with those at $\sigma=-\infty$:

$$u(\boldsymbol{k}_i, \boldsymbol{p}_j, \boldsymbol{q}_k) = \int (\boldsymbol{k}_i, \boldsymbol{p}_j, \boldsymbol{q}_k | S | \boldsymbol{k}_i', \boldsymbol{p}_j', \boldsymbol{q}_k') \times u^0(\boldsymbol{k}_i', \boldsymbol{p}_j', \boldsymbol{q}_k') \Pi d\boldsymbol{k}_i' \Pi d\boldsymbol{p}_j' \Pi d\boldsymbol{q}_k'. \quad (1)$$

On the other hand it turns out that the integral equation for a bound state is given by

$$u(\boldsymbol{p}_1, \boldsymbol{p}_2) = \int (\boldsymbol{p}_1, \boldsymbol{p}_2 | R | \boldsymbol{p}_1', \boldsymbol{p}_2') \times u(\boldsymbol{p}_1', \boldsymbol{p}_2') d\boldsymbol{p}_1' d\boldsymbol{p}_2', \quad (2)$$

where $R=S-1$. The other elements $(\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots | R | \boldsymbol{p}_1', \boldsymbol{p}_2')$ should vanish in order that the system becomes stationary. This will give a further restriction for the energy eigenvalues.

It is a merit in the eq. (2) to show clearly the relation between the bound state problem and the S-matrix and therefore the renormalization procedure. In this respect the author is indebted to Mr. C. Hayashi for his kind instruction.

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On Gauge Invariance in Electrodynamics and the Self-energy Problems of the Photon

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February 21, 1953

Although the gauge invariance in electrodynamics and the self-energy problem of the photon have been discussed by many authors, the situation has not yet been fully cleared. For example, satisfactory answer has not been given to questions such as what is the consequence of the gauge invariance, why the interaction of the electromagnetic field must be introduced in a gauge invariant way, or how this requirement is related to the self-energy problem of the photon. It is the purpose of this short note to try to throw some light on these problems.

The first thing that we want to point out is that the gauge transformation is a transformation which, conserving the transverse waves, alters the homogeneous parts of the longitudinal and scalar components of the electromagnetic field. Therefore, the fact that the whole scheme of electrodynamics is invariant under gauge transformations is another expression of the requirement that all physically significant results must not depend upon these parts of the electromagnetic field. The situation will be made clearer by secondly pointing out that the gauge transformation can be regarded as a sort of canonical transformation, and so the homogeneous parts of the longitudinal and scalar components lie within the arbitrariness of describing the field which is allowed by the canonical transformation. This is even the case when the electromagnetic field is treated as external, where a suitable canonical transformation applied to the particle system just reproduces the gauge transformation.

Thus, although it is generally believed that the self-energy of the photon must vanish by the requirement of gauge invariance, this conclusion does not seem compelling to us. We think that there are insufficient points in the current discussions of this problem both in the analysis of the induced current and of the theory of relativity. Indeed we can show the existence of a gauge invariant and still non-vanishing induced current by a straight forward calculation. Of course it is true that we can not avoid mathematical ambiguities inherent in present quantum-

electrodynamics, but these ambiguities can be reduced by attributing small mass to the photon, which enables us to use its rest system in the actual calculation. The continual approach of such field to the electromagnetic field has already been shown by Glauber.[1] The induced current thus obtained is

$$\delta j_\mu = (\text{quadratically and logarithmically divergent coefficient}) \times \left(A_\mu - \frac{1}{\varkappa^2} \partial_\mu \partial_\nu A_\nu \right),$$

where $\varkappa$ is the rest mass attributed to the photon. This result is obviously invariant under a generalized gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$(\Box - \varkappa^2)\Lambda = 0,$$

and satisfies the conservation law of charge on account of

$$(\Box - \varkappa^2) A_\mu = 0.$$

The important conclusion that can be drawn from this form of δj_μ is that the additional term to the interaction Hamiltonian caused by the interaction between δj_μ and A_μ can never be interpreted as giving an alteration to the rest mass of the quanta. In order that such an interpretation is possible, δj_μ must be of the form (const.) $\times A_\mu$, which is in violent contradiction to the requirement of the gauge invariance of δj_μ.

Thus, the condition of gauge invariance imposes a severe restriction to the physical interpretation of the interaction energy, and we are faced to a serious difficulty in confronting with energy which is hard to interpret. The only way out of this difficulty seems to renormalize the light velocity as proposed in our previous paper. In fact, if we reformulate Lorentz transformation following this idea, all difficulties concerning with the self-energy problem of the photon can be solved very naturally. We want to consider this fact as a proof of the necessity of renormalizing the light velocity which might seem rather strange at first sight.

Details will be published in a later issue of this journal.

1) R. J. Glauber, Prog. Theor. Phys. **9** (1953), No. 3, in press.
2) O. Hara and H. Shimazu, Prog. Theor. Phys. **8** (1952), 265.

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Remarks on the Formulation of Quantum Mechanics with Classical Pictures and on Relations between Linear Scalar Fields and Hydrodynamical Fields

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The formulation of quantum mechanics in terms of the picture of trajectory ensemble or of the hydrodynamical picture, developed in a previous article, is further supplemented for some points. Individual trajectories for some typical non-stationary state are explicitly obtained, illustrating that they generally show complicated fluctuations due to the action of quantum potential. Next, emphasis is placed on the fact that the Schrödinger field admits a mechanical model. Also the Klein-Gordon field is represented with a quasi-irrotational flow under some additional scalar field. This representation of Klein-Gordon field suggests its possible generalization to include the vortical motion. Such treatments of interacting Klein-Gordon and electromagnetic fields involve the formalism of Dirac's recent classical electrodynamics as a limiting case. Some remarks are given on this Dirac's theory. Finally, the picture of trajectory ensemble is generalized so as to represent wave equations with sources. In an appendix criticism is given on the attempt to interpret quantum mechanics in terms of Markoff process.

Introduction and summary

Quantum mechanics, since it was first found out by Heisenberg through the 'Umdeutung' of kinematical and mechanical relations, consists of a kinematics essentially different from that of the classical theory, admitting no classical pictures as real thing. Nevertheless, it is possible to let some classical pictures formally correspond to the Schrödinger equation of quantum mechanics[1) 2) 3)] by separating a part of essentially kinematical effect so as to be able to regard it as due to dynamical action where Planck's constant plays the role of the coupling constant. This enables an alternative formulation of quantum mechanics in terms of the classical pictures within certain limited range of applicability; and on this formulation we have developed a detailed analysis in the previous article[3)] (hereafter referred to as FQM).

Now the purpose of the present article is twofold, that is, on the one hand to provide some supplementary remarks for this formulation of quantum mechanics, and on the other (though formally in close connection with the former) to represent complex scalar fields, such as Schrödinger field and Klein-Gordon field, including the interaction with electromagnetic field and also the interaction due to source, each in some mechanical picture, investigating the implications of such representations.

First we briefly recapiturate the above-mentioned formulation of quantum mechanics for the simplest case, together with miscellaneous remarks for this formulation (§ 1). Further we explicitly obtain the individual trajectories belonging to an ensemble for some

typical example, in order to illustrate how the trajectories deviate, due to quantum force, from purely classical ones, generally showing very complicated fluctuations in non-stationary cases (§ 2). In this connection we shall criticize the attempt[4] to interpret quantum mechanics in terms of Markoff process (Appendix B).

We started from the fact that the original Schrödinger equation corresponds to a picture of trajectory ensemble or to a hydrodynamical picture. This correspondence was extended in FQM to the generalized Schrödinger equation of more general system including quantum Bose fields, leading to the representation of a quantum-mechanical motion of such system with an ensemble picture.

Now the above correspondence can also be extended in another direction, i.e., to the cases of wave equations with sources. Here the ensemble picture must be subject to some important generalization (§ 7). This extension is in connection with the re-examination of the consistency of the ensemble picture in general cases (§ 3).

Further it is emphasized that the original correspondence of the Schrödinger equation to the ensemble picture implies for the Schrödinger field regarded as a classical wave field to admit the hydrodynamical model (Appendix D). Generally a linear field such as is accepted to describe some sort of elementary particles is essentially different from a hydrodynamical field, which implies motions of some *mechanical continuous matter*, to be described with density and velocity variables. Nevertheless, for a *linear complex scalar field* we can let it correspond to certain hydrodynamical motion. Here the first clue is given from the correspondence between the charge conservation for the former and the matter conservation for the latter.

For such hydrodynamical representation of wave fields, we must remark the following: The interactions in terms of external potential, which retain the linearity of the original wave equation, enter the picture merely as the corresponding classical potential; whereas the interaction, which acts as source for the original field and thus destroys its linearity, requires an essential generalization of the picture, (due to what was stated in the foregoing paragraph).

Next, we are on the other hand required also to re-examine the applicability of the statistical formulation for the case of Klein-Gordon equation (§ 4), the treatment of which in FQM was insufficient. For this purpose, we first explain the classical relativistic mechanics for a particle in general scalar field. We can then identify under certain condition the Klein-Gordon equation (represented in terms of amplitude and phase), with the 'Hamilton-Jacobi' equation in some additional scalar field and the equation of continuity.

This analysis also verifies that the classical Klein-Gordon field can be represented in a relativistic hydrodynamical form. Here continuous distribution of charged matter flows quasi-irrotationally both under electromagnetic interaction and also certain internal scalar field. The latter is now taken as a particular density-dependent pressure potential or as due to internal mechanical stress. In this representation further we can describe the interacting Klein-Gordon and electromagnetic fields solely with matter's variables.

The hydrodynamical representation also suggests a fairly natural way of generalization of Klein-Gordon field, i.e., the inclusion of the vortical flow. This leads to an essentially

new type of non-linear field equations. We do not investigate whether such field might have any analogue in the nature, yet we may suspect that it might perhaps be of use to describe mesons. In these equations we can again eliminate electromagnetic variables by means of the generalized Clebsch's transformation. In this connection the similar generalization of non-relativistic Schrödinger field (proposed in FQM) is somewhat supplemented (Appendix E).

The above formulation for the interacting Klein-Gordon and electromagnetic fields, which is so to speak a sort of 'electrodynamical hydrodynamics', involves the formalism of 'new classical theory of electrons'[5] recently presented by Dirac, as the limit of vanishing quantum potential; and we give some remarks on this theory of Dirac (§ 6). As a generalization of this theory, we also formulate a classical theory of mixed fields where the source is considered as mechanical continuous matter, (Appendix G).

§ 1. Introductory remarks on the formulation of quantum mechanics in terms of classical pictures

We begin with summarizing the formulation of quantum mechanics associated with classical pictures for the simplest case of a single particle without spin under the force potential $V(x)$: It is based upon the simultaneous equations of motion:

$$\begin{cases} \dfrac{\partial S}{\partial t}+\dfrac{1}{2m}(\nabla S)^2+V-\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}=0, & (1\cdot1) \\ \dfrac{\partial P}{\partial t}+\operatorname{div}(P\nabla S/m)=0, \quad (P\equiv R^2), & (1\cdot2) \end{cases}$$

which are derived by inserting

$$\psi=Re^{iS/\hbar} \qquad (R,\ S:\ \text{real}) \tag{1·3}$$

into the Schrödinger equation,

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(x,\ \frac{\hbar}{i}\nabla\right)\right\}\psi=0. \tag{1·4}$$

We can let the following picture correspond to eqs. (1·1) and (1·2): We regard P as the density of the probability distribution for particle position, and S as velocity potential in the sense that it yields the velocity to be taken by particle reaching the point x at time t by

$$v(x,t)=\nabla S/m. \tag{1·5}$$

Then from (1·1), by taking the gradient, we obtain

$$m\frac{dv}{dt}=-\nabla\left(V-\frac{\hbar^2}{2m}\frac{\Delta R}{R}\right). \tag{1·6}$$

Eqs. (1·1) and (1·2) are now regarded as representing an ensemble of particle trajectories, which belong to single velocity potential S by the relation (1·5), and are determined

by the classical dynamics assuming the external potential $V(\boldsymbol{x})$ and moreover the extra 'quantum' potential $V'=-(\hbar^2/2m)\Delta R/R$ (that is in turn dependent upon the density distribution of the ensemble). Here eq. (1·6) expresses the Newtonian equation of motion for individual trajectory.

If we are given the initial velocity field $\boldsymbol{v}(\boldsymbol{x}, 0)$ *irrotationally*, $S(\boldsymbol{x}, 0)$ is determined, by (1·5), up to an arbitrary additive constant, (which does not enter (1·1) and (1·2)), and if in addition given the initial density distribution $P(\boldsymbol{x}, 0)$, then their later distributions $\boldsymbol{v}(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$ are uniquely determined by (1·1) and (1·2) (cf. § 7).

Such '*gauge-invariancy' of the picture* holds for the case of a particle in general electromagnetic field $(\boldsymbol{A}, \phi)$. Here the Schrödinger equation,

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t}+\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-\frac{e}{c}\boldsymbol{A}\right)^2+e\phi+V\right\}\psi=0 \tag{1·7}$$

is transformed, with the substitution (1·3), into the form:

$$\begin{cases}\dot{S}+\dfrac{1}{2m}\left(\nabla S-\dfrac{e}{c}\boldsymbol{A}\right)^2+e\phi+V-\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}=0, & (1·8)\\[2ex] \dot{P}+\operatorname{div}\left\{P\left(\nabla S-\dfrac{e}{c}\boldsymbol{A}\right)\Big/m\right\}=0. & (1·9)\end{cases}$$

By imparting to the gauge invariant quantity

$$\boldsymbol{v}(\boldsymbol{x}, t)=\left(\nabla S-\frac{e}{c}\boldsymbol{A}\right)\Big/m \tag{1·10}$$

the meaning of particle velocity, eqs. (1·8) and (1·9) are considered to be the equations of motion for an ensemble of trajectories under the actions of Lorentz force and quantum potential (FQM § 3). Another gauge-invariant quantity

$$E(\boldsymbol{x}, t)=-\dot{S}-e\phi \tag{1·11}$$

represents particle energy in our picture.

We come back to the simplest case, (1·1) and (1·2). These equations are also written respectively as

$$\begin{cases}\dfrac{dS}{dt}=\dot{L}=\dfrac{1}{2m}(\nabla S)^2-V+\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}, & (1·1')\\[2ex] \dfrac{1}{P}\dfrac{dP}{dt}=-\Delta S/m. & (1·2')\end{cases}$$

The latter is equivalent to

$$\frac{1}{\delta\tau}\frac{d(\delta\tau)}{dt}=\operatorname{div}\boldsymbol{v}=\Delta S/m, \tag{1·12}$$

which means merely that in the velocity field $\boldsymbol{v}$, the volume $\delta\tau$ of a small portion of the 'cloud' varies 'substantially' at the rate $\operatorname{div}\boldsymbol{v}$, and shows that the cloud is locally either extending or contracting according as $\Delta S\gtrless 0$.

We shall next remark the following: The *limit of vanishing quantum force*

$$(\hbar^2/2m)|\nabla(\Delta R/R)|/|\nabla V| \to 0, \qquad (1\cdot13)^{*)}$$

which is somewhat more general than the condition of the simple *classical limit*, $\hbar \to 0$, does not immediately reduce our picture to that of classical particle mechanics; because the effect of quantum mechanics still remains in the statistical blurring of trajectories, although each of which is now purely classical.

In place of the statistical picture thus far considered, the foregoing equations can also be understood in a hydrodynamical picture. Here they are regarded as representing an irrotational flow defined by the velocity potential S in a compressible perfect fluid with a particular pressure potential $-(\hbar^2/2m)\Delta R/R$, or with the mechanical stress tensor $\sigma_{ik}=(\hbar^2/4m)P\partial^2(\log P)/\partial x_i \partial x_k$ (see FQM § 3 and Appendix A).

In our pictures the density field reacts upon particle with the quantum potential $V' \doteqdot -(\hbar^2/2m)\Delta R/R$, where Planck's constant $\hbar$ is regarded as playing the role of the coupling constant**); indeed $\hbar$ occurs in our fundamental equations (1·1) and (1·2) only here and moreover solely in its square. This is partly due to that we have chosen $\hbar$ for the unit of the phase in the substitution (1·3); but as a result, $\hbar$ appears in the *first power* in the *subsidiary condition*

$$\oint dS = n\hbar, \qquad (n: \text{integer}) \qquad (1\cdot14)$$

which is required corresponding to the single-valuedness of ψ (see FQM, § 7). Thus in our formulation Planck's constant must play the parts of coupling constant connecting the particle to its density field, and of the '*quantum of modulus*' at the same time.

Next, in the statistical formulation the ensemble can be represented in *phase space* by a density distribution $P(\boldsymbol{x})$ on the curve $\boldsymbol{p}=\nabla S(\boldsymbol{x})$ at each instant (FQM § 7), so that it has the definite particular distribution function:

$$f(\boldsymbol{x},\boldsymbol{p})=P(\boldsymbol{x})\delta(\boldsymbol{p}-\nabla S(\boldsymbol{x})). \qquad (1\cdot15)$$

The picture evidently stands upon simultaneously determined position and momentum of particle that mean 'hidden variables', and was capable of representing the quantum-mechanical equation of motion (Schrödinger equation). But it leads, as to the problems such as the probability distributions of various physical quantities (other than particle position), to different results from those due to ordinary quantum mechanics, as long as we merely adopt the values that the picture directly yields (see FQM §§ 5 8). Such a circumstance is, however, quite generally valid for any theory involving hidden variables, according to what was proved by von Neuman. Thus in order that the statistical formulation should attain the agreement with ordinary quantum mechanics as to probability distributions, it becomes necessary that we distinguish, for physical quantities such as momentum, the value

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* For stationary state this condition reduces to

$$(\hbar^2/2m)\,\Delta R/R\,|E-V| \to 0. \qquad (1\cdot13')$$

** This was remarked to the author by Dr. T. Inoue.

to be observed from the 'real' value, by some reasonable theory of measurement. Bohm[2] was, in fact, led to such a procedure for the statistical picture.

The substantial rate of change of the distribution function in phase space (1·15) is given by

$$\frac{df}{dt} = -f \operatorname{div} \boldsymbol{p}/m = -f \Delta S/m , \tag{1·16}$$

which is analogous with (1·2′). This relation replaces the Liouville's theorem and shows a difference of our ensemble from Gibbs' one where df/dt vanishes.

§ 2. Fluctuation of trajectories

(a) In our statistical picture the particle on its individual trajectory is acted on by the additional quantum potential, $-(\hbar^2/2m)\Delta R/R$, which, in general non-stationary states, explicitly depends on the time t, and so the individual trajectory is of non-conservative classical system, even though the system is originally conservative, i.e., $\partial V/\partial t = 0$. Thus its energy

$$H_1(\boldsymbol{p}, \boldsymbol{x}, t) = \frac{\boldsymbol{p}^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\Delta R}{R} \tag{2·1}$$*)

varies along the trajectory by

$$\frac{dH_1}{dt} = \frac{\partial H_1}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial t}\left(\frac{\Delta R}{R}\right), \tag{2·2}$$

which means the energy law for individual trajectory, to be combined with the momentum law (1·6). The avarage value of H_1 over the ensemble of trajectories naturally conserves, in agreement with the quantum-mechanical mean energy, (FQM § 5). As the result of such time-dependent quantum force, a trajectory of particle will generally show a complicated fluctuation.

When we explicitly know any possible solution (S, P), we may further obtain individual trajectories belonging to this ensemble by integrating the differential equation :

$$d\boldsymbol{x}/dt = \boldsymbol{v}(\boldsymbol{x}, t) = \nabla S/m. \tag{2·3}$$

We can in this way get the trajectories of number of ∞^3 belonging to a single 'elementary' ensemble (S, P) in the form $\boldsymbol{x}(\boldsymbol{x}_0, t)$, involving a parameter $\boldsymbol{x}_0$, the initial value of position.**)

Usually, in practice, we first get the wave function ψ by solving the Schrödinger equation, and then obtain S by

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* $H_1(\boldsymbol{p}, \boldsymbol{x}, t)_{\boldsymbol{p}=\nabla S(\boldsymbol{x})} = E(\boldsymbol{x}, t)$. Here $E(\boldsymbol{x}, t)$ is defined by (1·11) or FQM, Eq. (5·6).

** For a stationary state (of energy E), we have $S = S'(\boldsymbol{x}) - Et$, and $\boldsymbol{v} = \nabla S'/m$ which does not involve t explicitly, so (2·3) takes the form

$$d\boldsymbol{x}/dt = \boldsymbol{v}(\boldsymbol{x}) = \nabla S'/m. \tag{2·3′}$$

$$S=(\hbar/2i)\log(\psi/\psi^*), \tag{2·4}$$

and so $\boldsymbol{v}$ by

$$\boldsymbol{v}(\boldsymbol{x},t)=(\hbar/m)\,\mathrm{Im}(\nabla\psi/\psi). \tag{2·5}$$

We can say that a problem in quantum mechanics is perfectly solved by obtaining S and R, without requiring further to get the individual trajectories belonging to this ensemble explicitly by solving*) (2·3), which is generally a very complicated mathematical problem; yet it may be very illustrative to carry it out for some typical example.

Now, we shall solve (2·3) for a typical non-stationary state in free space,

$$\psi=\psi_1+a\psi_2, \qquad (a=ae^{i\delta}\,;\ a,\ \delta:\ \text{real}) \tag{2·6}$$

produced by the superposition of two plane waves:

$$\psi_i=\exp i\left(\boldsymbol{k}_i\boldsymbol{x}-\frac{\hbar k_i^2}{2m}t\right), \qquad (i=1,2) \tag{2·7}$$

each of which means, in our picture, an ensemble where particles move in constant velocity $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$, with $\boldsymbol{v}_i=\hbar\boldsymbol{k}_i/m$. For the state (2·6) the equation of trajectories becomes

$$\frac{d\boldsymbol{x}}{dt}=\frac{\hbar}{m}\,\mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right)=\frac{\hbar}{m}\,\frac{\boldsymbol{k}_1+a^2\boldsymbol{k}_2+a(\boldsymbol{k}_1+\boldsymbol{k}_2)\cos\xi}{1+a^2+2a\cos\xi}, \tag{2·8}$$

where

$$\xi=(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{x}-\frac{\hbar}{2m}(k_1^2-k_2^2)t-\delta. \tag{2·9}$$

In one-dimensional case, (2·8) can be integrated into

$$\sin\xi+\frac{k_1-k_2}{2a}\left\{(1+a^2)x-\frac{\hbar}{m}(k_1+a^2k_2)t\right\}=\text{const.} \tag{2·10}$$

This is an equation of Keplerian type between ξ and t, and is solved to be

$$x-x_0=\bar{v}t+\frac{1}{k_1-k_2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}J_n\left(\frac{2a}{1+a^2}n\right)\sin(n\omega t), \tag{2·11}$$

where $\bar{v}=\dfrac{v_1+a^2v_2}{1+a^2}$, $\omega=\dfrac{1-a^2}{1+a^2}\dfrac{\hbar}{2m}(k_1-k_2)^2$, and J_n's are Bessel functions.

---

* To get the trajectories $\boldsymbol{x}(\boldsymbol{x}_0,t)$ from the solution (S,R) corresponds to the transition from the Eulerian solution to the Lagrangian solution in the hydrodynamical language. In order to get the latter solution $\boldsymbol{x}(\boldsymbol{x}_0,t)$ directly, we are to start from, in place of (1·1) and (1·2), the Lagrangian equation of motion:

$$\begin{cases} m\dfrac{\partial\boldsymbol{v}}{\partial t}\dfrac{\partial\boldsymbol{x}}{\partial x_{0i}}=-\dfrac{\partial}{\partial x_{0i}}\left(V-\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}\right), & (\boldsymbol{v}=\partial\boldsymbol{x}(\boldsymbol{x}_0,t)/\partial t) \\ P\dfrac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}=P_0, \end{cases}$$

under the subsidiary condition: $\sum_i[\nabla_0 v_i\times\nabla_0 x_i]=0$.

Eq. (2·11) means the motion of constant velocity $\bar{v}$ superposed with oscillating fourier components of frequencies $n\omega$. Thus in the ensemble produced by the superposition of two constant-velocity ensembles, particle fluctuates rapidly about the motion of constant velocity (that is equal to the weighted mean of two component velocities), the amplitude becoming smaller and the frequency larger, the larger is the difference $v_1 - v_2$. The fluctuating part of motion represents how the trajectory deviates by the presence of quantum potential from purely classical one (which must, in free space, be a motion of constant velocity). This is what we referred to as the 'interference' of two trajectory ensembles (FQM § 7). For special cases for this example, see Appendix A.

(**b**) In our statistical picture particle trajectories generally take very complicated fluctuations in non-stationary cases due to the time-dependent quantum potential, as was illustrated in (a). On the other hand, since it implies conceptual difficulties to interpret this potential as some *real* interaction within the density distribution of the ensemble, it may be natural to inquire into the possibility that such effects might be attributed to some external action. Thus Bohm[2] directly assumed this potential as the external force field. But this interpretation seems to raise another difficulties (FQM § 4). Instead of doing this, it might also be conceivable somehow to deduce the fluctuation of particle trajectories due to the quantum potential through a mechanism similar to that of the Brownian movement; that is, we regard the fluctuation as produced by the random action from the outside medium. In this viewpoint we must then postulate a virtual medium (or, so to speak, an 'aether'); and therefore, 'hidden variables' are introduced not only as the simultaneously defined particle position and velocity as in the original statistical picture, but also as the freedoms of the medium.

It is, however, not expected that such a mechanism of stochastic process would be able to result in the 'quantum-theoretical' fluctuation of particle trajectories; because this fluctuation, though may be very complicated, is not at random but is perfectly determined *for each individual trajectory*, and must be, at any rate, of such type as is produced by the force $(\hbar^2/2m)\Gamma(\Delta R/R)$; whereas in a stochastic process the particle is acted on by a fluctuating force, *uncorrelated* at every successive time and naturally again independent of the probability distribution P.

Nevertheless, as is well known, quantum-mechanical motion has some resemblances to the diffusion phenomena as the result of the formal similarity between the Schrödinger equation and the diffusion equation. Starting from this analogy, Fényes[4] recently attempted to interpret quantum mechanics as some Markoff process. But both theories are essentially different in spite of some resemblances, and we can anticipate, from what we above stated using our statistical formulation of quantum mechanics, that such attempts would not succeed. We show this point more explicity in Appendix B.

§ 3. Consistency of the ensemble picture

In FQM we called the action function S, in case of the statistical picture, as 'Hamilton-Jacobi function'. It is true, as pointed out by Halpern[6], that Eq. (1·1) is

not perfectly the same thing with the Hamilton-Jacobi equation originally treated in classical mechanics, because the function R defining the extra potential is related with S by the other equation (1·2). But this circumstance implies no more difficulties for the statistical picture, than that an individual trajectory be modified by the extra potential determined by the very density distribution of the ensemble. (Difficulties in the *interpretation* of this latter fact were stated in FQM § 4).

When we solve the simultaneous equations (1·1) and (1·2), surely the same integration constants a_i's generally enter both S and R, and yet we can put (1·5) as in the original Hamilton-Jacobi scheme. Indeed we must not further put $\partial S/\partial a_i = \text{const.}$; nevertheless, by integrating (1·5) (i.e. (2·3)), we can perfectly obtain individual trajectories of number of ∞^3 belonging to the ensemble defined by a set of values of a_i's, as was illustrated in § 2. For convenience we shall still call the S-function as 'Hamilton-Jacobi function' occasionally*). In exceptional cases where the quantum force vanishes, the S-function is a genuine Hamilton-Jacobi function, which coincides with the classical one. These cases were treated in FQM § 14.

The consistency of the statistical picture based upon equations (1·1), (1·2) and (1·5) is evident in one-body problems[7] from its correspondence to the hydrodynamical picture. To make sure, we shall further explain this point for the general case treated in FQM § 11 (a): We consider a system with Hamiltonian H quadratic in each momentum p_i:

$$H=\sum_{ij} 2^{-1}(a_{ij}p_ip_j+p_ip_ja_{ij})+\sum_i 2^{-1}(b_ip_i+p_ib_i)+V, \tag{3·1}$$

where each a_{ij}, b_i, and V may be real functions of coordinates q_i's. Then its quantum-mechanical equation of motion is written in our formulation as

$$\begin{cases} \dfrac{\partial S}{\partial t}+H(\cdots q_i, \partial_i S, \cdots)+V'[R(\cdots q_i\cdots)]=0, & (3·2) \\ \dfrac{\partial P}{\partial t}+\sum_i \partial_i\left[P\left(\dfrac{\partial H}{\partial p_i}\right)_{p_i=\partial_i S}\right]=0, \quad \left(\partial_i \equiv \dfrac{\partial}{\partial q_i}\right) & (3·3) \end{cases}$$

where

$$V'=-\frac{\hbar^2}{2R}\sum_{ij}\{a_{ij}\partial_i\partial_j R+\partial_i\partial_j(a_{ij}R)\}. \tag{3·4}$$

Here we assume the S-function as the 'velocity potential' in the sense that from S be derived the velocity components in configuration space, dq_k/dt, through the relations:

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* The substitution (1·3) means a sort of phase transformation for ψ. If we take the genuine Hamilton-Jacobi function for the shift of phase S in (1·3), the 'density' function R——which is no longer necessarily a real quantity——must satisfy

$$\partial R^2/\partial t+\operatorname{div}\{R^2(\nabla S-e\boldsymbol{A}/c)/m\}=i(\hbar/m)R\nabla^2 R,$$

in place of (1·9). In this separation of the Schrödinger equation, we have an alternative picture in terms of 'quantum source', $(i\hbar/m)R\nabla^2 R$, in place of the quantum potential. This procedure can also be applicable to the Dirac equation.

$$p_j = \partial_j S, \tag{3·5}$$

and

$$\frac{dq_k}{dt} = \sum_j (a_{kj} + a_{jk}) p_j + b_k . \tag{3·6}$$

Then, by integrating (3·6) with the substitution (3·5), we can determine the trajectories belonging to a S-function (which in turn is obtained by solving (3·2) and (3·3)). We have thus the kinematical picture of 'irrotational' velocity field in the configuration space corresponding to a S-function. The time-rate of change of the quantity p_k along a trajectory is given by

$$\frac{dp_k}{dt} = \frac{\partial p_k}{\partial t} + \sum_i \partial_i p_k \frac{dq_i}{dt} . \tag{3·7}$$

Now, differentiating (3·2) with respect to q_k, and employing (3·5), we obtain

$$\begin{aligned} &\frac{\partial p_k}{\partial t} + \sum_{ij} \{\partial_k a_{ij} \cdot p_i p_j + a_{ij} (\partial_i p_k \cdot p_j + p_i \partial_j p_k)\} \\ &+ \sum_i (\partial_k b_i \cdot p_i + b_i \partial_i p_k) + \partial_k V + \partial_k V' = 0, \end{aligned} \tag{3·8}$$

which is further transformed, by using (3·6) and (3·7), into

$$-\frac{dp_k}{dt} = \sum_{ij} \partial_k a_{ij} \cdot p_i p_j + \sum_i \partial_k b_i \cdot p_i + \partial_k V + \partial_k V' . \tag{3·9}$$

Here we see that this equation, together with (3·6), coincides with the classical equation of motion*) for the system with Hamiltonian

$$\bar{H} = \sum_{ij} a_{ij} p_i p_j + \sum_i b_i p_i + V + V' = H + V', \tag{3·10}$$

namely,

$$\begin{cases} \dfrac{dq_k}{dt} = \dfrac{\partial \bar{H}}{\partial p_k} = \dfrac{\partial H}{\partial p_k}, & (3·11) \\ -\dfrac{dp_k}{dt} = \dfrac{\partial \bar{H}}{\partial q_k}. & (3·12) \end{cases}$$

Thus the kinematical picture which we have let to correspond to (3·2) is understood dynamically, i.e., it just represents the ensemble of classical motions, referring to single S-function, of the system with additional quantum potential (3·4). Furthermore, taking into account (3·5) and (3·6), (i.e, (3·5) and (3·11)), eq. (3·3) is identified with the equation of probability conservation, with P as probability density of the ensemble.

In the above procedure the additional potential V' is determined only by the density function P, but we also encounter the cases where V' depends upon the S-function, too. It is possible to generalize the picture so as to fit to such cases. The examples are the

* Eq. (3·9), from which are eliminated p_i's by the use of (3·6), is regarded as the Newtonian equation of motion for such system.

case of multi-component wave function referring to spin which was already treated in FQM § 11 (c), and the case of inhomogeneous wave equation (though here the wave function does not mean a state vector) which will be treated in § 7. Also cf. the case of Appendix B, eq. (B·10).

§ 4. Ensemble picture for Klein-Gordon equation

As was explained in FQM § 11, our statistical picture for quantum mechanics loses its applicability to electron when relativity, spin, and exclusion principle are taken into account. The explanation for the case of Klein-Gordon equation (FQM § 11 (d)), taken up as a relativistic modification of one-particle Schrödinger equation, was however partly incorrect, and so will be supplemented in this section.

The Klein-Gordon equation:

$$\left\{\left(\frac{\hbar}{i}\partial_\mu-\frac{e}{c}A_\mu\right)^2+m^2c^2\right\}\psi=0, \tag{4·1}$$

(where A_μ is electromagnetic potential, $\partial_\mu=(\nabla, \partial/\partial ct)$, and $\mu=1, 2, 3, 0$), is transformed, by the usual substitution

$$\psi=Re^{iS/\hbar}, \tag{4·2}$$

into

$$\begin{cases}(\partial_\mu S-eA_\mu/c)^2+m^2c^2-\hbar^2\Box R/R=0, & (4·3)\\ \partial_\mu\{P(\partial^\mu S-eA^\mu/c)\}=0, \quad (P\equiv R^2). & (4·4)\end{cases}$$

Here, it is not allowed simply to regard eqs. (4·3) and (4·4) as the relativistic 'Hamilton-Jacobi' equation under the additional scalar potential $-\hbar^2\Box R/R$, and the continuity equation with the rest density P, respectively. This is because, for the solution (S, P) of the simultaneous equations (4·3) and (4·4), the quantities $u^\mu=(\partial^\mu S-eA^\mu/c)/m$ do not satisfy the condition to be 4-velocity:

$$u_\mu u^\mu=-c^2, \tag{4·5}$$

unless $\Box R/R=0$; furthermore, in this connection, the quantity $j^0=P(\partial^0 S-eA^0/c)/m$ is not positive definite, and so cannot be regarded as a density function*).

Now we shall find Hamilton-Jacobi equation for classical relativistic motion of a particle moving under a scalar field U as well as the electromagnetic potential A_μ. Here the scalar field affects the motion as if it produced the shift of mass $\delta m=fU/c^2$, the Lagrangian being

$$L=\frac{1}{2}(m+fU/c^2)(u_\mu u^\mu-c^2)+(e/c)u_\mu A^\mu, \tag{4·6}**)$$

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* If $u^0=(\partial^0 S-eA^0/c)/m$ could be taken as the 0-th component of a 4-velocity, then $u^0\geqq c$, and j^0 would be positive definite.

** We might as well take the form, $L=-(mc+fU/c^2)(-u_\mu u^\mu)^{1/2}+(e/c)u_\mu A^\mu$.

where $u_\mu = x^\mu = dx^\mu/d\tau$ $[cd\tau = (-dx_\mu dx^\mu)^{\frac{1}{2}}]$ is 4-velocity, to be restricted by (4·5), and f is coupling constant. The 4-momentum p_μ, conjugate to x^μ, is

$$p_\mu = \frac{\partial L}{\partial u^\mu} = (m + fU/c^2)u_\mu + eA_\mu/c, \qquad \text{i.e.,} \quad u_\mu = \frac{p_\mu - eA_\mu/c}{m + fU/c^2}. \tag{4·7}$$

The Hamiltonian is

$$H = p_\mu u^\mu - L = \frac{1}{2}\left\{\frac{(p_\mu - eA_\mu/c)^2}{m + fU/c^2} + (mc^2 + fU)\right\}. \tag{4·8}$$

The equations of motion derivable from (4·6) (or (4·8)) are written as

$$\dot{p}_\mu = \frac{\partial L}{\partial x^\mu} = \frac{e}{c} u_\nu \partial_\mu A^\nu - f\partial_\mu U, \tag{4·9}$$

i.e.,

$$m\frac{d}{d\tau}\left\{\left(1 + \frac{fU}{mc^2}\right)u^\mu\right\} = \frac{e}{c} u_\nu F^{\mu\nu} - f\partial^\mu U, \tag{4·10}$$

or

$$\left(m + \frac{fU}{c^2}\right)\dot{u}^\mu = \frac{e}{c} u_\nu F^{\mu\nu} - f\left(\frac{u^\mu u^\nu}{c^2}\partial_\nu U + \partial^\mu U\right). \tag{4·11}$$

This is consistent with (4·5), since eq. (4·9) obviously has the integral $u_\mu u^\mu = \text{const.}$, as L does not involve τ explicitly. This const. is put to be equal to $-c^2$. Considering this, from (4·8) we get the Hamilton-Jacobi equation:

$$(\partial_\mu S - eA_\mu/c)^2 + (mc + fU/c)^2 = 0. \tag{4·12}$$

From the solution S of this equation, momentum is given by $p_\mu = \partial_\mu S$, and so the velocity by $u^\mu = (\partial^\mu S - eA^\mu/c)/(m + fU/c^2)$, which evidently satisfies (4·5)*.

The equation of continuity to be combined with (4·12) should be of the form,

$$\partial_\mu\left\{P_0\frac{\partial^\mu S - eA^\mu/c}{m + fU/c^2}\right\} = 0, \tag{4·13}$$

where P_0 means the density function in the rest frame.

Now in order that eqs. (4·12) and (4·13) agree with (4·3) and (4·4) respectively, we are to put

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* The Lagrangian (4·6) can be replaced by the Lagrangian of the three-dimensional form,

$$L^* = L(1 - \boldsymbol{v}^2/c^2)^{\frac{1}{2}} = -(mc^2 + fU)(1 - \boldsymbol{v}^2/c^2)^{\frac{1}{2}} + (e/c)\boldsymbol{A}\boldsymbol{v} - e\phi, \tag{4·6′}$$

where $\boldsymbol{v} = d\boldsymbol{x}/dt$ is the usual velocity, and $A^\mu = (\boldsymbol{A}, \phi)$. The canonical momentum is

$$\boldsymbol{p} = (m + fU/c^2)\boldsymbol{v}(1 - \boldsymbol{v}^2/c^2)^{-\frac{1}{2}} + (e/c)\boldsymbol{A}, \tag{4·7′}$$

and the Hamiltonian becomes

$$H^* = \{(\boldsymbol{p}c - e\boldsymbol{A})^2 + (mc^2 + fU)^2\}^{\frac{1}{2}} + e\phi. \tag{4·8′}$$

The equations of motion to be derived from (4·6′) or (4·8′) are found to be identical with the first three ones of (4·11), and also the Hamilton-Jacobi equation derived from (4·8′) to be identical with (4·12).

$$fU=mc^2(\gamma-1), \qquad P_0=\gamma P, \tag{4·14}$$

with

$$\gamma=(1-\lambda_c^2\Box R/R)^{\frac{1}{2}}, \qquad \lambda_c\equiv\hbar/mc.$$

In order the above form of U and P_0 be physically admitted, we require

$$\lambda_c^2\Box R/R\leq 1, \quad \text{everywhere.} \tag{4·15}$$

This condition is not always fulfilled; for instance, it breaks down, when the rate of fluctuation of R in the space-time region of the order of Compton wave-length is not small, or when $\Box R/R$ diverges in the region $R\to 0$.

The 4-velocity is now

$$u^\mu=(\partial^\mu S-eA^\mu/c)/m\gamma, \tag{4·16}$$

whose 0-th component u^0, and so $j^0=P_0u^0$ are certainly positive-definite, provided (4·15) be satisfied.

Consequently, the Klein-Gordon equation corresponds to the picture of ensemble of relativistic particle trajectories, in the additional scalar field, referred to single 'Hamilton-Jacobi' function, under the condition (4·15); here the additional field U is related to the rest density of the ensemble through (4·14). The equation of motion for the trajectory, such as is consistent with the 'Hamilton-Jacobi' equation (4·3) is obtained, by inserting (4·14) into (4·10), i.e

$$m\frac{d}{d\tau}(\gamma u^\mu)=\frac{e}{c}u_\nu F^{\mu\nu}+\frac{\hbar^2}{2m\gamma}\partial^\mu\left(\frac{\Box R}{R}\right). \tag{4·17}$$

The correspondence of the above picture to the Klein-Gordon equation is imperfect in that it fails when the condition (4·15) breaks down. Indeed, this fact does not immediately mean a difficulty for the statistical formulation of quantum mechanics itself, since the Klein-Gordon equation cannot, in fact, be regarded originally as one for the probability amplitude for a particle*). But the ensemble picture fails also for Dirac equation, as was stated in FQM.

Where the rate of the density fluctuation is small such that $\lambda_c^2|\Box R/R|\ll 1$**), the extra scalar field and the rest density reduce respectively to

$$fU\simeq-(\hbar^2/2m)\Box R/R, \qquad P_0\simeq P-\frac{1}{2}\lambda_c^2R\Box R, \tag{4·18}$$

which correspond to the first approximation with respect to $\lambda_c^2|\Box R/R|$, and in this case the condition (4·15) is always satisfied; further in the *zero-th approximation*, (4·3) reduces to the purely classical Hamilton-Jacobi equation

* For the Klein-Gordon field as a quantum Bose field, its quantum-mechanical motion can be pictured as an ensemble of classical motions of the assembly of field oscillators under certain extra potential, in a similar way as was shown for cases of Schrödinger field etc. in FQM.

** This is also written as $(\hbar^2/m)|\Box R/R^2|\ll mc^2$, which means the limit of vanishing quantum potential, corresponding to the condition (1·13) for the non-relativistic case.

$$(\partial_\mu S - eA_\mu/c)^2 + m^2c^2 = 0 \qquad (4\cdot19)$$

and P coincides with the rest density.

§ 5. Hydrodynamical representation of Klein-Gordon field

(a) Hydrodynamical form. Elimination of electromagnetic variables

As was previously stated, a hydrodynamical picture can be associated with the Schrödinger equation; this fact implies for the Schrödinger field regarded as a *classical wave field* to admit such *mechanical model**). This property of Schrödinger field is very remarkable in contrast, e.g., to Maxwell field which is well known to admit no mechanical model.

The situation is similar as to the other linear complex scalar field, the one due to Klein and Gordon**): As was stated, we can associate an ensemble of relativistic trajectories under some additional scalar potential with the Klein-Gordon equation; and this fact implies for the *classical Klein-Gordon field* that it can have such mechanical model, which is taken also as the hydrodynamical one (though in this case the model has real meaning only under the condition (4·15)). The detailed explanation for this point is given in Appendix F, but anyhow in the hydrodynamical picture, we consider a relativistic quasi-irrotational flow involving the stress such as to correspond to the internal scalar potential $fU = mc^2(\gamma - 1)$. Here the rest density***) is P_0, and the term 'quasi-irrotational' means that the velocity field is derivable from S by (4·16), i.e., it should satisfy

$$\partial_\mu \bar{u}_\nu - \partial_\nu \bar{u}_\mu = -(e/mc)F_{\mu\nu}, \qquad (5\cdot1)^{****}$$

where $F_{\mu\nu}$ is electromagnetic field strength, and

$$\bar{u}_\mu = \gamma u_\mu = (\partial_\mu S - eA_\mu/c)/m. \qquad (4\cdot16)$$

The relation (5·1) implies that $\bar{u}_\mu$, referring to 4-velocity of matter, can at the same time be regarded as *the potential of the electromagnetic field.*

Now the flow (4·16) with the rest density P_0 produces the current

$$j^\mu = cP_0 u^\mu = cP\bar{u}^\mu.$$

This current is to be taken as the real one for the Klein-Gordon field, not only due to our hydrodynamical model, but also because it is derivable from (D·4) of the general

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* For this point further see, Appendix D.

** As to a condition that we should be able to represent any wave field with a hydrodynamical field, see Appendix $F(b)$.

*** We choose the dimension of ψ such that the Lagrangian density for the Klein-Gordon field becomes

$$\mathcal{L}_K = \frac{1}{2m}\left(\frac{\hbar}{i}\partial_\nu + \frac{e}{c}A_\nu\right)\psi^*\left(\frac{\hbar}{i}\partial^\nu - \frac{e}{c}A^\nu\right)\psi - \frac{1}{2}mc^2\psi^*\psi, \qquad (5\cdot2)$$

giving the current vector by

$$j_\mu = \frac{e\hbar}{2mi}(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*) - \frac{e^2}{2mc}A_\mu\psi^*\psi,$$

which turns into the form $j_\mu = eR^2(\partial_\mu S - eA_\mu/c)/m$ through the substitution (4·2).

**** This is the relativistic generalization of the condition curl $\boldsymbol{v} = -c\boldsymbol{H}/mc$, (see FQM Eq. (3·6), and also Appendix C).

theory. Then this current reacts upon the electromagnetic field according to the Maxwell equation :

$$\partial_\nu F^{\mu\nu} = (e/c) P\bar{u}^\mu. \tag{5·3}$$

that is,

$$\partial_\mu \partial_\nu A^\nu - \Box A_\mu = (e/mc) P(\partial_\mu S - eA_\mu/c). \tag{5·4}$$

Eq. (5·4) and

$$(\partial_\mu S - eA_\mu/c)^2 + m^2c^2 - \hbar^2 \Box R/R = 0 \tag{4·3}$$

constitute the fundamental equations*) for *interacting electromagnetic field and classical Klein-Gordon field* expressed in a hydrodynamical form. (Here the continuity equation (4·4) is omitted because it is now derivable from (5·4)). These equations are derivable from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}{}^2 - \frac{\hbar^2}{2m} (\partial_\nu R)^2 - \frac{1}{2} R^2 \{(\partial_\nu S - eA_\nu/c)^2/m + mc^2\}. \tag{5·5}$$

Thus the interacting Klein-Gordon and electromagnetic fields correspond to a sort of 'electrodynamical hydrodynamics', i.e., to a quasi-irrotational flow of continuous charged matter, each portion of which is mutually interacting by means of electromagnetic field and at the same time by an internal scalar field. This latter field is perfectly determined by the distribution of matter, without any freedoms of its own (differing from electromagnetic field), and is of a type fluctuating with the variation of density. Suppose we here replace this scalar field by some *cohesive* one, then we shall have a system such as was considered to construct the model of the electron in classical theory**) (cf. § 6 and Appendix G).

The former set of equations ((5·4) and (4·3)) is also expressed, in terms of $\bar{u}_\mu$, as (5·1), (5·3), and

$$\bar{u}_\mu \bar{u}^\mu = -\gamma^2 c^2 = -c^2 + \frac{\hbar^2}{m^2} \frac{\Box R}{R}. \tag{5·8}$$

This form represents the theory in terms of quantities such as velocity and field strengths, having direct physical significances, in place of their potentials S and A_μ.

It is however also regarded as the expression in the particular gauge where $-k\bar{u}_\mu = A_\mu - (c/e)\partial_\mu S$, $(k \equiv mc/e)$, plays the role of electromagnetic potential***) : That is, we take, as fundamental equations, (5·8) and

$$\partial_\mu \partial_\nu \bar{u}^\nu - \Box \bar{u}_\mu = -r_0 P \bar{u}_\mu, \qquad (r_0 \equiv e^2/mc^2) \tag{5·9}$$

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* In the usual representation this theory consists of the original Klein-Gordon equation (4·1) combined with the Maxwell equation

$$\Box A_\mu - \partial_\mu \partial_\nu A^\nu = -(e\hbar/2mci)(\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^*) + (e^2/mc^2) A_\mu \psi^* \psi. \tag{5·6}$$

They are derivable from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}{}^2 + \mathcal{L}_K \tag{5·7}$$

with $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. By the usual substitution (5·7) turns into (5·5).

** For instance Poincaré[7a] really introduced such scalar cohesive pressure.

*** In the similar fashion we can elimitate the gauge variable also for the case of Schrödinger equation, (see Appendix C).

which is the form of (5·4) in this particular gauge. They are derivable from

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}{}^2-m\left\{(\hbar^2/2m)(\partial_\nu R)^2+\frac{1}{2}R^2(\bar{u}_\nu{}^2+c^2)\right\}, \tag{5·10}$$

(where the first term is to be understood as $-\frac{1}{4}F_{\mu\nu}{}^2=-\frac{1}{4}\hbar^2(\partial_{[\mu}\bar{u}_{\nu]})^2$).

We can consider that in this form the electromagnetic variables disappear, and the motion of the matter field, including the electromagnetic interactions, is described solely in terms of matter variables, R and $\bar{u}_\mu$. The original Klein-Gordon equation was written as (5·8) and the continuity equation

$$\partial_\mu(P\bar{u}^\mu)=0. \tag{5·11}$$

The latter is now replaced by (5·9), which includes (5·11) as a consequence, so as to include the motion of interacting electromagnetic field implicitly. We can in this way represent the *quasi-irrotational* motion of certain charged continuum, including the mutual interaction intermediated by the electromagnetic field having its own freedoms of motion, merely in terms of matter variables, the formulation thus bearing the appearance of a sort of unitary theory.*)

In FQM we represented the Schrödinger field in the hydrodynamical form and then tried to quantize it; likewise now the quantization of the Klein-Gordon field in its hydrodynamical form, together with the electromagnetic field, is required. This will lead to an equivalent to the Pauli-Weisskopf theory (including the interaction with the electromagnetic field) which is the direct quantization of the Klein-Gordon field.

(b) Generalization of Klein-Gordon field

Again, in FQM § 13 we generalized the irrotational flow that corresponds to the Schrödinger field, so as to include the rotational; likewise the hydrodynamical form of Klein-Gordon field suggests to generalize its original quasi-irrotational relativistic flow to the 'rotational'. For doing so, we take, in place of (4·3) (i.e. (5·8) and (5·1)), the equation of motion (4·17) with $\bar{u}_\mu$ as field variables, i.e.,

$$m\bar{u}_\nu\partial^\nu\bar{u}^\mu=\frac{e}{c}\bar{u}_\nu F^{\mu\nu}+\frac{\hbar^2}{2m}\partial^\mu\left(\frac{\Box R}{R}\right) \tag{5·12}$$

that is the relativistic version of FQM (3·3), together with the constraint (5·8). We have besides the Maxwell equation (5·3) with $F_{\mu\nu}=\partial_{[\mu}A_{\nu]}$.

In this case, though $\bar{u}_\mu$ cannot itself play the part of electromagnetic potential concurrently, it is still possible to eliminate electromagnetic variables by means of kinematical quantities for matter $\bar{u}_\mu$, ξ, and η where the additional quantities ξ and η refer to the vortical freedoms of motion. It is an application of the method due to Dirac[8], and corresponds to the Clebsch's transformation in non-relativistic hydrodynamics (see Appendix E).

* But the Eq. (5·9) is a 2nd-order differential equation in u_μ, at variance with a proper hydrodynamical equation which is to be first order (see e.g. (5·13)); and accordingly the motion is not uniquely determined by specifying the initial values of P and u_μ, unless we further specify $\partial u_\mu/\partial t$ initially.

Now we shall briefly state the procedure: eq. (5·8) yields $\bar{u}_\nu \partial_\mu \bar{u}^\nu = (\hbar^2/2m)\partial_\mu \times$ $\times (\Box R/R)$. Combined with this, eq. (5·12) can be replaced by

$$\bar{u}_\nu(\partial^\mu \bar{u}^\nu - \partial^\nu \bar{u}^\mu) = -\frac{e}{mc}\bar{u}_\nu F^{\mu\nu}, \tag{5·13}$$

that is also written as

$$\bar{u}_\nu f^{\mu\nu} = 0, \quad \text{or} \quad u_\nu f^{\mu\nu} = 0, \tag{5·13′}$$

with

$$f^{\mu\nu} = F^{\mu\nu} + k(\partial^\mu \bar{u}^\nu - \partial^\nu \bar{u}^\mu), \qquad (k \equiv mc/e) \tag{5·14}$$

which means roticity. This quantity on the other hand, since it satisfies (5·13′), can be written as

$$f_{\mu\nu} = \partial_\mu(\xi \partial_\nu \eta) - \partial_\nu(\xi \partial_\mu \eta) = \frac{\partial(\xi, \eta)}{\partial(x^\mu, x^\nu)} \tag{5·15}$$

by use of two quantities ξ and η satisfying

$$\bar{u}^\nu \partial_\nu \xi = \bar{u}^\nu \partial_\nu \eta = 0. \tag{5·16}$$*)

Fundamental equations thus reduce to (5·8), (5·3) and (5·16), with

$$F^{\mu\nu} = \frac{\partial(\xi, \eta)}{\partial(x_\mu, x_\nu)} - k(\partial^\mu \bar{u}^\nu - \partial^\nu \bar{u}^\mu). \tag{5·17}$$

They are derivable from the Lagrangian density (5·10) that is now to be regarded as a functional**) with respect to R, $\bar{u}_\mu$, ξ, and η, on account of (5·17).

In this representation the potential A^μ is written, from (5·14) and (5·15), as

$$(e/c)p^\mu = A^\mu + k\bar{u}^\mu = \xi \partial^\mu \eta + \partial^\mu s, \tag{5·18}***)$$

where p^μ is the canonical momentum defined in (4·7). Further, by taking suitable gauge, we have

$$A_1^\mu + k\bar{u}^\mu = \xi \partial^\mu \eta. \tag{5·18′}$$

On the other hand ξ and η can be subject to the transformation $\xi, \eta \to \xi', \eta'$ such that it leaves $f^{\mu\nu}$ invariant, i.e, satisfies $\partial(\xi', \eta')/\partial(\xi, \eta) = 1$. This is a *contact transformation* that is regarded as produced by means of a generator $s^*(\eta, \eta')$ such that

$$\xi = -\frac{\partial s^*}{\partial \eta}, \qquad \xi' = \frac{\partial s^*}{\partial \eta'}. \tag{5·19}$$

Considering this transformation to be combined with the gauge transformation $A_1^\mu \to A_1'^\mu = A_1^\mu + \partial^\mu s^*$, the relation (5·18′) is left invariant.

It is to be noted that for the Lagrangian and Hamiltonian formulations of hydro-

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* This corresponds to (E·7) in the non-relativistic case.

** The equations of motion can equally well be derived from the Lagrangian density (5·10) regarded as a functional with respect to A_μ, R, ξ, and η, through the relation $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ and (5·18′).

*** This corresponds to the Clebsch transformation (E·3) in non-relativistic case.

dynamics it is essential that we should introduce suitable 'potential variables' such as ξ, η, and s (involving the arbitrariness of gauge), just as in the case of electrodynamics (further see Appendix E).

The quantization of our non-linear field may be difficult*), and it would lead to a far wider theory than the Pauli-Weisskopf's: Since the particular case of quasi-irrotational flows corresponds to Pauli-Weisskopf field, so the rotational flow must imply some other things (just as was stated for the non-relativistic case in FQM § 13).

The above generalization of Klein-Gordon field to include the vortical motion is not any inevitable one, as it stands merely upon the hydrodynamical *analogy*, lacking any evidences to necessitate it. But perhaps it might serve to describe mesons; and for this purpose it would be necessary to introduce the interaction with nucleon that acts as source (cf. § 7).

§ 6. On the Dirac's 'new classical theory of electrons'

We shall now take up the *limit of vanishing quantum potential* for the theory stated in the preceding section. This corresponds to the 'WKB-approximation' referred to in FQM § 15. In this case, first, the Klein-Gordon equation reduces to

$$(\partial_\mu S - eA_\mu/c)^2 + m^2c^2 = 0, \tag{4·19}$$

and (4·4), the condition (4·15) always satisfied. We have then the picture of a quasi-irrotational flow of compressible fluid with the rest density P, the velocity potential S, and with vanishing internal stress. The fundamental equations are (4·19) and (5·4), or in terms of $F^{\mu\nu}$ and u_μ (the latter is now identical with $\bar{u}_\mu$ and so plays itself the role of electromagnetic potential),

$$\begin{cases} u_\mu u^\mu = -c^2, & (6·1) \\ \partial_\mu u_\nu - \partial_\nu u_\mu = -(e/mc)F_{\mu\nu}, & (6·2) \\ \partial_\nu F^{\mu\nu} = (e/c)Pu^\mu. & (6·3) \end{cases}$$

This set of equations**) is nothing but the equations introduced in Dirac's 'new classical theory of electrons'[5)9)], putting $\lambda = mP$, and $k = mc/e$. If one adopts this classical field of quasi-irrotational flow, which is the limit of vanishing internal stress for Klein-Gordon the reasonable representation of *classical* state of motion of *electrons*, and expects the field, as reasonable theory of electrons in some proper quantization of this form of motion, it corresponds just to the Dirac's point of view in his above theory.

The simpleness of this electrodynamics is essentially due to the assumption to regard electrons as continuum and to restrict its motion to the quasi-irrotational, where the state

* cf. Appendix E for non-relativistic case.

** If we omit the $\hbar^2$-term in (5·7), introducing $u_\nu = (\partial_\nu S - eA_\nu/c)/m$ and $\lambda = mR^2$, then we get

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}{}^2 - \frac{1}{2}\lambda(u_\nu{}^2 + c^2)$$

that is identical with the Dirac's Lagrangian.

of motion of electrons is described with two continuous field variables P and S. In Dirac's form the theory is further expressed in terms of quantities having direct physical meanings. This fact can also be regarded as choosing a particular gauge, and is of rather formal significance; Dirac's theory revises the model for matter, but introduces no essential alteration to the scheme of electrodynamics, at least as long as one remains within classical theory.

As in § 5 (a), we can further eliminate $F^{\mu\nu}$ and express the equations solely in terms of matter variables R and u_μ; that is, we take (6·1) and

$$\partial_\mu \partial_\nu u^\nu - \Box u_\mu = -r_0 P u_\mu. \tag{6·4}$$

Next, for the theory of Dirac, the generalization such as to include the rotational flow is a necessity, in so far as we consider the classical real charged fluid. We take then, in place of (4·19), the Euler-Lorentz equation

$$u_\nu \partial^\nu u^\mu = (e/mc) u_\nu F^{\mu\nu} \tag{6·5}$$

that is the special case of (5·12), together with the constraint (6·1), and the Maxwell equation (6·3) with $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. Recently, Dirac[8] himself introduced the rotational flow, where he has shown the way to diminish the variables (cf. § 5, (b)).

As for the quatization of the theory of Dirac, it is to be remarked that this corresponds, for quasi-irrotational case, to starting with the Klein-Gordon field which is transformed into the hydrodynamical form and then modified by the removal of the quantum term $-\hbar^2 \Box R/R$, so it will deviate from the Pauli-Weisskopf theory; and further that the quantizatian of the general Euler's non-linear field would again lead to a far wider theory than that of Pauli-Weisskopf's type.

The intent of Dirac was to propose an improved classical theory of *electrons* with this theory, based on his view that 'the troubles of the present quantum electrodynamics should be ascribed primarily to our working from a wrong classical theory.' We do not consider this view of Dirac be fully legitimate; nevertheless the very problem of classical motions and their quantization of *charged mechanical continuous matter* interacting with electromagnetic field would be worth while to be fully investigated, and may serve to describe certain sort of states of an assembly of charged particles approximately.

This theory is a *pure* electrodynamics in the sense that it does not *assume* the point charge and only treats continuous charge distribution without any non-electromagnetic interaction, but on account of this it must bear also an *unstable* character. In the classical electron theory that takes up point charge, one is led to the divergence of self-energy, which is now avoided in the stage of and within the scope of classical theory by taking continuous model; but in the classical self-energy difficulty it is also involved that the stable distribution of charge is forbidden due to mutual electrostatic repulsion within the charge distribution, unless we assume some other cohesive force field. This same instability is taken over into the Dirac theory, and it is doubtful that this difficulty would be eliminated by quantization.

A factor that would contribute to leading the theory to stabilization is introducing another charge of reverse sign, as nature is so constructed. In order to deal with this

charge consistently in the framework of Dirac theory, we should regard it also as a continuum. When there coexist more than two kinds of charged matter, that are assumed simultaneously to occupy the space (contradicting with the classical principle of inpenetrability of matter), their motions are to be described by the following equations:

$$\begin{cases} u_{i\nu}\partial^{\nu}u_i^{\mu}=k_i^{-1}u_{i\nu}F^{\mu\nu}, & (6\cdot6)\\ u_{i\nu}u_i^{\nu}=-c^2, & (6\cdot7)\\ \partial_{\nu}(\lambda_i u_i^{\nu})=0, & (6\cdot8)\\ \partial_{\nu}F^{\mu\nu}=c^{-1}\sum_i \lambda_i u_i^{\mu}, & (6\cdot9) \end{cases} \qquad (i=1, 2, \cdots n)$$

with $F^{\mu\nu}=\partial^{[\mu}A^{\nu]}$, where $\lambda_i=e_iP_i$ and $k_i=m_ic/e_i$. Eq. (6·6) with (6·7) shows that each 4-velocity u_i^{μ} can be written as

$$k_iu_i^{\mu}=-A^{\mu}+\partial^{\mu}s_i+\xi_i\partial^{\mu}\eta_i, \qquad (6\cdot10)$$

by use of (ξ_i, η_i) satisfying

$$u_i^{\mu}\partial_{\mu}\xi_i=u_i^{\mu}\partial_{\mu}\eta_i=0. \qquad (6\cdot11)$$

Here the suitable choice of gauge enables only one of s_i's to vanish. Fundamental set of equations now becomes (6·11), (6·7), (6·8), and (6·9) for the variables $(A_{\mu}, \lambda_i, s_i, \xi_i, \eta_i)$; here $F^{\mu\nu}$ and u_i^{μ} are derived quantities through $F^{\mu\nu}=\partial^{[\mu}A^{\nu]}$ and (6·10) respectively. If one would consider A_{μ} as a derived quantity by one of (6·10) so as to eliminate the electromagnetic variables, it results in the useless destruction of the symmetry in the above set of equations. The equations of continuity (6·8) must also be retained, only one of these being derivable from (6·9)*).

It is also to be noted that such motions of several mutually interpenetrating continuous charges in the interaction with the electromagnetic field cannot be reduced, even in the special case of 'equivalent matters', $k_1=k_2=\cdots=k$, to those of a single continuum.

Finally, as a generalization of the Dirac theory in favor of stabilization, we may consider to mix a scalar field into the electrodynamics of mechanical continua (see Appendix G).

§ 7. Wave equations with sources

The possibility of representing a wave equation by the picture of trajectory ensemble such as was treated thus far depends on the linear and homogeneous property of the original wave equation. The picture of the same kind is therefore no longer applicable to wave equations with *sources*; nevertheless it is still possible to represent these cases with the similar picture generalized in such a way that there the distribution density no longer conserves along a trajectory and moreover the additional force to act on particle explicitly

* The eliminations of electromagnetic variables and of continuity equation are rather of a formal significance also for the interacting Klein-Gordon field stated in § 5 (a).

depends upon the *velocity potential*. The picture is just of similar type to the two-fluid picture for the case of wave equation with spin (FQM § 11 (c)).

As an example*) we shall treat for simplicity the Schrödinger equation with source:

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t}+H\left(x,\frac{\hbar}{i}\nabla\right)\right\}\psi=\hbar\eta. \tag{7·1}$$**)

This is transformed by $\psi=R\exp(iS/\hbar)$ and $\eta=\rho\exp(i\sigma/\hbar)$ (ρ, σ: real) into

$$\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+V-\frac{\hbar^2}{2m}\frac{\Delta R}{R}=\hbar\frac{\rho}{R}\cos\frac{S-\sigma}{\hbar}, \tag{7·3}$$

$$\frac{\partial P}{\partial t}+\operatorname{div}(P\nabla S/m)=2\rho R\sin\frac{S-\sigma}{\hbar}. \tag{7·4}$$

Suppose we regard here as before S as the velocity potential (i.e., velocity is derived by $mv=\nabla S$) and P as the distribution density, then these equations describe certain sort of motion of an irrotational velocity field of matter; and this kinematical picture further allows the following dynamical interpretation: First, the gradient of (7·3) reads

$$m\frac{dv}{dt}=-\nabla V+\nabla\left(\frac{\hbar^2}{2m}\frac{\Delta R}{R}\right)+\hbar\nabla\left(\frac{\rho}{R}\right)\cos\frac{S-\sigma}{\hbar}-\frac{\rho}{R}\sin\frac{S-\sigma}{\hbar}(mv-\nabla\sigma). \tag{7·5}$$

Eq. (7·4) is understood that the density field has source (or sink)

$$Q=2\rho R\sin\frac{S-\sigma}{\hbar}, \tag{7·6}$$

which is proportional to the original source density for the Schrödinger field, coupled with the fluctuating multiplier $\sin[(S-\sigma)/\hbar]$ dependent on the velocity potential; while eq. (7·3) shows that this sources reacts upon the particle through another new additional potential

$$V_a=-\hbar\frac{\rho}{R}\cos\frac{S-\sigma}{\hbar}=-\frac{\hbar}{2P}\{(2\rho R)^2-Q^2\}^{\frac{1}{2}}, \tag{7·7}$$

which involves a 'damping force' (as is observed in (7·5)),

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* Similar considerations are also applicable to the Klein-Gordon equation with source.

** Here the wave function is regarded as some classical charged field that will be quantized later on according to Bose statistics. The source function η is suitably determined by the wave function of the field that is interacting with ψ-field; the latter naturally reacts on the former but we neglect this and consider η as a given space time function.

The kernel equation for the Schrödinger wave function belongs to a special case of (7·1) having imaginary source only at a single space time point; that is, the kernel transferring the wave function from t_0 to t is determined by[10]

$$\{(\hbar/i)\partial/\partial t+H\}K(x,t;x_0,t_0)=(\hbar/i)\delta(t-t_0)\delta(x-x_0), \tag{7·2}$$

the solutions of which were investigated in FQM § 14.

$$-\frac{\rho}{R}m\boldsymbol{v}\sin\frac{S-\sigma}{\hbar}=-\frac{Q}{2l'}m\boldsymbol{v}. \tag{7·8}$$

Thus, in this picture, the interaction expressed with the potential V—more generally this may be the interaction due to electromagnetic field—merely acts on particle as the original classical force, whereas the *interaction that occurs as a source* η for the original field, acts (coupled with the velocity potential) as the source for the density field, and at the same time as a reaction upon particle*).

It is a remarkable fact that in the equation of motion (7·5) there enters the velocity potential explicitly**). The extent to which this circumstance requires to generalize classical-mechanical scheme is especially signified in the following fact: Given the initial velocity field $\boldsymbol{v}(\boldsymbol{x}, 0)$ irrotationally, $S(\boldsymbol{x}, 0)$ is determined, from $\nabla S(\boldsymbol{x}, 0)=m\boldsymbol{v}(\boldsymbol{x}, 0)$, up to an additive constant S_0, the arbitrariness of which must compensate with the arbitrariness of σ_0 in $\sigma(\boldsymbol{x}, 0)$. Therefore, given the initial distributions $\boldsymbol{v}(\boldsymbol{x}, 0)$ and $P(\boldsymbol{x}, 0)$, *together with the phase difference constant* $S_0-\sigma_0$, which is determined if in addition the initial value of the source Q is also given, the later distributions $\boldsymbol{v}(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$ are uniquely determined by (7·3) and (7·4)***).

When $\boldsymbol{v}(\boldsymbol{x}, t)$ is thus obtained, we can further get the irrotational ensemble of trajectories by integrating $d\boldsymbol{x}/dt=\boldsymbol{v}(\boldsymbol{x}, t)$ as before, though the density of particles moving along each individual trajectory is not conserved.

Finally we remark that the situation is essentially the same for the case of wave equation with spin. Putting $\psi_p=R_p\exp(iS_p/\hbar)$ for two-component wave function (we consider in non-relativistic approximation according to FQM § 11 (c)), we get the equations which correspond to the two-fluid picture. They explicitly involve S_p's but solely in the form S_1-S_2. (This is because the theory is invariant to the 'gauge transformation' $S_p\to S_p+\alpha$). The initial conditions that we must specify in order to determine later state of motion by means of the equations of motion, consist not only of $\boldsymbol{v}_p(\boldsymbol{x}, 0)$ and $P_p(\boldsymbol{x}, 0)$ but also of $S_{10}-S_{20}$, (the difference of arbitrary additive constants in $S_i(\boldsymbol{x}, 0)$), which is determined if the rate of transformation between the two fluids at initial time is given.

I have treated my subjects at some length, yet the analysis may require further developments in some points.

I wish to express my sincere thanks to Prof. S. Sakata and Dr. T. Inoue for helpful interests in this work, and also to Dr. O. Hara, and Dr. S. Nakajima for valuable discussions. I also wish to thank Prof. D. Bohm*) for the detailed explanation of his view

* When the source function is of singular type:

$$\eta=a\delta(\boldsymbol{x}-\boldsymbol{x}_0), \tag{7·9}$$

the source for the density field and its reaction occur only at $\boldsymbol{x}_0$.

** Seemingly, this circumstance is somewhat similar to the case of classical electrodynamics; there the equations, when expressed in terms of canonical variables for charged particles, involve not only field strengths but also their potentials, though the theory is gauge invariant.

*** Eq. (7·3) and (7·4) are invariant with respect to a 'gauge transformation' $S\to S+a$, $\sigma\to\sigma+a$.

on my previous work (FQM), though the points he discussed are not treated in the present paper.

Appendix

A. Special cases for superposed plane waves

We shall consider the trajectories in special cases for a non-stationary state in free space (2·6).

(i) Put $k_1=0$, then (2·6) reduces to

$$\psi=1+a\exp i\left(kx-\frac{\hbar k^2}{2m}t\right). \tag{A·1}$$

If we here consider a be small, (A·1) is the solution corresponding to a sound wave**) in the hydrodynamical picture:

$$\psi\simeq(1+a\cos\eta)e^{ia\sin\eta},\quad\left(\eta=kx-\frac{\hbar k^2}{2m}t\right), \tag{A·2}$$

and for this case the trajectories reduce to the fluctuating part such that

$$x-x_0\simeq-\frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}J_n(2an)\sin\left(n\frac{\hbar k^2}{2m}t\right). \tag{A·3}$$

(ii) For equal-weight superposition, $a=1$, (2·9) or (2·11) gives

$$x=x_0+\frac{\hbar}{2m}(k_1+k_2)t\,.$$

But in this special case we can readily solve the problem three-dimensionally: The ensemble consists of a density wave (i.e. interference fringes) in the $\boldsymbol{k}_1-\boldsymbol{k}_2$ direction:

$$P\propto 1+\cos\left\{(\boldsymbol{k}_1-\boldsymbol{k}_2)\boldsymbol{x}-\frac{\hbar}{2m}(k_1^2-k_2^2)t-\delta\right\}, \tag{A·4}$$

where particle moves with the constant velocity $\hbar(\boldsymbol{k}_1+\boldsymbol{k}_2)/2m$.

Incidentally, the stress in this density wave is obtained as $\sigma_{ij}=-(\hbar^2/2m)K_iK_j$, where $\boldsymbol{K}=\boldsymbol{k}_1-\boldsymbol{k}_2$. Taking the x-axis in $\boldsymbol{K}$-direction, the stress components vanish except for the component $\sigma_{xx}=-(\hbar^2/2m)K^2$. Thus there acts only a constant non-isotropic pressure in the $\boldsymbol{K}$-direction, which makes no contribution to *force*. It was already seen in FQM (14·33) that the distributions such as (A·4) are free from extra forces.

(iii) If $k_1=-k_2(=k)$, then ψ becomes a stationary state:

$$\psi=(e^{ikx}+ae^{-i(kx-\delta)})e^{-i\omega t},\quad(\omega=E/\hbar=\hbar k^2/2m).$$

The trajectories (which may now be determined by (2·3′)), are

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* Private communication to the author.

** See FQM Appendix B.

$$x-x_0=\frac{1-\alpha^2}{1+\alpha^2}\frac{\hbar k}{m}t+\frac{1}{2k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}J_n\left(\frac{2\alpha}{1+\alpha^2}n\right)\sin\left(n\frac{1-\alpha^2}{1+\alpha^2}4\omega t\right),$$

which correspond to the trajectories in such cases when a plane wave interferes with its reflected wave due to some potential hill.

B. On the attempted interpretation of quantum mechanics in terms of Markoff process

In Markoff process the probability distribution function $P(x, t)$ for the stochastic variable x directly develops with time by means of the transition probability function $T(x_0, t_0; x, t)$ such that

$$P(x, t)=\int P(x_0, t_0)T(x_0, t_0; x, t)dx_0\,, \qquad (B\cdot 1)$$

whereas in quantum mechanics the probability *amplitude* $\psi(x, t)$, which yields the probability distribution for the quantity x by the square of its absolute value, develops with time by means of the kernel of a unitary transformation such that

$$\psi(x,t)=\int K(x, t; x_0, t_0)\psi(x_0, t_0)dx_0. \qquad (B\cdot 2)$$

These relations show the general similarity and difference in the laws of temporal unfolding for both theories. The conservation relation for probability naturally holds in both theories, expressible in respective *differential* equation of continuity under certain conditions. For Markoff process it is involved in the Fokker-Planck equation:

$$\frac{\partial P}{\partial t}=-\sum_i\frac{\partial}{\partial x_i}\left\{a_iP-\sum_k\frac{\partial}{\partial x_k}(b_{ik}P)\right\}, \qquad (B\cdot 3)$$

where a_i are the first moments and b_{ik} the second moments (dispersion tensor), higher moments assumed to vanish. On the other hand in quantum mechanics the probability conservation is expressed in Eq. (1·2), in the representation that makes the positional coordinates x diagonal.

In Markoff process we shall hereafter consider the coordinates of a particle x to be the stochastic variables; then (B·3) resembles (1·2). The quantity

$$j_i=a_iP-\sum_k\frac{\partial}{\partial x_k}(b_{ik}P) \qquad (B\cdot 4)$$

means the probability current, and further we define a sort of 'local mean velocity',

$$v_i^*=j_i/P=a_i-\frac{1}{P}\sum_k\frac{\partial}{\partial x_k}(b_{ik}P). \qquad (B\cdot 5)$$

Now, in order that some Markoff process might represent quantum-mechanical motion of a particle, the Fokker equation (B·3) should correspond to (1·2) in conformity with Fényes[4], and hence v^* to $v=\nabla S/m$ of our former statistical picture though their physical

meanings are different for both quantities. This requires $\boldsymbol{v}^*$ should have its potential, and accordingly

$$\boldsymbol{a}=\nabla\bar{S}/m,\ b_{ik}=D\delta_{ik}\quad(D=\text{const.}) \tag{B·6}$$

or

$$\boldsymbol{v}^*=\nabla S^*/m,\quad S^*=\bar{S}-mD\log P. \tag{B·7}$$

Fokker's equation is now written as

$$\partial P/\partial t+\operatorname{div}(P\boldsymbol{a})-D\Delta P=0. \tag{B·8}$$

This is seen to be identical with the diffusion equation for a Brownian particle with the diffusion constant D and the drift velocity $\boldsymbol{a}=\nabla\bar{S}/m$.

In quantum mechanics, however, the state is described by the probability amplitude which involves the phase S besides the probability function P, and as the result we have the other (simultaneous) equation (1·1). In order that Markoff process should reproduce the Schrödinger equation, we must thus have the equation,

$$\dot{\bar{S}}+\frac{1}{2m}(\nabla\bar{S})^2+V-D\nabla\bar{S}\frac{\nabla R}{R}-\left(2mD^2+\frac{\hbar^2}{2m}\right)\frac{\Delta R}{R}=0,\quad(R\equiv P^{\frac{1}{2}}) \tag{B·9}$$

for the quantity $\bar{S}$. Taking the gradient we obtain

$$m\left(\frac{\partial\boldsymbol{a}}{\partial t}+(\boldsymbol{a}\nabla)\boldsymbol{a}\right)=-\nabla V+\nabla\left\{D\boldsymbol{a}\frac{\nabla R}{R}+\left(2mD^2+\frac{\hbar^2}{2m}\right)\frac{\Delta R}{R}\right\}. \tag{B·10}$$

Surely in Markoff process the first moment $\boldsymbol{a}$ may well be a function of space and time, and yet it is inexplicable that this quantity be connected with P also by the other relation (B·10)*) besides the Fokker equation.

According to the model of a diffusing Brownian particle, $\boldsymbol{a}$ means the medium velocity, and we have the picture of a Brownian particle diffusing with reference to the medium which itself flows irrotationally. Here eq. (B·10) requires that this flow depends on P in such a way as if the medium flowed under the actions of the increased quantum potential $-2m\{D+(\hbar/2m)^2\}\Delta R/R$ and also the other additional potential $-D\boldsymbol{a}\nabla R/R$. Thus it is again unreasonable that the force acting on medium in any individual experiment be determined by the probability distribution of particle for the ensemble of similar experiments repeated.

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* Fényes did not deduce this equation from some physically reasonable model; instead, he merely showed that one can derive the Schrödinger equation from the variational principle if one postulates the Lagrangian:

$$L=-\int\left\{P\dot{S}^*+\frac{1}{2m}P(\nabla S^*)^2+PV+\frac{1}{2}mD^2\frac{(\nabla P)^2}{P}\right\}d\boldsymbol{x}, \tag{B·11}$$

with

$$D=\hbar/2m. \tag{B·12}$$

But the expression (B·11) (with (B·12)) coincides with Eq. (D·10), i.e., the Lagrangian to derive the equation of motion (1·1) and (1·2). Thus to assume (B·11) involves to assume (B·9), and gives no answer to the problem.

Formally speaking, the above procedure consists of the transformation from (1·2) and (1·1) to (B·8) and (B·10) through (B·7). We could have eliminated the usual quantum potential with the choice $D=i\hbar/2m$; such imaginary diffusion constant is, however, not admitted for the Fokker equation to hold physical meaning.

As regards the attempt of Markoff-process representation of quantum mechanics, it is next to be remarked that this is based on the concept of hidden variables, and hence from the general reason remarked in § 1 it cannot lead as it is, in respect to the probability distributions of various dynamical quantities (other than particle position), to the results identical with those due to quantum mechanics. Thus to overcome this discrepancy it would be again necessary to distinguish reasonably the value to be observed from the 'real' value for a quantity such as momentum.

In the diffusion model we have a sort of 'uncertainty relation' between particle position x and the part referring to diffusion of 'local mean velocity', $\boldsymbol{u}=-D\nabla\log P$, such that

$$\Delta x_i \Delta u_i \geq D, \tag{B·13}$$

with

$$(\Delta x_i)^2 \equiv \langle (x_i-\langle x_i\rangle)^2\rangle, \quad (\Delta u_i)^2=\langle u_i^2\rangle \quad (\langle u_i\rangle=0).$$

This relation is formally identical with the uncertainty relation in quantum mechanics[11] (taking into account (B·12).), and is equivalent to the inequality for 'self-potential',

$$\langle V'/2m\rangle^{\frac{1}{2}}\cdot\Delta x \geqq \hbar/4m, \qquad \text{(FQM (5·8))}$$

in our former statistical picture, since $\langle \boldsymbol{u}^2\rangle = D^2\int P^{-1}(\nabla P)^2 d\boldsymbol{x}=\langle V'\rangle/2m$. Here it is, however, to be noted that this $\boldsymbol{u}$ (as well as $\boldsymbol{v}^*$) not only differs from the 'real' velocity of the particle in individual experiment but also cannot be regarded as the value to be measured for velocity.

Besides the 'uncertainty relation', Brownian process has some outward resemblances to quantum phenomena such as the diffusion of wave packet or the penetration of potential barriers.

Finally, we shall make a remark on the distribution function in phase space. Generally, the Boltzmann equation implies that the quantity $\dfrac{\partial f}{\partial t}+\dfrac{\partial H}{\partial \boldsymbol{p}}\dfrac{\partial f}{\partial \boldsymbol{x}}-\dfrac{\partial H}{\partial \boldsymbol{x}}\dfrac{\partial f}{\partial \boldsymbol{p}}$ taken for the one-particle distribution function $f(\boldsymbol{x},\boldsymbol{p},t)$ does not vanish (where H is the Hamiltonian for the particle plus the *external* force only), as the result of reducing the Gibbs ensemble in many-particle phase-space to the distribution in one-particle phase-space. Brownian process, which may be regarded also as a Markoff process in phase space[12], obeys the phase space Fokker equation, which is of similar nature with Boltzmann equation. On the other hand *in our statistical picture* for quantum mechanics the phase-space distribution function for the ensemble also does not obey the Liouville's theorem as was shown in eq. (1·16). It is however to be noted that the non-vanishing right-hand expressions are quite different in both cases. At any rate Markoff process and quantum phenomena are of essentially different characters in spite of some similarities.

C. Elimination of the gauge variable from the Schrödinger equation

Schrödinger equation for a spinless particle in an external electromagnetic field $(\boldsymbol{A}, \phi)$ is represented in our formulation by Eqs. (1·8) and (1·9). Employing particle velocity $\boldsymbol{v}$ and energy E defined respectively by (1·10) and (1·11), they are re-written as

$$\begin{cases} E=\dfrac{1}{2}m\boldsymbol{v}^2+V-\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}, & \text{(C·1)} \\ \dot{P}+\operatorname{div}(P\boldsymbol{v})=0. & \text{(C·2)} \end{cases}$$

It is to be noted that there the potential $(\boldsymbol{A}, \phi)$ are treated as given space-time functions, namely we consider the electromagnetic field classically, and moreover ignore the reaction of the particle on the field (differing from the treatment of interacting Klein-Gordon field in § 5).

Now to use $\boldsymbol{v}$ and E in place of $\boldsymbol{A}$, ϕ and S is regarded as a particular gauge transformation from $(\boldsymbol{A}, \phi)$ to

$$\bar{\boldsymbol{A}}=-(mc/e)\boldsymbol{v}, \qquad \bar{\phi}=-e^{-1}E, \tag{C·3}$$

the field strengths $(\boldsymbol{E}, \boldsymbol{H})$ being derived by

$$\begin{cases} \boldsymbol{H}=\operatorname{curl}\bar{\boldsymbol{A}}, & \operatorname{curl}\boldsymbol{v}=-\dfrac{e}{mc}\boldsymbol{H}, & \text{(C·4)} \\ \text{i.e.,} & & \\ \boldsymbol{E}=-\dfrac{1}{c}\dot{\bar{\boldsymbol{A}}}-\nabla\bar{\phi}, & \nabla E+m\dot{\boldsymbol{v}}=e\boldsymbol{E}. & \text{(C·5)} \end{cases}$$

Thus, $\boldsymbol{v}$ and E have twofold meanings of particle velocity and energy, and of the potential for electromagnetic field.

Given the external field $\boldsymbol{E}$ and $\boldsymbol{H}$, the motion of electron is determined by solving Eqs. (C·1), (C·2), (C·4) and (C·5), with respect to $(P, \boldsymbol{v}, E)$. This is just possible because $\boldsymbol{E}$ and $\boldsymbol{H}$ must be given not independent, but restricted by the Maxwell equations

$$\operatorname{curl}\boldsymbol{H}-c^{-1}\dot{\boldsymbol{E}}=\boldsymbol{j}^{\mathrm{ex}}, \qquad \operatorname{div}\boldsymbol{E}=\rho^{\mathrm{ex}},$$

where ρ^{ex} and $\boldsymbol{j}^{\mathrm{ex}}$ (meaning charge and current densities other than those due to the electron under consideration), are, in turn, to be given restricted by $\dot{\rho}^{\mathrm{ex}}+\operatorname{div}\boldsymbol{j}^{\mathrm{ex}}=0$.

D. Hydrodynamical model for Schrödinger field

According to the general theory of classical fields, the equations of motion are derived by means of some Lagrangian density function $\mathcal{L}(\psi_\sigma, \partial_\mu\psi_\sigma)$, such that

$$\frac{\partial\mathcal{L}}{\partial\psi_\sigma}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\sigma)}=0. \tag{D·1}$$

Further, with the Lagrangian density, are defined the canonical energy-momentum tensor T^μ_ν and the current density vector j^μ, which satisfy respective conservation laws; that is, T^μ_ν is

$$T^{\mu}_{\nu}=-\Big(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi_{\sigma})}\partial_{\nu}\psi_{\sigma}+\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi_{\sigma}^{*})}\partial_{\nu}\psi_{\sigma}^{*}\Big)+\delta_{\mu\nu}\mathcal{L}\,, \tag{D·2}$$

satisfying

$$\partial_{\mu}T^{\mu}_{\nu}=\frac{\partial\mathcal{L}}{\partial(x^{\nu})}, \tag{D·3}$$

where the right-hand side means the derivative of $\mathcal{L}$ with respect to x^{ν}, for constant ψ_{σ} and $\partial_{\mu}\psi_{\sigma}$; while j^{μ} is

$$j^{\mu}=-i\varepsilon\Big(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi_{\sigma})}\psi_{\sigma}-\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi_{\sigma}^{*})}\psi_{\sigma}^{*}\Big), \tag{D·4}$$

and satisfies

$$\partial_{\mu}j^{\mu}=0, \tag{D·5}$$

assuming the gauge invariance of the first kind. Thus the general theory consistently determines for the field, the equations of motion and quantities such as energy, momentum, and current; but it does not in itself further provide the field with any *mechanical model*.

Now, for the Schrödinger field, the Lagrangian density is known to be

$$\mathcal{L}=\frac{i\hbar}{2}(\psi^{*}\dot{\psi}-\dot{\psi}^{*}\psi)-\frac{\hbar^{2}}{2m}\nabla\psi^{*}\nabla\psi-V\psi^{*}\psi\,, \tag{D·6}$$*)

from which we obtain j_{μ}'s according to (D·3), such that

$$\begin{cases} \rho\equiv j^{0}/c=e\psi^{*}\psi, & \text{(D·7)}^{**)} \\ j^{k}=\dfrac{e\hbar}{2mi}(\psi^{*}\partial_{k}\psi-\psi\partial_{k}\psi^{*}), & \text{(D·8)} \\ \dfrac{\partial\rho}{\partial t}+\partial_{k}j^{k}=0. & \text{(D·9)} \end{cases}$$

Since, in this case, $\psi^{*}\psi$ is positive definite, it is possible to regard, not only ρ and j^{k} as charge and current, but also ρ/e and j^{k}/e as density and velocity***). This fact is well known,[13] but seems not to have been pushed forward; namely, this picture was not developed so as to explain the equation of motion or the expressions of field energy and momentum, *mechanically*.

Now, the Schrödinger equation in our hydrodynamical form (1·1) and (1·2) are derived from the Lagrangian density

$$\mathcal{L}=-P\Big\{\dot{S}+\frac{1}{2m}(\nabla S)^{2}+V\Big\}-\frac{\hbar^{2}}{8m}\frac{(\nabla P)^{2}}{P}, \tag{D·10}$$

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* We take up, for simplicity, the case where only the scalar potential V present.

** For the Schrödinger field relativistic indices are superficial, but may still be employed.

*** It is to be remarked that now ψ should not be normalized.

which is nothing but the re-expression of (D·6) by the substitution (1·3)*). From (D·6) or (D·10) we can also get, according to the general formula (D·2) and (D·3), the energy-momentum tensor*) :

$$\begin{cases} T^0{}_k/c \equiv \mathcal{G}_k = mPv_k\,, \qquad (v_k = \partial_k S/m) & \text{(D·11)} \\ T^k{}_l = mPv_kv_l + (\hbar^2/4m)P^{-1}\partial_k P\partial_l P + \delta_{kl}\mathcal{L}^{\mathrm{e}}\,, & \text{(D·12)} \\ -T^0{}_0 \equiv \mathcal{H} = P\left(\frac{1}{2}mv^2 + V\right) + (\hbar^2/8m)P^{-1}(\nabla P)^2, & \text{(D·13)} \\ -cT^k{}_0 = \mathcal{S}_k = -(P\dot{v}_k S + (\hbar^2/4m)P^{-1}\partial_k P\cdot\dot{P}), & \text{(D·14)} \end{cases}$$

the conservation relation for momentum :

$$\dot{\mathcal{G}}_l + \partial_k T^k{}_l = \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial(x^l)} = -P\partial_l V, \qquad \text{(D·15)}$$

and the energy conservation :

$$\dot{\mathcal{H}} + \partial_k \mathcal{S}_k = -\frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial(t)} = P\dot{V}. \qquad \text{(D·16)}$$

The charge and current, (D·7) and (D·8), are re-expressed as

$$\rho = eP, \qquad j^k = ePv_k\,, \qquad \text{(D·17)}$$

and the conservation relation for charge coincides with the second of our equations of motion, (1·2).

The first of our equations of motion, (1·1), is understood, as previously stated, according to the picture which not only takes P and $\nabla S/m$ as density and velocity but also introduces the mechanical stress, $\sigma_{kl} = (\hbar^2/4m)P\partial_k\partial_l(\log P)$. This picture is further consistent with the above forms for field momentum and energy. First, in virtue of the relation, $\mathcal{G}_k = (m/e)j^k$, field momentum agrees with what is given by the picture. Taking into account that for actual motions $\mathcal{L}^{\mathrm{e}} = -(\hbar^2/4m)\Delta P$, T^k_l is re-written as

$$T^k_l = mPv_kv_l + (\hbar^2/4m)(P^{-1}\partial_k P\partial_l P - \delta_{kl}\Delta P). \qquad \text{(D·12′)}$$

Inserting this, the momentum conservation relation (D·15) is expressed as

$$m\frac{\partial}{\partial t}(Pv_k) + m\partial_l(Pv_kv_l) = -P\partial_k V + \partial_l\sigma_{kl}\,, \qquad \text{(D·15′)}$$

the form given in FQM, appendix A. Next, the expression of field energy, (D·13), can be undrstood by considering the accumulation of *stress energy* in the density $(\hbar^2/8m)\times(\nabla P)^2/P = (\hbar^2/2m)(\nabla R)^2$ due to the occurring of the stress σ_{kl}.

Thus the hydrodynamical picture may be taken as sufficiently real for the classical Schrödinger field.

After the quantization of this field according to Bose statistics, the hydrodynamical form can be retained in certain measure, and the hydrodynamical picture is to be appreciated in the correspondence-theoretical significance.

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* See FQM, § 12.

E. Canonical formalism for the generalized Schrödinger field including the vortical flow

We have generalized the Schrödinger field so as to include the vortical motion by assuming the equations,

$$\begin{cases} m\dfrac{d\boldsymbol{v}}{dt}=-\nabla\left(V-\dfrac{\hbar^2}{2m}\dfrac{\Delta R}{R}\right), & \text{(E·1)} \\ \dfrac{\partial P}{\partial t}+\operatorname{div}(P\boldsymbol{v})=0, & \text{(E·2)} \end{cases}$$

that were derivable from the Schrödinger equation, to be the fundamental field equations without the condition (1·5) (FQM § 13). Now we shall somewhat supplement the previous treatment of this field.

First we attain the Lagrangian formalism for this field, utilizing the Clebsch transformation in usual hydrodynamics[14]. This is similar to what was adopted in the similar generalization of Klein-Gordon field (§ 5 (b)). Putting

$$\boldsymbol{v}=-\nabla s-\xi\nabla\eta, \qquad \text{(E·3)}$$

we transform the components of $\boldsymbol{v}$ to the new variables ξ, η, and s. This expression (E·3) is invariant to the '*canonical transformation*' $(\xi, \eta, s)\to(\xi', \eta', s')$ such that

$$\xi=-\frac{\partial w}{\partial\eta}, \qquad \xi'=\frac{\partial w}{\partial\eta'}, \qquad s'=s-w. \qquad \text{(E·4)}$$

This, of course, also leaves $\partial s/\partial t+\xi\partial\eta/\partial t$ invariant. The variables ξ and η refer to vortical motion; thus from (E·3) the roticity is expressed to be

$$\operatorname{curl}\boldsymbol{v}=[\nabla\xi\times\nabla\eta], \qquad \text{i.e.} \qquad \operatorname{curl}_i\boldsymbol{v}=\frac{\partial(\xi,\eta)}{\partial(x_k, x_l)}. \qquad \text{(E·5)}^{*)}$$

If either ξ or η be constant, the motion reduces to the irrotational.

Now in the new variables the equations of motion are written as

$$\begin{cases} \dfrac{ds}{dt}=-\dfrac{1}{2}\boldsymbol{v}^2+V/m-\dfrac{\hbar^2}{2m^2}\dfrac{\Delta R}{R}, & \text{(E·6)} \\ \dfrac{d\xi}{dt}=\dfrac{d\eta}{dt}=0 & \text{(E·7)} \end{cases}$$

together with (E·2). It is because (E·6) and (E·7) turn into (E·1) through the relation

$$\frac{d\boldsymbol{v}}{dt}=-\nabla\left(\frac{ds}{dt}+\frac{1}{2}\boldsymbol{v}^2\right)-\xi\nabla\frac{d\eta}{dt}-\nabla\xi\frac{d\eta}{dt}, \qquad \text{(E·8)}$$

which is derived from (E·3).

---

* (i, k, l) is an even permutation of (1, 2, 3).

Here, the eqs. (E·2), (E·6), and (E·7) are derivable from the variational principle, by taking the following Lagrangian density function :

$$\mathcal{L}[P, s, \xi, \eta] = mP\left(-\frac{1}{2}\boldsymbol{v}^2 + \frac{\partial s}{\partial t} + \xi\frac{\partial \eta}{\partial t} - V/m\right) - \frac{\hbar^2}{8m}\frac{(\nabla P)^2}{P}, \qquad (E\cdot 9)$$

$$= mP\left(\frac{1}{2}\boldsymbol{v}^2 + \frac{ds}{dt} + \xi\frac{d\eta}{dt} - V/m\right) - \frac{\hbar^2}{8m}\frac{(\nabla P)^2}{P}, \qquad (E\cdot 9')^{*)}$$

which is obtainable from $\mathcal{L}$ for the irrotational case, (D·10), by replacing $\partial_\mu S/m$ with $-(\partial_\mu s + \xi\partial_\mu \eta)$.

By the use of (E·9), we can calculate the energy-momentum tensor T_ν^μ ; for instance, the energy density is

$$\mathcal{H} = P\left(\frac{1}{2}m\boldsymbol{v}^2 + V\right) + \frac{\hbar^2}{8m}\frac{(\nabla P)^2}{P}, \qquad (D\cdot 13)$$

and momentum density :

$$\mathcal{G} = mP\boldsymbol{v}. \qquad (D\cdot 11)$$

Other components T_l^k and $\mathcal{S}_k$ are also obtained. They are all in agreement with the respective expressions for the irrotational case (D·11)—(D·14), except for the fact that now $\dot{S}/m$ in the previous expressions is replaced with $-(\dot{s} + \dot{\xi}\dot{\eta})$. Again for actual motions $\mathcal{L} = -(\hbar^2/4m)\Delta P$, and the conservation relation for momentum agrees with (D·15′).

Next we make transition to the Hamiltonian formalism : From (E·9), we find the canonical momenta to be

$$\begin{cases} \pi_s = mP, & \pi_P = 0, \\ \pi_\xi = 0, & \pi_\eta = mP\xi. \end{cases} \qquad (E\cdot 10)$$

The Hamiltonian (D·13), which is regarded as a functional of s, η, π_s, and π_η, with P, ξ, π_P and π_ξ eliminated, such that

$$\mathcal{H}[s, \eta, \pi_s, \pi_\eta] = \pi_s\left\{\frac{1}{2}\left(\nabla s + \frac{\pi_\eta}{\pi_s}\nabla\eta\right)^2 + V/m\right\} + \frac{\hbar^2}{8m}\frac{(\nabla\pi_s)^2}{\pi_s} \qquad (E\cdot 11)$$

again reproduces (E·2, 6, 7) as its Hamilton equations.

The first step of quantization is formally to put the commutation relations :

$$[s, \pi_s'] = [\eta, \pi_\eta'] = i\hbar\delta(x - x'). \qquad (E\cdot 12)$$

In respect to the current vector,

$$\boldsymbol{j} = mP\boldsymbol{v} = \mathcal{G} = -\pi_s\nabla s - \pi_\eta\nabla\eta, \qquad (E\cdot 13)$$

we can further calculate the commutation relations between J and j_k's First we get

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* For this $\mathcal{L}$, we can also vary P, S, ξ, η, and x independently, then we get the equations of motion together with (E·3).

$$[\boldsymbol{j}, P']=(\hbar/i)P\nabla\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{E·14}$$

from which is derived

$$[\boldsymbol{j}, P']+[\boldsymbol{j}', P]=i\hbar\nabla P\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{E·15}$$

Further we get

$$[j_i, j_k']+[j_i', j_k]=i\hbar\,\mathrm{curl}_{ik}\boldsymbol{j}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{E·16}$$

The relations (E·15) and (E·16) coincide with what were obtained by London[15].

Appendix

F. Pressure potential in the hydrodynamical representation of Klein-Gordon field, etc.

(a) We shall consider the explicit representation of the Klein-Gordon field in hydrodynamical form. This is possible because, first the ensemble of particle trajectories under a scalar field, considered in § 4, is formally equivalent to a motion of continuous matter under the scalar field; and second, the latter motion for the present type of scalar force agrees with a particular case of relativistic hydrodynamics: The equation of motion of a continuous matter, acted on by a scalar field U and at the same time by the electromagnetic field $F^{\mu\nu}$, is obtained from (4·10) as

$$\frac{d}{d\tau}\{(1+f_1U/c^2)u^\mu\}=\frac{e_1}{c}u_\nu F^{\mu\nu}-f_1\partial^\mu U, \tag{F·1}$$

where e_1 and f_1 are specific values*) of the coupling constants with the electromagnetic and scalar fields respectively. We have besides the equation of continuity,

$$\partial_\nu(\mu_0 u^\nu)=0, \tag{F·2}$$

with μ_0 as the rest mass density*). On the other hand the relativistic flow of fluid with pressure potential Π is described by the Euler equation,

$$(1+\Pi/c^2)\dot{u}_\mu=-(\partial_\mu\dot{\Pi}+u_\mu\Pi/c^2), \tag{F·3}$$[16]

and by (F·2). Now eq. (F·1) agrees with (F·3) with the pressure potential $\Pi=f_1U$, also assuming the Lorentz force acting. In our case U is determined by (4·14), and therefore we can regard Eqs. (4·3) and (4·4) as representing a relativistic 'quasi-irrotational' flow with the rest mass density $\mu_0=mP_0=m\gamma P$ and the pressure potential $\Pi=c^2(\gamma-1)$.

Now both eqs. (F·3) and (F·2) can be derived from the equation,

$$\partial_\nu\{\mu_0(1+\Pi/c^2)u^\mu u^\nu\}+\mu_0\partial^\mu\Pi=0, \tag{F·5}$$

which means the equation of energy-momentum conservation. As for our case, (4·3) (4·4) are derived from

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* $e_1=e/m$, $f_1=f/m$, $\mu_0=mP_0$.

$$\partial_\nu(mP\bar{u}^\mu\bar{u}^\nu)=P\left\{\frac{e}{c}\bar{u}_\nu F^{\mu\nu}+\frac{\hbar^2}{2m}\partial^\mu\left(\frac{\Box R}{R}\right)\right\}, \tag{F·6}$$

and the condition (5·1). Here

$$T^{\mu\nu}_{(M)}=mP\bar{u}^\mu\bar{u}^\nu=mP_0\gamma\, u^\mu u^\nu \tag{F·7}$$

means the energy-momentum tensor for matter. Eq. (F·7) can be rewritten as

$$\partial_\nu(mP\bar{u}^\mu\bar{u}^\nu)\,P_0\frac{e}{c}u_\nu F^{\mu\nu}+\partial_\nu\sigma^{\mu\nu}, \tag{F·8}$$

with

$$\sigma^{\mu\nu}=(\hbar^2/2m)\,(R\partial^\mu\partial^\nu R-\partial^\mu R\partial^\nu R)=(\hbar^2/4m)P\partial^\mu\partial^\nu(\log P). \tag{F·9}$$

Thus we can also use the *stress tensor* $\sigma^{\mu\nu}$ in place of the pressure potential Π for the hydrodynamical picture of Klein-Gordon field.

Next we shall obtain the energy-momentum tensor for the case of the hydrodynamical field generalized as in § 5 (b). From the Lagrangian density (5·10) (with (5·17)) we can obtain this tensor T^μ_ν (symmetrized by adding a divergence-free tensor $k\partial_\lambda(F^{\mu\lambda}\bar{\eta}_\nu)$),

$$T^\mu_\nu=T^\mu_{K\nu}+T^\mu_{M\nu}+T'^\mu_\nu \tag{F·10}$$

satisfying the energy-momentum conservation $\partial_\mu T^\mu_\nu=0$, where

$$T^\mu_{K\nu}=F^{\mu\sigma}F_{\nu\sigma}-\delta_{\mu\nu}F^2_{\sigma\rho}/4 \tag{F·11}$$

is the energy-momentum tensor for the electromagnetic field, $T^\mu_{M\nu}$ was given by (F·7), and

$$T'^\mu_\nu=(\hbar^2/m)[\partial^\mu R\partial_\nu R-\delta_{\mu\nu}(2^{-1})\{(\partial_\lambda R)^2+R\Box R\}],$$

which agrees with $-\sigma^\mu_\nu$ (Eq. (F·9)) apart from a divergence-free part. Thus the relation $\partial_\mu T^\mu_\nu=0$ agrees with (F·8). (b) We now consider a general condition to let a hydrodynamical field correspond to any linear field. For such correspondence it is first required that we must be able to introduce density and velocity variables which satisfy the matter conservation (F·2). (cf., however, § 7 and FQM § 11 (c)). Now a linear complex field possesses the current vector j^μ, which is defined by (D·4) and satisfies the charge conservation relation (D·5). We are then to reinterpret this relation as the matter conservation by introducing the 4-velocity field u^μ as the 'normalized' quantity for j^μ; i.e., we define u^μ by suitably multiplying j^μ by a factor (which is to be identified with the reciprocal of the rest density P_0) such that $u_\mu u^\mu=-c^2$, i.e.,

$$u^\mu=cj^\mu/(-j_\mu j^\mu)^{1/2}, \qquad P_0=\text{const}\,(-j_\mu j^\mu)^{1/2}. \tag{F·12}$$

In case of Klein-Gordon field we have $j^\mu=P(\partial^\mu S-eA^\mu/c)$, and so

$$(-j_\mu j^\mu)^{1/2}=P[-(\partial_\mu S-eA_\mu/c)^2]^{1/2}=mcP(1-\lambda_C^2\Box R/R)^{1/2}$$

in virtue of (4·3). Hence we obtain $u^\mu=(\partial^\mu S-eA^\mu/c)/m\gamma$ and $P_0=\gamma P$, the same result as before.

On the other hand, in case of the Dirac field for example, we have current vectors:

$j_\mu \propto \bar{\psi}\gamma_\mu\psi$, $j_\mu^{(0)} \propto \left(\frac{\hbar}{i}\partial_\mu + \frac{e}{c}A_\mu\right)\bar{\psi}\cdot\psi - \bar{\psi}\left(\frac{\hbar}{i}\partial_\mu - \frac{e}{c}A_\mu\right)\psi$, or $j_\mu^{(1)} \propto \partial^\nu(\bar{\psi}\gamma_\mu\gamma_\nu\psi)$ with $\bar{\psi}=\psi^*\gamma_4$; and so we can as well define u_μ and P_0 by (F·12), using any one of these current vectors. Given a solution of the Dirac equation, we can further obtain the trajectories of fluidal particles by integrating $dx^\mu/d\tau = cj^\mu/(-j_\mu j^\mu)^{1/2}$, in similar manner as in § 2 (a). Such a hydrodynamical field, however, does not represent an equivalent of the original Dirac field, because the former consists of four independent variables whereas the latter involves eight independent (real) variables. In case of a complex scalar field, it involves only two independent (real) variables and so was able just to correspond to an *irrotational* hydrodynamical field.

G. Classical theory of mixed fields for continuous matter

We shall mix a scalar field with the electromagnetic field, both of which are interacting with continuous mechanical matter. We have previously shown that the Klein-Gordon field is represented with a motion of continuous matter interacting with the electromagnetic field and also with an extra scalar force field. But in this case the scalar field was interpretable as the internal stress of matter without its own freedoms of motion; whereas the field that we are going to introduce is a scalar field which itself obeys the *real* Klein-Gordon equation, inducing the short-range attraction between sources.

First we shall consider the equation of motion of continuous matter acted on both by the electromagnetic field $F^{\mu\nu}$ and any scalar field U. But this is already obtained as (F·1) and (F·2). Now introducing

$$\gamma = 1 + f_1 U, \qquad \bar{n}^\mu = \gamma n^\mu, \qquad \mu^* = \mu_0/\gamma, \tag{G·1}$$ *)

and applying the method similar to that of § 5 (b), these equations of motion can be transformed into

$$\begin{cases} \bar{u}^\nu\partial_\nu\xi = \bar{u}^\nu\partial_\nu\eta = 0, & \text{(G·2)} \\ \partial_\nu(\mu^*\bar{n}^\nu) = 0, & \text{(G·3)} \\ \bar{n}_\nu^0 = -\gamma^0, & \text{(G·4)} \end{cases}$$

with the relation

$$\bar{n}^\nu = -e_1 A^\nu + \xi\partial^\nu\eta + \partial^\nu s. \tag{G·5}$$

They are derivable from the Lagrangian density

$$\mathcal{L}_M[\mu^*, \xi, \eta, s] = -\frac{1}{2}\mu^*(\bar{n}_\nu^0 + \gamma^0). \tag{G·6}$$

Here we consider the scalar field U obeys the Klein-Gordon equation, then we are to take the total Lagrangian density as

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* In this subsection we use the unit such that $c=1$.

with

$$\left.\begin{aligned} &\mathcal{L}=\mathcal{L}_R+\mathcal{L}_S+\mathcal{L}_M \\ &\mathcal{L}_R=-\frac{1}{4}(F^{\mu\nu})^2, \qquad \mathcal{L}_S=-\frac{1}{2}((\partial_\nu U)^2+\varkappa_S^2U^2), \end{aligned}\right\} \tag{G·7}^{*)}$$

where $\mathcal{L}$ is considered to be a functional with respect to P, ξ, η, A^μ, and U, through (G·5), in which the last term $\partial^\nu s$ can now be omitted. The variation yields

$$(\Box-\varkappa_S^2)\,U=f_1\mu_0, \tag{G·8}$$

$$\partial_\nu F^{\mu\nu}=e_1\mu_0u^\mu, \tag{G·9}$$

together with (G·2, 3, 4), i.e., (5·1) and (5·2), as the fundamental set of equations. From (G·7) we also obtain the symmetrized energy-momentum tensor as

$$T^\mu_\nu=T^\mu_{M\nu}+T^\mu_{R\nu}+T^\mu_{S\nu}, \tag{G·10}$$

where $T^\mu_{M\nu}$ is given by (5·7) (i.e., $T^\mu_{M\nu}=\mu^*\bar{u}^\mu\bar{u}^\nu$), $T^\mu_{R\nu}$ by (F·11), and

$$T^\mu_{S\nu}=\partial^\mu U\,\partial_\nu U-\delta_{\mu\nu}(1/2)\,((\partial_\lambda U)^2+\varkappa_S^2U^2). \tag{G·11}$$

This formalism is different from the so-called C-meson theory[17)18)] where the source is also represented by some linear field equation to describe some sort of elementary particle according to the usual formalism, while in the present theory the source is treated as a hydrodynamical matter. In this theory, the limit, where the mass distribution contracts to a point, corresponds to the case treated by Stueckelberg[18)19)] as a classical model of electron. Here the self-energy of electron becomes finite if one put $e^2=f^2$. Similarly in the case of general distribution of mass, we may expect some stabilizing effect.

It might also be conceivable to apply this type of formalism to nucleon in the viewpoint which considers the nucleon as extended continuous source, and this would imply somewhat similar picture to the strong-coupling theory.

Note added in proof, to § 2 (b). In connection with the idea of a stochastic mechanism, we may recall the method of the classical electrodynamics of Wheeler and Feynman[20)]. This theory resorts to absorbers, which may be regarded, in a sense, as *hidden variables*; and it analyses the *radiative reaction* upon an accelerated charge, $(2e^2/3c^3)\,\ddot{v}$, into the *statistical* result of the advanced action due to absorber. We might desire also in our case to subject, in similar fashion, the quantum potential $-(\hbar^2/2m)\,\Delta R/R$ to further analysis,**) but this seems not to be feasible from the same reason stated in § 2 (b).

By the way, the Wheeler-Feynman theory eliminates the freedoms of the field perfectly (not merely the scalar and longitudinal components), degrading them to the retarded and advanced *potentials*; but it must in return introduce absorbers, which may be hidden variables for the usual electrodynamics. On the other hand for the Wheeler-Feynman theory the freedoms of the electromagnetic *field* may be said to be hidden variables.

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* Here the interaction with the scalar field is involved in $\mathcal{L}_M$ as $\mathcal{L}'=-\mu^*f_1U\left(1+\frac{1}{2}f_1U\right)=-\mu_0f_1U\times\frac{1+f_1U/2}{1+f_1U}$.

** This was remarked to the author by Prof. S. Sakata.

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New Formulation of One-body Problem in Quantum Electrodynamics

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One electron states at time t are reasonably defined and the space spanned by them is denoted by $\mathfrak{H}_1^e(t)$. One electron problem is clearly formulated using $\mathfrak{H}_1^e$. The motion of the projection of a given state Ψ onto $\mathfrak{H}_1^e$ is described by the "wave function". If Ψ is in $\mathfrak{H}_1^e(-\infty)$ the "wave function" satisfies the equation given by Schwinger. The probability of staying in one-electron state and the expectation values of physical quantities are written as the generalized form of ordinary quantum mechanics so as to indicate the non-local character of our theory. The expectation value of energy has the imaginary part concerned with the damping property. The application to the problem of Lamb shift and line breadth is made for example. Divergence difficulties are not concerned.

§ 1. Introduction

Recently, the relativistic problem of bound systems has been treated by several authors. Bethe and Salpeter[1] and Kita[2] obtained the relativistic equation for two-body problems by a method analogous to Feynman's one. The same equation was deduced by Gell-Mann and Low[3] from the ordinary field theory and also by Schwinger in his new theory of the Green function.[4] Common characteristics of these works are the appearance of the wave function and the kernel or the Green function both of which are the functions of coordinates of particles of definite number. But the physical meaning of these quantities is not so obvious in their works, and, for instance, we do not know to what boundary condition the wave function must be subjected or how to calculate physical quantities from it. No doubt, the concept of the wave function in these theories corresponds to what was introduced in the instructive work of Feynman.[5] But, as it seems to us, his interpretation of this concept is only applicable to the case of scattering problems. Further investigations are required in order to apply it to the problem of bound systems, although some authors look on it simply as that of the ordinary point quantum mechanics. These circumstances must have been realized by Kita, who based his theory on Heisenberg's S-matrix theory. In his theory the wave function is nothing but a mathematical tool and there is no need of physical interpretation.

In this paper we shall attempt to investigate the corpuscular aspect of electrons in quantum electrodynamics and then to give a reasonable wave function which will turn out

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to be identical with the above one.*)

Now, the prototype of quantum field theory is the quantized Schrödinger field,[6] whose corpuscular aspect was most elegantly shown by Fock.[7] To obtain a reasonable meaning of the wave function in the more modern theories, we shall only need to generalize Fock's treatment. In this sense, the present paper may be said to be a generalization of Fock's work.

One of the characteristics of Fock's theory is that the number of particles is the constant of motion. Owing to this fact the norm of the wave function of the definite number of particles is conserved. This does not generally hold in more general theories, which admit creation and annihilation of particles. The hydrogen atom, for instance, which is composed of one electron and one proton, always has the possibility of photon emission, so that the probability of staying in the two-particle state (one electron, one proton and no photon) will be damped. The stationary character cannot appear unless transitions to other number states are forbidden by the conservation law of energy or other absolute selection rules.

In case of relativistic field another complication appears. The construction of the rigorous conception of the position of a particle seems to be impossible even if no interaction with other fields is present.[8] So it may be more convenient to use the momentum representation, because the momentum of a particle has a sharp meaning under very general conditions.

The construction of the "wave function" from quantum field theory was first attempted by Gell-Mann and Low[8]. But there seems to be some contradictions in their theory. As their work is closely related to Schwinger's work, we shall here give a brief survey of Schwinger's work and then discuss the work of Gell-Mann and Low.**)

Schwinger[9] defined the Green function $G(xt, x't')$ and deduced an equation for it by his new formalism.[9] Following Gell-Mann and Low, and Utiyama et al,[10] we shall write down the definition and the equation in the following way:

$$G(x, t;\ x', t') = i(\Psi_0,\ T(\psi_H(x, t)\bar{\psi}_H(x', t'))\Psi_0)$$
$$= \frac{i}{S_v}(\Phi_0,\ T(S(\infty, -\infty)\psi_I(x, t)\bar{\psi}_I(x', t'))\Phi_0), \qquad (1.1)$$

where

$$S_v = (\Phi_0,\ S(\infty, -\infty)\Phi_0)$$

and

$$\left[\gamma^\mu(-i\partial_\mu - e\langle A_\mu(x)\rangle) + m + ic\gamma^\mu\frac{\delta}{\delta f_\mu(x)}\right]G(x, x') = \delta(x-x'), \qquad (1.2)$$

where

*) We shall treat here one-body problem in a given static external field. The restriction to the static external field may be thought to be unsatisfactory from the relativistic point of view. But as the concept of the external field is only conventional, we do not care about time dependent external fields. Formal generalization to include it is not difficult. (See footnote on p. 226).

**) For simplicity we translate their work to one-body problem.

$$\langle A_\mu(x)\rangle = (\Psi_0, A_{H\mu}(x)\Psi_0) = \frac{1}{S_v}(\Phi_0, T(S(\infty,-\infty)A_{I\mu}(x))\Phi_0). \tag{1.3}$$

Here suffices I and H denote the interaction and Heisenberg representation. Φ_0 and Ψ_0 are the free- and true-vacuum state, and S is the transformation operator in the interaction representation. We use Schwinger's γ^μ and the natural unit.

If we assume the adiabatic switch on and off process at the remote past and future respectively, it holds[10]

$$\left(m+ie\gamma^\mu\frac{\delta}{\delta J_\mu(x)}\right)G(x,x') = \int M(x,x'')G(x'',x')d^4x''. \tag{1.4}$$

Therefore, using matrix notations treating x and x' as indices, (1.2) is rewritten in a simple form

$$(\gamma\Pi + M)G = 1. \tag{1.5}$$

Then, Shwinger introduced the wave function $\chi(x)$ as a function satisfying the corresponding homogeneous equation

$$(\gamma\Pi + M)\chi = 0. \tag{1.6}$$

Gell-Mann and Low assume the bound state in question as the eigen-state Ψ_n of the total energy and define the wave function as follows;

$$\chi_n(\boldsymbol{x}, t) = (\Psi_0, \psi_H(\boldsymbol{x}, t)\Psi_n). \tag{1.7}$$

Then, using the completeness of Ψ_n and the special limiting process (which we call "L-operation"), they obtained from eq. (1.5) the very equation (1.6) for χ_n.

We shall now discuss some contradictions in their theory. By virtue of the definition (1.7), χ_n oscillates harmonically in time, and yet (1.6) seems to admit damping solutions in some cases. (See Appendix B). Furthermore, the direct deduction from the definition (1.7) leads to

$$\left[\gamma^\mu(-i\partial_\mu - e(\Psi_0, A_{H\mu}(x)\Psi_n)) + m + ie\gamma_\mu\frac{\delta}{\delta J_\mu(x)}\right]\chi(x) = 0, \tag{1.8}$$

which is not coincident with (1.6). These contradictions seem to be due to the following fact. The essential point of their deduction is the commutativity of the "L-operation" and the mass operator M of (1.5). This is justified because the "L-operation" is related to the time interval (from -2∞ to $-\infty$) whereas M is related to the nonoverlapping one (from $-\infty$ to $+\infty$). Now, M is only defined assuming the adiabatic switch-on process, so the orthogonality of χ_n obtained by the "L-operation" refers to the states before switching. Unless many to one- or one to one-correspodence between free states and true states is ensured, the "L-operation" is not capable of obtaining the equation for a single χ_n.

Although the stationary treatment is justified in case of scattering where we can assume a constant supply of incident particles, it is natural to take into account the damping character to be able to discuss the physical bound states as asserted before.

§ 2 One-electron (positron) staet

The ordinary wave function of the point quantum mechanics is interpreted as the probability amplitude of the existence of the particle at a given point. To define a reasonable wave function in our theory, we must first construct the state of an electron existing at point x at time t, using the concept of quantum electrodynamics. It is natural to require for such a state the following conditions: (i) It is the eigen-state of the total charge operator Q belonging to the eigen-value $-e$; (ii) it admits some intuitive interpretation of existence at x; and (iii) the concept of this state vector should not contradict with the uniformity of space-time.

As the theory is usually treated by the perturbation method, it might appear that we should adopt $\psi_{I\alpha}^*(x,t)\Phi_0$ for such a state. Obviously, it satisfies our first condition. As the second and third conditions, we first envisage the expectation value of the charge density at another point x' and at the same time t in order to get a clue to the required interpretation. This expectation value turns out to depend only on $x-x'$ not on t and to be nearly equal to zero for $|x-x'|>\hbar/mc$ (See Appendix A.), so that our second and third postulates are satisfied. It includes, however, no effect of the electro-magnetic field, or, in other words, it represents only a "bare" electron existing at x and t. A fourth condition must be added expressing the electron be a "physical" electron: (iv) The state must include the electromagnetic effect in such a manner that when the charge constant is zero it reduces to $\psi_{I\alpha}^*(x,t)\Phi_0$.

If we assume the existence of the true vacuum state Ψ_0,*) which is the common eigen-state of the total energy-momentum operator $(H, \boldsymbol{P})$ and the total charge operator Q belonging to the lowest eigen-value of H and the null eigen-value of Q, a most favorable candidate for the conditions (i)—(iv) seems to be the state $\psi_{H\alpha}^*(x,t)\Psi_0$. Apparently, it satisfies the condition (i) and (iv). Next, we shall consider the expectation value of the charge density operator;

$$\langle\rho(x',t)\rangle=(\Psi_0,\psi_{H\alpha}(x,t)\rho_H(x',t)\psi_{H\alpha}^*(x,t)\Psi_0). \tag{2.1}$$

It is easily proved that it satisfies the following relations:

$$\frac{\partial}{\partial t}\langle\rho(x',t)\rangle_{x,t}=0,\quad\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\langle\rho(x',t)\rangle_{x,t}=0. \tag{2.2}$$

This shows that the condition (iii) is also satisfied. Concerning the condition (ii), a complete proof will be hopeless because we must calculate $\langle\rho\rangle$ exactly. But, as the analogous expression to (2.1) in case of non-relativistic Schrödinger field**) is always δ-func-

*) In case of very strong external fields the existence of Ψ_0 may not be ensured. If time-dependent external field is included, H is time dependent and we can not take the common Ψ_0 for every $H(t)$. But as we may be able to use the true vacuum at particular time, our procedure will be applied to the case of more general external field though it is meaningless for rapidly varying fields. The first relation of (2.2) and time-independence of $A(p)$ (§ 3.) are destroyed.

**) In Schrödinger case, Ψ_0 is not the lowest energy state but the state belonging to the null eigen-value of H.

tion regardless the existence of electro-magnetic field, so it seems to be very probable that $\langle\rho\rangle$ is not essentially modified by the electro-magnetic effect. This is ascertained to the first several terms of the perturbation calculus (Appendix A).

We shall hereafter call the state $\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0$ the state of physical electron with "polarization" α existing at $\boldsymbol{x}$ and t. Here the word "polarization" is introduced to distinguish four states for given $\boldsymbol{x}$ and t. It is natural to define the space of one electron state at time t (denoted as $\mathfrak{H}_1^e(t)$) as the space spanned by these $\psi_{H\alpha}(\boldsymbol{x}, t)\Psi_0$'s. $(\alpha=1, 2, 3, 4\,; -\infty<\boldsymbol{x}<+\infty)$.

A state

$$\sum_{\alpha}\int\psi_{H\alpha}^*(\boldsymbol{x}, t)\,\Psi_0 d^3\boldsymbol{x}\, f_\alpha(x) \tag{2.3}$$

is interpreted as representing the physical electron at t being distributed with the "weight" $f(\boldsymbol{x})$.

It is worth while to show here the essential difference of the Schrödinger- and Dirac-field stated in § 1. The state $\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0$ is not orthogonal to the state $\psi_{H\alpha}^*(\boldsymbol{x}'t)\Psi_0$, so that there is no operator of which the set of eigen-states coincides with that of all $\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0$'s. On the contrary $\psi_H^*(\boldsymbol{x}, t)\Psi_0$ in Schrödinger case is apparently the eigen-state of the position operator $\boldsymbol{X}=\int\psi_H^*(\boldsymbol{x}, t)\boldsymbol{x}\,\psi_H(\boldsymbol{x}, t)\,d^3\boldsymbol{x}$. That $\langle\rho\rangle$ is approximately zero for $|\boldsymbol{x}-\boldsymbol{x}'|>\hbar/mc$ seems to correspond to the well-known uncertainty of the position of a Dirac electron deduced from the thought experiment using the γ-ray microscope.[8]

The space of one-positron state at time t (denoted as $\mathfrak{H}_1^p(t)$) may be analogously constructed. We need only to replace the state $\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0$ by the state $\psi_{H\alpha}(\boldsymbol{x}, t)\Psi_0$ representing a physical positron with polarization α existing at $\boldsymbol{x}$ and t. The important fact to be noted is that $\mathfrak{H}_1^e$ is orthogonal to $\mathfrak{H}_1^p$ because they belong to the different eigen-values of the total charge operator Q. The requirement of the existence of Ψ_0 is strong enough to establish the distinction of the electron and the positron.

In order to discuss further about $\mathfrak{H}_1^e$, it is more convenient to translate the above results to the "momentum" representation. We take $(1/\sqrt{2\pi})^3\delta_\alpha^\beta\exp(i\boldsymbol{p}\cdot\boldsymbol{x})$ for $f_\beta(\boldsymbol{x})$ in (2.3), then we get the state of the physical electron with polarization α distributed with the weight $\exp(i\boldsymbol{p}\cdot\boldsymbol{x})$. It becomes to

$$\frac{1}{\sqrt{(2\pi)^3}}\sum_{\beta}\int\psi_{H\beta}^*(\boldsymbol{x}, t)\Psi_0 d^3\boldsymbol{x}\delta_\alpha^\beta\exp(i\boldsymbol{p}\cdot\boldsymbol{x})=\psi^{H\alpha*}(\boldsymbol{p}, t)\Psi_0. \tag{2.4}$$

It may be more appropriate to call it "the state of a physical electron with momentum $\boldsymbol{p}$ and polarization α existing at t".

In contrast with the case of $\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0$, it is the eigen-state of the total momentum operator $\boldsymbol{P}$, because of the relation

$$(\boldsymbol{P}-\boldsymbol{P}_0)\psi_{H\alpha}^*(\boldsymbol{p}, t)\Psi_0=\boldsymbol{p}\psi_{H\alpha}^*(\boldsymbol{p}, t)\Psi_0, \tag{2.5}$$

where $\boldsymbol{P}_0$ is the eigen-value of $\boldsymbol{P}$ belonging to Ψ_0 and where we have assumed an admissible condition $\lim_{|\boldsymbol{x}|\to\infty}\psi_{H\alpha}^*(\boldsymbol{x}, t)\Psi_0=0$. As $\boldsymbol{P}$ is hermitian, the states $\psi_{H\alpha}^*(\boldsymbol{p}, t)\Psi_0$ and

$\psi_{H\beta}{}^*(\boldsymbol{p}', t)\Psi_0$ are orthogonal if $\boldsymbol{p} \neq \boldsymbol{p}'$. So their product should have $\delta(\boldsymbol{p}-\boldsymbol{p}')$ as a factor, *i. e.*;

$$(\Psi_0,\ \psi_{H\alpha}(\boldsymbol{p}, t)\psi_{H\beta}{}^*(\boldsymbol{p}', t)\Psi_0) = \Lambda^+_{\alpha\beta}(\boldsymbol{p})\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{2.6}$$

Time independence and hermitian character of Λ^+ can be easily proved.

If we define Λ^- by

$$(\Psi_0, \psi_{H\beta}{}^*(\boldsymbol{p}', t)\psi_{H\alpha}(\boldsymbol{p}, t)\Psi_0) = \Lambda^-_{\alpha\beta}(\boldsymbol{p})\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{2.6'}$$

the relation

$$\Lambda^+(\boldsymbol{p}) + \Lambda^-(\boldsymbol{p}) = 1$$

is easily obtained. The precise form of $\Lambda^\pm$ cannot be obtained until we solve quantum electrodynamics completely. But we need not bother about it. We shall now construct the projection operator $\Gamma_+(t)$ onto $\mathfrak{H}^e_1(t)$. As Λ^+ is hermitian, there exists a unitary matrix $U(\boldsymbol{p})$, which transform Λ^+ to the diagonal form;

$$\sum U_{\alpha\beta}(\boldsymbol{p})\Lambda^+_{\beta\gamma}(\boldsymbol{p})U^*{}_{\gamma\delta}(\boldsymbol{p}) = C_\alpha(\boldsymbol{p})\delta_{\alpha\delta}, \tag{2.7}$$

where some C_α's may be zero.

We introduce $\phi_\alpha(\boldsymbol{p}t)$ by

$$\phi(\boldsymbol{p}t) = U(\boldsymbol{p})\psi(\boldsymbol{p}, t),\ \phi^*(\boldsymbol{p}, t) = \psi^*(\boldsymbol{p}, t)U^*(\boldsymbol{p}). \tag{2.8}$$

Then, the states $\phi_\alpha{}^*(\boldsymbol{p}, t)\Psi_0 (C_\alpha \neq 0, -\infty < p < +\infty)$ are mutually orthogonal and it is apparent that they also span $\mathfrak{H}^e_1(t)$.

The projection operator onto $\mathfrak{H}^e_1(t)$ is now easily constructed as

$$\Gamma(t) = \sum_{C\alpha\neq 0}\int d^3\boldsymbol{p}\frac{1}{C_\alpha(\boldsymbol{p})}\phi_\alpha{}^*(\boldsymbol{p}, t)\Psi_0\Psi_0{}^*\phi_\alpha(\boldsymbol{p}, t). \tag{2.9}$$

Returning to ψ, we get

$$\Gamma_+(t) = \sum_{\beta\gamma}\int d^3\boldsymbol{p}\psi_\beta{}^*(\boldsymbol{p}, t)\Psi_0\Omega^+_{\beta\gamma}(\boldsymbol{p})\Psi_0{}^*\psi_\gamma(\boldsymbol{p}, t), \tag{2.10}$$

where Ω^+ is a hermitian operator given by;

$$\Omega^+_{\beta\gamma}(\boldsymbol{p}) = \sum_{C\alpha\neq 0}\frac{U_{\beta\alpha}{}^*(\boldsymbol{p})U_{\alpha\gamma}(\boldsymbol{p})}{C_\alpha(\boldsymbol{p})}. \tag{2.11}$$

It satisfies the relations

$$\left.\begin{aligned}\Omega^+\Lambda^+\Omega^+ &= \Omega^+,\\ \Lambda^+\Omega^+\Lambda^+ &= \Lambda^+.\end{aligned}\right\}$$

If all C_α equal to unity or zero, that is, if Λ^+ has projective character, we get

$$\Omega^+ = \Lambda^+. \tag{2.12}$$

The coordinate representation of Γ_+ is given by

$$\Gamma_+(t) = \sum_{\beta\gamma}\iint d^3\boldsymbol{x}d^3\boldsymbol{x}'\psi_\beta{}^*(\boldsymbol{x}, t)\Psi_0\Omega^+{}_\gamma(\boldsymbol{x}-\boldsymbol{x}')\Psi_0{}^*\phi_\gamma(\boldsymbol{x}', t), \tag{2.13}$$

where

$$\Omega^{+}_{\beta\gamma}(x-x')=\frac{1}{(2\pi)^3}\int d^3p\exp(ip\cdot(x-x'))\,\Omega^{+}_{\beta\gamma}(p). \tag{2.14}$$

The projection operator $\Gamma_{-}(t)$ onto $\mathfrak{H}_1^{p}(t)$ is similarly defined;

$$\left.\begin{aligned}\Gamma_{-}(t)&=\sum_{C\alpha\neq 1}\int d^3p\frac{1}{1-C_\alpha(p)}\phi_\alpha(p,t)\Psi_0\Psi_0{}^{*}\phi_\alpha{}^{*}(p,t)\\&=\sum_{\beta\gamma}\int d^3p\psi^{\gamma}(p,t)\Psi_0\Omega^{-}_{\beta\gamma}(p)\Psi_0{}^{*}\psi_\beta{}^{*}(p,t)\\&=\sum_{\beta\gamma}\iint d^3x d^3x'\psi_\gamma(x',t)\Psi_0\Omega^{-}_{\beta\gamma}(x-x')\Psi_0{}^{*}\psi_\beta{}^{*}(x,t)\end{aligned}\right\} \tag{2.15}$$

where

$$\left.\begin{aligned}\Omega^{-}_{\beta\gamma}(p)&=\sum_{C\alpha\neq 1}\frac{U^{*}_{\beta\alpha}(p)U_{\alpha\gamma}(p)}{1-C_\alpha(p)}\,,\\\Omega^{-}_{\beta\gamma}(x-x')&=\frac{1}{(2\pi)^3}\int d^3p\exp(ip\cdot(x-x'))\,\Omega^{-}_{\beta\gamma}(p).\end{aligned}\right\} \tag{2.16}$$

We can construct $\Lambda^{\pm}(p)$ and then $\Omega^{\pm}(p)$ from the Green function (§ 3).

§ 3. Formulation of one electron (positron) problem

Let Ψ be a state in $\mathfrak{H}_1(t')$; then it takes the form of (2.3). We now consider the probability amplitude χ_α of the transition from Ψ to the state of one-electron existing at x and t. It is, of course, given by $(\psi_{H\alpha}{}^{*}(x,t)\Psi_0,\Psi)$. We get

$$\chi(x,t)=\frac{1}{i}\int G(x,t\,;x',t')\beta f(x')d^3x'\quad(t>t') \tag{3.1}$$

where we put

$$G(x,t\,;x',t')=i(\Psi_0,\psi_H(x,t)\bar{\psi}_H(x',t')\Psi_0)\quad(t>t'). \tag{3.2}$$

G is identical with Schwinger's Green function (in the region $t>t'$). We may call χ to be the "wave function" of one electron. From the way of our construction, the physical meaning of χ seems to be apparent, so that the boundary condition for χ may be established without ambiguity. It is, however, meaningless to call χ the "wave function", until we show the method of calculating physical quantities in terms of it. We shall now make use of the concept introduced in § 2.

We assume that Ψ is normalized to unity.

Consider the probability $P(t)$ of our system staying in one-electron state at time t. From the general law of the quantum mechanics, it is natural to define this probability by the norm of $\Gamma_{+}(t)\Psi$. By virtue of (2.13), it reads

$$P(t)=(\Psi,\Gamma_{+}(t)\Psi)=\iint\chi^{*}(x,t)\Omega^{+}(x-x')\chi(x',t)d^3xd^3x'. \tag{3.3}$$

From the definition, it is evident that $P \leqq 1$.

We shall now attempt to translate any operator of the field theory in our scheme. Take two states Ψ' and Ψ'' belonging to $\mathfrak{H}_1^c(t')$ and $\mathfrak{H}_1^c(t'')$, respectively, *i. e.*;

$$\Psi' = \int \psi_H^*(x', t')\Psi_0 d^3x' g(x'),$$

$$\Psi'' = \int \psi_H^*(x'', t'')\Psi_0 d^3x'' f(x'').$$

The matrix element* of an operator $O(t')$ between these states is given by

$$\langle g, t' \mid O(t') \mid f, t'' \rangle \equiv (\Psi', O_H(t')\Psi'')$$
$$= \iint g^*(x')(\Psi_0, \psi_H(x', t')O_H(t')\psi_H^*(x'', t'')\Psi_0) f(x'') d^3x' d^3x''. \quad (3.4)$$

Assume that the operator $O(t)$ has Ψ_0 as an eigenstate (belonging to the eigenvalue O_0) and satisfies the relation:

$$[\psi_H(x, t),\ O_H(t)] = \mathfrak{D}\psi_H(x, t). \quad (3.5)$$

Here $\mathfrak{D}$ is a c-number operator operating spinor indices and coordinates. Then, (3.4) reduces to a simple form:

$$\langle g, t' \mid O(t') - O_0 \mid f, t'' \rangle = \int g^*(x')\mathfrak{D}\chi(x', t')d^3x'. \quad (3.6)$$

The former assumption expresses that operator O does not disturb the vacuum state Ψ_0, that is,

$$[O_H(t), H]\Psi_0 = 0. \quad (3.7)$$

The latter property (3.5) is characteristic of the total energy-momentum operator and is also possessed by the operators of following form:

$$O_H(t) = \int \psi_H^*(x, t)\mathfrak{D}\psi_H(x, t)d^3x + \text{Electromagnetic Terms.}$$

These two properties which are essential for the construction of the corpuscular aspect seem to be possessed by the physically important observables.

We now define the "one electron expectation value" of $O(t)$ for a state $\Psi \in \mathfrak{H}_1^c(t)$ by

$$\langle O(t) \rangle = (\Gamma_+(t)\Psi, O(t)\Psi)/P(t). \quad (3.8)^{**}$$

By the relations (3.6) and (3.7), this also reduces to

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*) It is to be noticed that the relation $\overline{AB} = \bar{A}\,\bar{B}$ does not always hold where $\bar{A}$ is the one-electron submatrix of an operator A. If A and B commute with Γ, the above relation is satisfied.

**) If O commute with Γ_+, then $\langle O \rangle$ is equal to $(\Gamma_+\Psi, O\Gamma_+\Psi)/P$ which is real because of the hermitian character of O. See note added in proof.

$$\langle O(t)\rangle - O_0 = \frac{1}{P(t)} \iint \chi^*(x,t)\,\Omega^+(x-x')\,\mathfrak{D}\chi(x',t)\,d^3x d^3x'. \qquad (3.9)$$

The physically significant expectation value of O is given by the expression of right hand side of (3.9), because we can observe only deviation from the vacuum expectation value.

That we can express the probability and the expectation value by the overlapping integrals of χ and χ^* may be considered to show the corpuscular aspect of quantum electrodynamics.

In view of (3.3) and (3.9), it is now reasonable to call χ the wave function and $\mathfrak{D}$ the representation of O in the corpuscular aspect.

The appearance of Ω is characteristic of our theory and is considered to represent the non-local character anticipated from the uncertainty of the position of an electron. The precise form of Ω has not been obtained, but for the discussion of the bound state energy and the decay constant it causes no trouble. (See Appendix B). In case of the absence of dynamical electro-magnetic field, $\Lambda^{\pm}$ is just the projection operator on the positive (negative) energy state (in the sense of the unquantized Dirac theory) and hence $\Lambda^{\pm} = \Omega^{\pm}$. This is the reason why the non-local character is not apparent in this case in spite of the uncertainty of position. It is very probable that our Λ may also be the projection operator on electron state in the sense of our corpuscular aspect, but we have failed to prove it.

We now consider about positron processes. As we do not use the charge conjugate representation, the positron process must be described as the inverse process of an electron with the negative energy.

Let Ψ be in $\mathfrak{H}_1^P(t')$;

$$\Psi = \int \psi_H(x',t')\,\Psi_0 d^3x' f^*(x'). \qquad (3.10)$$

We ask the probability of our system staying in one-positron space at time t and the one-positron expectation value of an operator at t. Using (2.15), we get

$$P(t) = (\Psi, \Gamma_-(t)\Psi) = \iint \chi^*(x,t)\,\Omega^-(x-x'')\chi(x'',t)\,d^3x d^3x'', \qquad (3.11)$$

$$\langle O(t)\rangle - O_0 \equiv (O(t)\Psi, \Gamma(t)\Psi)/P(t) - O_0$$

$$= \frac{-1}{P(t)} \iint \chi^*(x',t)\,\Omega^-(x'-x'')\,\mathfrak{D}\chi(x'',t)\,d^3x' d^3x'', \qquad (3.12)$$

where

$$\chi(xt) = -\frac{1}{i} \int G(x,t;x',t')\beta f(x')\,d^3x' \quad (t<t') \qquad (3.13)$$

and

$$G(x,t;x',t') = -i(\Psi_0, \bar{\psi}_H(x',t')\psi_H(x,t)\Psi_0) \quad (t<t'). \qquad (3.14)$$

G is again the Green function of Schwinger in the region $t<t'$.

If we put $t' = -\infty$ in (3.1) or $t' = +\infty$ in (3.13), such $\chi^+_{-\infty}$ and $\chi^-_{+\infty}$ both satisfy the wave equation given by Schwinger ;

$$(\gamma\Pi+M)\chi=0, \tag{1.6}$$

as follows from the equation of G:

$$(\gamma\Pi+M)G=1. \tag{1.5}$$

Since (1.6) is an integro-differential equation, it is very questionable whether our wave function (3.1) defined for $t>t'=$finite time could be so defined for $t<t'$ that the extended χ should satisfy (1.6). In other words, Schwinger's wave equation may probably be incapable to describe the finite time processes. The description of finite time processes is perhaps unnecessary from the physical point of view, though it is possible to treat such processes in our theory.

The general solution χ of (1.6) may be expressed by the sum of $\chi^{+}_{-\infty}$ and $\chi^{-}_{+\infty}$. When we get a solution of (1.6), it is necessary to separate the electron and positron parts of it in practical problems. In case of no dynamical electto-magnetic field, this separation is automatically done by $\Omega^{\pm}=\Lambda^{\pm}$ in (3.3) and (3.9). (In this case Eq. (1.6) reduces to that of the unquantized Dirac field.) Since the projective character of Ω has not generally been proved and no mathematical method of separation has been obtained, it is necessary to use physical considerations for this separation.

Finally we consider the expectation value of total energy operator H, which is given by

$$\langle H-E_0\rangle=\pm\frac{1}{P(t)}\iint\chi^*(\boldsymbol{x},t)\Omega^{\pm}(\boldsymbol{x}-\boldsymbol{x}')i\frac{\partial}{\partial t}\chi(\boldsymbol{x}',t)d^3\boldsymbol{x}d^3\boldsymbol{x}', \tag{3.15}$$

where E_0 is the vacuum eigenvalue of H. Since this quantity is generally complex, we put

$$\langle H-E_0\rangle=E-i\frac{\Gamma}{2}. \tag{3.16}$$

Here E and Γ are real functions of t. From (3.16) and properties of H and Γ we obtain

$$\Gamma=\mp\frac{1}{P(t)}\frac{\partial}{\partial t}P(t). \tag{3.17}$$

It is, therefore, evident that Γ gives formally the total probability per unit time of transition to all other particles states. However, the physically significant quantity is not Γ itself but the mean value of Γ over all time, because Γ may include unphysical decays occurring in finite interval of time. In this point of view it is important to consider the eigen-value problem

$$\mathcal{H}\chi=W\chi,$$

where $\mathcal{H}$ is the Hamiltonian of the corpuscular aspect and is given by rewriting (1.6) in the following form:

$$i\frac{\partial}{\partial t}\chi=\mathcal{H}\chi.$$

Since the eigenvalue W for this problem is obviously an expectation value of H and is independent of time, its imaginary part gives the physical deay constant. It is evident that the decay constant is positive for electron process and negative for positron process. The above mentioned separtion of χ will be achieved by checking its damping property.

The reasonable results have been obtained for the Lamb shift and the line breadth by solving (1.6) by the perturbation method. (See Appendix B).

§ 4. Conclusion

We have constructed the sub-spaces $\mathfrak{H}_1'(\mathfrak{H}_1'')$ and the projection operator onto it. Using these conceptions the "wave function" describing the motion of one electron (positron) state is defined. Our wave function includes the Schwinger's one as the special case. The expectation value of the physical quantities is expressed by (3.3) and (3.9). Therefore, it may be said that we have succeeded to show the corpuscular aspect of quantum electrodynamics. The non-locality (the appearance of $\mathfrak{Q}$) and the damping property ($P \leq 1$) are characteristics of our theory. Our only assumption is the existence of the true vacuum state Ψ_0.

In the above considerations we have not concerned the divergence difficulties. Although the existence of Ψ_0 is not ensured without the renormalization procedure, the essential points of the present paper will hold for the renormalized Ψ_0. Therefore, it may be admissible that we first renormalize G according to Utiyama et al[10] and then define χ using this G.

No essential difficulties will appear in the generalization of our theory to many-particle problem and other types of field. It will be discussed in subsequent papers.

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Appendix A. Charge distribution of a physical electron

We shall calculate the expectation value $\langle \rho(x', t) \rangle_{x,t}$ of the charge density of a physical electron by the perturbation theory. As shown by (2.1) it is possible to compute this expectation value from the two-electron Green function:

$$\langle \rho(x', t) \rangle_{x,t} = -e \sum_{\beta\rho\sigma} \beta_{\rho\beta}\beta_{\sigma\alpha} \lim_{t \to t'} \lim_{\xi, \eta \to 0} G_{\alpha\beta,\sigma\rho}\left(x + \frac{\xi}{2}, x' + \frac{\eta}{2}; x - \frac{\xi}{2}, x' - \frac{\eta}{2}\right),$$

where ξ and η tend to zero along the space-like direction.

The two-electron Green function in zeroth-order approximation consists of two terms G_0' and G_0'':

$$G_0'(x_1, x_2, x_3, x_4) = iS^F_{\alpha\sigma}(x_1 - x_3) iS^F_{\beta\rho}(x_2 - x_4),$$

$$G_0''(x_1, x_2, x_3, x_4) = -iS^F_{\alpha\rho}(x_1 - x_4) iS^F_{\beta\sigma}(x_2 - x_3).$$

The contribution $\langle\rho\rangle_0'$ of G_0' to the charge density becomes

$$\langle\rho\rangle_0' = e \lim_{\xi\to 0}[S^F(\xi)\beta]_{\alpha\alpha} \lim_{\eta\to 0} S_F[S(\eta)\beta],$$

which vanishes as easily shown. Hence we get

$$\begin{aligned}\langle\rho\rangle_0 &= -e \lim[S^F(x-x')\beta S^F(x'-x)\beta]_{\alpha\alpha},\\ &= -e\left[m^2\Delta^F(r,0)^2+\left(\frac{\partial}{\partial r}\Delta(r,0)\right)^2+im(\gamma_n)_{\alpha\alpha}\frac{\partial}{\partial r}\Delta^F(r,0)^2\right],\end{aligned}$$

where $r=|x-x'|$ and $\gamma_n=\sum_{k=1}^{3}\gamma^k\frac{\partial r}{\partial x^k}$. This expression has the asymptotic form $(1/r^3)\times\exp(-2mr)$ for large r, because the first term of $\langle\rho\rangle_0$ is predominant.

The two-electron Green function of the second order consists of the terms:

$$\begin{aligned}G'_2(x_1,x_2x_3,x_4) &= e^2\iint d^4y\,d^4y'\,[iS^F(x_1-y)\gamma^\mu iS^F(y-x_3)]_{\alpha\sigma}\\ &\quad\times iD^F(y-y')[iS^F(x_2-y')\gamma^\mu iS^F(y'-x_4)]_{\beta\rho},\\ G_3''(x_1,x_2;x_3,x_4) &= -e^2\iint d^4y\,d^4y'[iS(x_1-y)\gamma^\mu iS^F(y'-x_4)]_{\alpha\rho}\\ &\quad\times iD^F(y-y')[iS^F(x_2-y)\gamma^\mu iS^F(y-x_3)]_{\beta\sigma},\end{aligned}$$

where we are not concerned with the self-energy terms. Since the terms without derivatives are predominant, it is sufficient to compute only the following integrals:

$$I'=\iint d^4y\,d^4y'\,D^F(y-y')\Delta^F(x-y)\Delta^F(y-x)\Delta^F(x'-y')\Delta^F(y'-x'),$$

$$I''=\iint d^4y\,d^4y'\,D^F(y-y')\Delta^F(x-y')\Delta^F(y'-x')\Delta^F(x'-y)\Delta^F(y-x).$$

For simplicity we shall discard all numerical factors. Using the integral representations

$$\Delta^F(\xi)=\int_0^\infty\exp\left[i\xi^2u-i\frac{m^2}{4u}\right]du,$$

$$D^F(\xi)=\int\frac{1}{k^2}\exp[ik\xi]\,d^4k$$

and the formula

$$\int\exp[iby+iy^2a]\,d^4y=\frac{i\pi}{a|a|}\exp\left[-i\frac{b^2}{4a}\right],$$

we can perform the integration with respect to y, y' and k'. Then the above integrals become

$$\begin{aligned}I' &= \frac{1}{(x-x')^2}\iiiint_0^\infty\frac{du\,d\beta\,d\gamma\,d\delta}{(u+\beta)^2(\gamma+\delta)^2}\exp\left[-i\frac{m^2}{4}\left(\frac{1}{u}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\right]\\ &\quad\times\exp\left[i(x-x')^2\frac{(u+\beta)(\gamma+\delta)}{u+\beta+\gamma+\delta}\right],\end{aligned}$$

$$I''=\frac{1}{(x-x')^2}\iiint_0^\infty \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(\alpha+\beta)^2(\gamma+\delta)^2}\exp\left[-\frac{im^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\right]$$
$$\times\exp\left[i(x-x')^2\left\{\frac{\alpha\beta}{\alpha+\beta}+\frac{\gamma\delta}{\gamma+\delta}+\left(1+\frac{\beta}{\alpha+\beta}+\frac{\delta}{\gamma+\delta}\right)\frac{(\alpha+\beta)(\gamma+\delta)}{\alpha+\beta+\gamma+\delta}\right\}\right],$$

where m^2 has a small negative imaginary part.

If we use the 4-dimensional polar coordinate in the space of the parameters α, β, γ and δ, the above integrals are written as follows;

$$I=\frac{1}{(x-x')^2}\iint_{\alpha,\beta,\gamma,\delta>0}\frac{dR\,d\Omega}{RA(\Omega)}\exp\left[-i\frac{m^2}{4R}B(\Omega)+i(x-x')^2C(\Omega)R\right],$$

where R and Ω are respectively the radial variable and the solid angle in 4-dimensional Euclidian space. Here $A(\Omega)$ and $C(\Omega)$ are order of magnitude 0 1 and $B(\Omega)$ is order of magnitude $1\sim\infty$. For $r^2=(x-x')^2>0$ we get

$$I=\frac{1}{r^2}\iint_{\alpha,\beta,\gamma,\delta>0}\frac{dR\,d\Omega}{RA(\Omega)}\exp\left[-\frac{m^2}{4R}B(\Omega)-r^2C(\Omega)R\right].$$

Since the integrand is positive definite, we can use the meanvalue theorem for the integral with respect to Ω. Then we get

$$I=\frac{1}{r^2}\int_0^\infty\frac{dR}{RA(R)}\exp\left[-\frac{m^2}{4R}B'(R)-r^2C'(R)R\right],$$

where it is easily shown that the quantities A', B' and C' are slowly varying functions of R, being order of magnitude of unity. It is evident that the above integral rapidly vanishes outside of the region $r<\frac{1}{m}$. We can, in fact, show by the steepest descent method that the asymptotic form of the above integral is $\exp(-amr)$.

By continuing the above procedure we may conclude in any order of the perturbation that the expectation value of the charge density $\rho(x')$ of a physical electron rapidly vanishes outside the region where $|x-x'|$ is smaller than the Compton wave length of the electron provided that the usual procedure of renormalization has been performed.

Appendix B. Lamb shift and line breadth

We shall discuss here a physical electron in an external field $B_\mu(x)$ by the perturbation theory, in which the zeroth-order term and the perturbation term are given by the following expressions:

$$\mathcal{H}_0=-i\beta\gamma^k\frac{\partial}{\partial x^k}-e\beta\gamma^\mu B_\mu(x)+m\beta,$$

$$V\chi=-e\beta\gamma^\mu\mathfrak{A}_\mu(x)+\beta\int\Sigma^*(x,x')d^4x'\chi(x'),$$

where

$$\mathfrak{A}_\mu(x') = \langle A_\mu(x) \rangle - B_\mu(x)$$

and

$$\Sigma^*(x,x') = ie^2\gamma^\mu \int G(x,y)\, \Gamma^\nu(y,x',y')\, \mathfrak{G}_{\nu\mu}(y',x)\, d^4y d^4y'.$$

Here Γ^ν and $\mathfrak{G}^{\nu\mu}$ are the quantities given by Schwinger[4] and Utiyama et al[10]. We shall calculate the first-order correction of the eigenvalue $E^0{}_A$ of the equation

$$\mathcal{H}_0\, \chi^0{}_A = E^0{}_A\, \chi^0{}_A.$$

If the external field $B_\mu(x)$ are so weak that it is impossible to create pairs of electron and positron, we can neglect the first term in comparison with the term Σ^* in the first-order calculation. Using the lowest order approximation for Γ^ν and $\mathfrak{G}_{\nu\mu}$, we get for the first-order correction of $E^0{}_A$

$$W' = -e^2 \int d^3\boldsymbol{x}\, \bar{\chi}^0{}_A(\boldsymbol{x},t) \int d^4x' \gamma_\mu G^{(B)}(x,x') \gamma^\mu D^F(x-x')\, \chi^0{}_A(\boldsymbol{x}',t'),$$

where

$$G^{(B)}(x,x') = i \sum_{+n} \bar{\chi}^0{}_n(\boldsymbol{x},t)\chi^0{}_n(\boldsymbol{x}',t'), \quad t > t',$$

$$= -i \sum_{-n} \bar{\chi}^0{}_n(\boldsymbol{x},t)\chi^0{}_n(\boldsymbol{x}',t'), \quad t < t'.$$

Here $\pm n$ denote the summations over positive and negative energy states, respectively. Using this expression of $G^{(B)}$ and the momentum representation of D_F we get

$$W^1_A = \Delta E - i\frac{\Gamma}{2},$$

where

$$\Delta E = \frac{e^2}{16\pi^3} P \int \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{+n} \sum_{\mu} \frac{|(A|\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})|\, n)|^2}{E_A{}^0 - E_n{}^0 - |\boldsymbol{k}|}$$

$$+ \frac{e^2}{16\pi^3} P \int \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{-n} \sum_{\mu} \frac{|(A|\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})|\, n)|^2}{E_A{}^0 - E_n{}^0 + |\boldsymbol{k}|},$$

$$\Gamma = \frac{e^2}{8\pi^2} \int \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{+n} \sum_{\mu} |(A|\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})|\, n)|^2 \delta(E_A{}^0 - E_n{}^0 - |\boldsymbol{k}|)$$

$$- \frac{e^2}{8\pi^2} \int \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{-n} \sum_{\mu} |(A|\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})|\, n)|^2 \delta(E_A{}^0 - E_n{}^0 + |\boldsymbol{k}|).$$

Here P denotes the Cauchy's principal value of the integral and

$$(A|\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})|\, n) = \int \bar{\chi}^0{}_A(\boldsymbol{x},0)\gamma^\mu \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\, \chi^0{}_n(\boldsymbol{x},0)\, d^3\boldsymbol{x}.$$

The summation over the polarizations of photon can be reduced to the sum of the two directions perpendicular to $\boldsymbol{k}$ as the result of the Lorentz condition. Subsequently the each integral in Γ is positive definite, so Γ is positive for positive energy states and negative for negative energy states as expected. Moreover, the expression of Γ explicitly describes

the total probability per unit time of transition from the state E_A^0 to all the states $E_n^0 < E_A^0$. These illustrate the features emphasized in § 1 and § 3. On the other hand it is evident that the real part ΔE of the correction W_A^1 gives the electro-magnetic shift (Lamb shift) of the energy level after subtraction of the infinite term of the self-energy.

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Note added in proof:

It is easily shown that the matrices

$$V^{\pm} = \Omega^{\pm} \Lambda^{\pm} = \Lambda^{\pm} \Omega^{\pm}$$

have the projective and hermitian character and satisfy the relations $[\mathfrak{H}, \Lambda^{\pm}] = 0$, and $V^{\mp}[\mathfrak{H}, \Omega^{\pm}] V^{\mp} = 0$. From the easily obtained relations

$$\sum_{\alpha} \int \psi_{\alpha}^*(\boldsymbol{p}, t)\, \Psi_0 f_{\alpha}(\boldsymbol{p})\, d^3\boldsymbol{p} = \sum_{\alpha\beta} \int \psi_{\alpha}^*(\boldsymbol{p}, t)\, \Psi_0 V_{\alpha\beta}^{+}(\boldsymbol{p}) f_{\beta}(\boldsymbol{p})\, d^3\boldsymbol{p},$$

$$\sum_{\alpha} \int \psi_{\alpha}(\boldsymbol{p}, t)\, \Psi_0 f_{\alpha}^*(\boldsymbol{p})\, d^3\boldsymbol{p} = \sum_{\alpha\beta} \int \psi_{\alpha}(\boldsymbol{p}, t)\, \Psi_0 V_{\beta\alpha}^{-}(\boldsymbol{p}) f_{\beta}^*(\boldsymbol{p})\, d^3\boldsymbol{p},$$

$\mathfrak{H}_1^e$ and $\mathfrak{H}_1^p$ are proved to be spanned by the states $(V^{+}\psi_H)^* \Psi_0$'s and $(V^{-}\psi_H)_{\alpha} \Psi_0$'s, respectively.

We should rather define the expectation value of any operator except energy by

$$\frac{1}{P(t)} (\Psi, \Gamma_{\pm}(t) O(t) \Gamma_{\pm}(t) \Psi) - O_0$$

than (3.8) and (3.12), whereas for the energy we prefer (3.8) and (3.12) to above definition in order to include the decay constant in it. This change of definition causes only the replacement of χ in (3.9) and (3.12) by $\chi^{\pm} = V^{\pm}\chi$.

If the relation $V^{+} + V^{-} = 1$ holds, we can use $V^{\pm}$ for the separation of the general solution of (1.6).

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Miscellanea in Elementary Quantum Mechanics, I

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In the course of lectures on quantum mechanics delivered for the undergraduate students of this university I have encountered a number of examples of more or less novelty and interest, which I want here in a series of papers to present before a wider circle. Since the familiarity with all kinds of problems leads eventually to a better understanding of quantum mechanics, just as we have been possessed with the complete master of classical mechanics only through the numerous exercises in schools, it would, I hope, be of some service for a future development of quantum theory. Some of matters I present are of older date and some of recent date.

I. Canonical wave packets. Examples of oscillating and circulating wave packets

§ 0. A recipe for constructing canonical wave packets

In quantum statistical mechanics one deals with the so-called density matrix:

$$\rho(x, x'|\beta) \equiv \sum_{k=0}^{\infty} \exp(-\beta E_k)\psi_k(x)\psi_k^*(x'), \tag{0·1}$$

where E_k, $\psi_k(x)$ are the eigenvalues and eigenfunctions of the Hamiltonian H of the mechanical system under consideration, and $\beta = 1/kT$ is the well-known temperature parameter. If we regard ρ as function of β and the first set of coordinates x, it will satisfy the so-called Bloch equation

$$\frac{\partial \rho}{\partial \beta} = -H\rho, \tag{0·2}$$

where the Hamiltonian H operates on the first set of coordinates x. The density matrix ρ is completely determined by the Bloch equation and the following "initial" condition

$$\lim_{\beta \to +0} \rho(x, x'|\beta) = \delta(x - x'), \tag{0·3}$$

which expresses the completeness of our eigenfunctions.

Now we extend the variability domain of β to the complex half plane and write $\beta + it$ instead of β. The function $\rho(x, x'|\beta + it)$ of x and the time variable t will satisfy the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = H\Psi, \qquad \Psi(x, t|x', \beta) \equiv \rho(x, x'|\beta + it). \tag{0·4}$$

Thus ρ, once constructed, will afford a time-dependent solution of the Schrödinger equation by a simple substitution. The solution obtained in this manner will simulate a wave-packet owing to the character (0·3) of the density matrix, so long as the parameter β and the time t remain small. The wave-packets constructed in this way will be called "canonical". Since it contains, two arbitrary parameters β and x', we may expect a wide variety of shape and behavior of the packets. Presumably, β would determine the breadth and x' the location of the center of the packet; but actually the situation turns out more complex. It is necessary to set x' complex in order to get a moving wave-packet.

The norm of our wave function Ψ, (0·4), is easily obtained by means of the rule of composition of the density matrix:

$$\int \rho(x, x'|\beta_1)\rho(x', x''|\beta_2)dx' = \rho(x, x''|\beta_1+\beta_2). \tag{0·5}$$

The result runs:

$$\int \Psi^*(x, t|x', \beta)\Psi(x, t|x', \beta)dx = \rho(x'\beta, x'|2\beta), \tag{0·6}$$

which is time-independent, as ought to be. If x' is complex-imaginary, the right hand side of (0·6) should be replaced by

$$\rho(x'^*, x'|2\beta). \tag{0·7}$$

§ 1. The rectilinear motion. Heisenberg's wave packets

Our first instance is the application of the recipe of the preceding section to the rectilinear motion of a particle. However, we first impose the the periodicity condition of period a on the wave functions, in order to avoid some confusion with normalization. The normalized eigenfunctions are

$$\exp(ikx)/\sqrt{a}, \qquad k=2\pi n/a, \qquad n=0, \pm 1, \pm 2, \ldots$$

and the eigenvalues $\hbar^2k^2/2m$. The corresponding density matrix (0·1) is computed as follows:

$$\begin{aligned}\rho(x, x'|\beta) &= \frac{1}{a}\sum_{n=-\infty}^{\infty}\exp\{i(x-x')k-\beta\hbar^2k^2/2m\}\\ &= \frac{1}{a}\{1+2\sum_{n=1}^{\infty}\exp(-\beta\hbar^2k^2/2m)\cos k(x-x')\}\\ &= \frac{1}{a}\vartheta_3\Big(\frac{\pi(x-x')}{a}\Big|2\pi i\frac{\beta\hbar^2}{ma^2}\Big),\end{aligned}$$

where ϑ_3 is a Jacobi theta function of imaginary period $\tau=2\pi i\beta\hbar^2/ma^2$. By virtue of the transformation formula of the theta functions, we have also

$$\rho(x, x'|\beta) = \frac{1}{a}\Big(\frac{2\pi\beta\hbar^2}{ma^2}\Big)^{-1/2}\exp\{-m(x-x')^2/2\beta\hbar^2\}\vartheta_3\Big(-i\frac{ma(x-x')}{2\beta\hbar^2}\Big|\frac{ima^2}{2\pi\beta\hbar^2}\Big)$$

$$=\left(\frac{m}{2\pi\beta\hbar^2}\right)\sum_{n=-\infty}^{+\infty}\exp\{-m(x-x'-na)^2/2\beta\hbar\}.$$

In the manner described in § 0 we easily get the corresponding canonical wave-packet, which in the limit $a\to\infty$ turns out to be

$$\Psi(x,\,t|x',\,\beta')=\left(\frac{m}{2\pi\hbar^2(\beta+it/\hbar)}\right)^{1/2}\exp\left(-\frac{m(x-x')^2}{2\hbar(\beta+it/\hbar)}\right), \tag{1·1}$$

where the peaks drop out except that at $n=0$. The parameter x' may now be chosen as complex, but, since the real part of x' will only contribute to an uninteresting static displacement of the center of the packet, we set $x'=ia$. Then the exponent of (1·1) becomes

$$-\frac{m(x-ia)^2}{2\hbar^2(\beta+it/\hbar)}=-\frac{m}{2\hbar^2(\beta^2+t^2/\hbar^2)}\left\{\beta\left(x-\frac{a}{\beta\hbar}t\right)^2-\frac{a^2(\beta^2+t^2/\hbar^2)}{\beta}+\text{imaginary terms}\right\}.$$

The probability distribution is Gaussian; the center is moving at a uniform velocity $a/\beta\hbar$; the fluctuation square

$$(\hbar^2\beta^2+t^2)/m\beta \tag{1·2}$$

is increasing, towards past as well as towards future. The normalization of the wave function can be effected by the method outlined at the end of § 0. The normalized wave-packet is

$$\left(\frac{m\beta}{\pi\hbar^2}\right)^{1/4}\frac{1}{(\beta+it/\hbar)^{1/2}}\exp\left\{-\frac{m\beta(x-at/\beta\hbar)^2}{2\hbar^2(\beta^2+t^2/\hbar^2)}+i\,\frac{m[(x^2-a^2)t/\hbar+2a\beta x]}{2\hbar^2(\beta^2+t^2/\hbar^2)}\right\}. \tag{1·3}$$

This is Heisenberg's wave packet, well known since early times.

§ 2. The harmonic oscillator. The generalized Schrödinger wave packet

The wave equation of a harmonic oscillator runs in "natural" units

$$-\frac{1}{2}\frac{d^2\psi}{dx^2}+\frac{1}{2}x^2\psi=E\psi,$$

of which the eigen-values and the eigen-functions are

$$\left.\begin{aligned}E_n&=n+\frac{1}{2},\\ \psi_n&=(2^n\cdot n!)^{-1/2}\pi^{-1/4}H_n(x)\exp(-x^2/2),\end{aligned}\right\}\quad n=0,\,1,\,2,\,\ldots \tag{2·1}$$

the latter being normalized with respect to x measured in "natural" units.

$H_n(x)$ are Hermite polynomials, defined by

$$H_n(x)=(-1)^n\exp(x^2)\frac{d^n}{dx^n}\exp(-x^2).$$

The corresponding density matrix has been summed previously,[1], by means of Mehler's formula,

$$\rho(x, x'|\beta) = \sum_{n=0}^{\infty} \exp\{-\beta(n+1/2)\} \cdot \frac{H_n(x) H_n(x')}{\pi^{1/2}\, 2^n\, n!} \exp\{-(x^2+x'^2)/2\}$$

$$= (2\pi \operatorname{sh}\beta)^{-1/2} \exp\left\{-\operatorname{th}\left(\frac{\beta}{2}\right)\cdot\left(\frac{x+x'}{2}\right)^2 - \operatorname{cth}\left(\frac{\beta}{2}\right)\cdot\left(\frac{x-x'}{2}\right)^2\right\}, \quad (2\cdot2)$$

where sh, th, cth are hyperbolic functions. Our rule gives the canonical wave-packet:

$$\Psi(x, t|x', \beta') = \{2\pi \operatorname{sh}(\beta+it)\}^{-1/2} \exp\left[-\frac{1}{4}\left\{\operatorname{th}\left(\frac{\beta+it}{2}\right)(x+x')^2 + \operatorname{cth}\left(\frac{\beta+it}{2}\right)(x-x')^2\right\}\right]. \quad (2\cdot3)$$

In this case the parameter x' may be chosen real without loss of generality. The expression within the bracket $\{\ \}$, namely

$$\frac{\operatorname{sh}\beta + i\sin t}{\operatorname{ch}\beta + \cos t}(x+x')^2 + \frac{\operatorname{sh}\beta - i\sin t}{\operatorname{ch}\beta - \cos t}(x-x')^2, \quad (2\cdot4)$$

has the real part

$$\frac{2\operatorname{sh}2\beta}{\operatorname{ch}2\beta - \cos 2t}\left(x - \frac{x'}{\operatorname{ch}\beta}\cos t\right)^2 + 2\operatorname{th}\beta\cdot x'^2. \quad (2\cdot5)$$

Hence the probability distribution is again Gaussian, the center is oscillating with the classical circular frequency 1 (in "natural" units) and with the amplitude $x'/\operatorname{ch}\beta$. The fluctuation square is

$$(\operatorname{ch}2\beta - \cos 2t)/2\operatorname{sh}2\beta, \quad (2\cdot6)$$

it attains its minimum values $\frac{1}{2}\operatorname{th}\beta$ at the extremities and its maximum value $\frac{1}{2}\operatorname{cth}\beta$ at the center of oscillation. Thus our oscillating wave-packet is at the same time pulsating, synchronously with the oscillation, but it contains no ever-increasing part, contrary to the case of rectilinear motion. This lack of monotonous diffusion may be accounted for on the ground of isochronism of the classical harmonic oscillation (every oscillating particle returns to its starting point within the same period, irrespective of its amplitudes. In the case of rectilinear motion, the faster particle will cover greater distances).

The oscillating wave-packet devised by Schrödinger in his early work[2] is based on the generating function

$$\exp(-s^2 + 2sx - x^2/2) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \exp(-x^2/2)\cdot H_n(x)$$

of the Hermite polynomials and is a special case of ours.

§ 3. The circular motion in a uniform magnetic field. The circulating wave-packet

Now we consider the motion on the x,y-plane of an electron when a uniform magnetic field H is present along the axis of z. The Schrödinger equation is set up as follows:

$$-\frac{\hbar^2}{2m_0}\Delta\psi - i\hbar\omega\frac{\partial\psi}{\partial\varphi}+\frac{1}{2}m_0\bar{\omega}^2r^2\psi=E\psi, \tag{3·1}$$

where we have introduced a harmonic binding potential $\frac{1}{2}m_0\omega_0^2r^2$, in order to assure the discreteness of the eigen-values, so that $\bar{\omega}$ in the equation is defined by

$$\bar{\omega}^2=\omega^2+\omega_0^2. \tag{3·2}$$

ω is the Larmor precession frequency, $\omega=eH/2m_0c$. m_0 is the (rest) mass of the electron, to be distinguished from the magnetic quantum number m, which appearly shortly.

The eigen-value problem (3·1) has been solved by Darwin.[3] By separation of variables $\psi=r^{-1/2}S(r)\cdot\exp(im\varphi)$, we get for the radial part

$$-\frac{\hbar^2}{2m_0}\left(\frac{d^2S}{dr^2}+\frac{1/4-m^2}{r^2}S\right)+\frac{1}{2}m_0\bar{\omega}^2r^2S=E_1\cdot S, \tag{3·3}$$

with

$$E_1=E-\hbar\omega m. \tag{3·4}$$

Again we introduce the natural units and write

$$r=\xi\cdot\sqrt{\frac{\hbar}{m_0\bar{\omega}}}, \qquad E_1=\varepsilon_1\cdot\hbar\bar{\omega}. \tag{3·5}$$

Then the solutions of (3·3) are given in terms of generalized Laguerre polynomials by

$$\left.\begin{aligned} &S=\exp(-\xi^2/2)\xi^{|m|+1/2}L_n^{|m|}(\xi^2),\\ &\varepsilon_1=2n+1+|m| \end{aligned}\right\} \qquad n=0,\ 1,\ 2,\ldots \tag{3·6}$$

The Laguerre polynomials $L_n^{|m|}$ are those defined by Szegö[1] and are detetmined by orthogonality relations

$$\int_0^\infty\exp(-\xi^2)\xi^{2\mu+1}L_n^\mu(\xi^2)L_{n'}^\mu(\xi^2)d\xi^2=\frac{1}{2}\mu!\binom{n+\mu}{\mu}\delta_{nn'}. \tag{3·7}$$

We first construct a partial density matrix, relating to the radial part only. Let $\mu=|m|$. The summation formula given in Szegö's book[4] leads to

$$\begin{aligned}\rho_\mu(\xi,\xi'|\beta)&=\sum_{n=0}^\infty\frac{2\cdot\exp(-\beta(2n+1+\mu))}{\mu!\binom{n+\mu}{n}}\cdot\exp(-\xi^2/2)\cdot\xi^\mu\\ &\qquad\cdot L_n^\mu(\xi^2)\exp(-\xi'^2/2)\xi'^\mu L_n^\mu(\xi')\\ &=\frac{1}{\operatorname{sh}\beta}\exp\left\{-\frac{\xi^2+\xi'^2}{2}\operatorname{cth}\beta\right\}\cdot I_\mu\left(\frac{\xi\xi'}{\operatorname{sh}\beta}\right),\end{aligned} \tag{3·8}$$

where I_μ is the Bessel function of imaginary argument.

Next we build the complete density matrix, the summation being effected by the well-known generating formula of Bessel functions,

$$\rho(\xi, \varphi, \xi', \varphi'|\beta) = \sum_{m=-\infty}^{\infty} (2\pi)^{-1} \exp\{im(\varphi-\varphi')\} \exp(-\beta m\omega/\bar{\omega}) \rho_{|m|}(\xi, \xi'|\beta)$$

$$= \frac{1}{2\pi \operatorname{sh} \beta} \exp\left\{-\frac{\xi^2+\xi'^2}{2} \operatorname{cth} \beta\right\} \cdot \sum_{m=-\infty}^{\infty} [\exp\{-\beta\omega/\bar{\omega} + i(\varphi-\varphi')\}]^m I_{|m|}\left(\frac{\xi\xi'}{\operatorname{sh} \beta}\right)$$

$$= \frac{1}{2\pi \operatorname{sh} \beta} \exp\left\{-\frac{\xi^2+\xi'^2}{2} \operatorname{cth} \beta + \frac{\xi\xi'}{\operatorname{sh} \beta} \cos(\varphi-\varphi' + i\beta\omega/\bar{\omega})\right\}. \qquad (3\cdot9)$$

From now on we put $\omega_0 = 0$, so that $\omega/\bar{\omega} = 1$ and the formula is greatly simplified. Then the exponent of the above formula may be written

$$-\frac{1}{2}\{\xi^2+\xi'^2-2\xi\xi' \cos(\varphi-\varphi')\} \operatorname{cth} \beta - i\xi\xi' \sin(\varphi-\varphi'). \qquad (3\cdot10)$$

Thus the density matrix depends, so far as the real part concerns, solely on the translationally invariant scalar product $(\boldsymbol{\xi}-\boldsymbol{\xi}')^2$, as it should be so in the absence of any potential ($\omega_0 = 0$) except the uniform magnetic field, whereas the phase factor contains the z-component of the vector product $\boldsymbol{\xi} \times \boldsymbol{\xi}'$, which is *not* translationally invariant. The last circumstance is related to the gauge transformation of the vector potential.

In deriving the canonical wave-packet from the above density matrix, we have to put the vector $\boldsymbol{\xi}'$ complex imaginary:

$$\boldsymbol{\xi}' = i\boldsymbol{a}, \qquad (\boldsymbol{a}: \text{ real vector}). \qquad (3\cdot11)$$

Then the wave function representing the wave-packet

$$\rho(\boldsymbol{\xi}, i\boldsymbol{a}|\beta + it)$$

has for the real part of its exponent

$$-\frac{(\boldsymbol{\xi}^2-\boldsymbol{a}^2) \operatorname{sh} 2\beta - 2(\boldsymbol{\xi}\cdot\boldsymbol{a}) \sin 2t}{2(\operatorname{ch} 2\beta - \cos 2t)} - (\boldsymbol{\xi} \times \boldsymbol{a})_z. \qquad (3\cdot12)$$

Hence the distribution of the electron clouds is Gaussian; its center lies at

$$\boldsymbol{b} \operatorname{cth} 2\beta + \frac{\boldsymbol{a} \sin 2t - \boldsymbol{b} \cos 2t}{\operatorname{sh} 2\beta}, \qquad (3\cdot13)$$

where the vector $\boldsymbol{b}$ is obtained by 90°-rotation in counter-clockwise sense of the vector $\boldsymbol{a}$. It circulates in circular orbit with the circular frequency 2 (*i.e.* 2ω, or twice the Larmor frequency, in the original units). The spreading of the wave-packet is given by its fluctuation square

$$(\operatorname{ch} 2\beta - \cos 2t)/2 \operatorname{sh} 2\beta, \qquad (3\cdot14)$$

and is independent on the direction, so that the cloud is isotropic. Here again we have

no ever-increasing diffusion of the cloud, which returns to its original extension every time the center describes a complete circle. This peculiarity is based on the isochronism of the circular motions of charged particles within a uniform field irrespective of their velocities (the principle of cyclotron).

The complete wave-function normalized according to the method of § 0 is

$$\Psi(\xi, t|\boldsymbol{a}, \beta) = \frac{(\mathrm{sh}\, 2\beta/2\pi)^{1/2}}{\mathrm{sh}\,(\beta+it)} \exp\left\{-\frac{1}{2}\mathrm{cth}\,(\beta+it)\cdot(\xi-i\boldsymbol{a})^2-(\xi\times\boldsymbol{a})_z-a^2\,\mathrm{cth}\,2\beta\right\}. \tag{3·15}$$

§ 4. Final remarks

We have treated the three cases of canonical wave-packets, which are all of simple and closed form. They are all of Gaussian type, and, once this type of solutions is conceived as possible, one can start from the *Ansatz*: $\psi \propto \exp(\text{quadratic function of } x)$, without entering into any confusing summation problem, This point of view will be adopted in a different context in my second report.

Our examples of oscillating and circulating wave packets are misleading in so far as they suggest the absence of ever-increasing diffusion be a rule. As I have emphasized, this is a peculiarity due to the isochronism of the underlying mechanisms, and is an exception rather than a rule.

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A Quantum Theory of Boson Assemblies, II

— *A Classical Approach to Many-Boson Problems* —

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Some collective behaviors of boson assemblies are described in terms of the classical density operator and the classical velocity operator. We introduce these two classical quantities by taking average of the quantized operators with respect to Heisenberg wave packets.

Equations of these two quantities are determined and we obtain the transport equations. Thus the assembly is described as a fluid having the quantum mechanical pressure tensor.

The collective behavior of the assembly in question is analysed applying the theory of sound waves. Such an approximate treatment enables us to regard the assembly in question as the assembly of oscillation quanta.

When a particle is projected into the assembly with the velocity v_0 which is larger than the sound speed of the medium, suffering the collective response of the remainder of the bosons, the particle is decelerated and emits sound quanta. These processes are analysed by a classical treatment.

Further we consider the interaction of every particle in the assembly with the field of the collective oscillation representing behaviors of the remainder of the assembly. The self-energy of the particle and the Møller interaction between particles are obtained according to the interaction representation of Tomonaga and Schwinger as well as the canonical transformation of Bloch and Nordsieck.

Introduction

In this paper we wish to describe the medium-like behavior of a dense boson gas. In a previous paper, hereafter referred to as I, we have studied the temporal transformation of the quantized phase space distribution function and introduced a quantum mechanical velocity operator.[1] The commutation relation between the density and the velocity involves the projection operators $\varepsilon(\lambda)$. This prevents us to find the canonical conjugate variable of the density operator. We have pointed out that this projection operator, together with its derivatives, can be eliminated inrestricting ourselves to such subspaces that the quantum numbers of the particle-number operators are sufficiently large everywhere. Another difficulty that the reciprocal operator of the density is not allowable prevents us from the further development of the reformulation given in I. As a matter of fact, the operators are elements of a ring algebra, but not of a division algebra.

On the other hand, the similiarity of the density matrix with the classical distribution function has been pointed out by Husimi[2] taking average of the density matrix with respect to Heisenberg wave packets. The classical density in phase is associated with the above classical distribution function. The classical current operator is also introduced by averaging the general current operator given in I. Hearafter we shall denote classical quantities by the symbol "$c-$" and the classical operator by "$\langle\ \rangle$". Then we define the

c–velocity operator $\langle v \rangle$ dividing the c–current operator $\langle J \rangle$ by c–density operator $\langle \rho \rangle$. It is proved that c– velocity $\langle v \rangle$ is identical with an average of the velocity operator if the fluctuation of the average value tends to zero. These c–quantities do not always commute with one another; they may, however, be regarded as commutative if the expectation value of the average density is very large. Hence we can describe many boson problems in terms of c–density and the c–velocity which are regarded as really c–number quantities.

From the temporal transformation of these classical quantities, we derive the transport equations such as the equation of the mass conservation, the momentum conservation and the energy conservation in terms of the c–density operator and the c–velocity operator. As an example, we consider a problem of coupled harmonic oscillators. The energy eigenvalues, the density distribution and the increment of the total energy with time are obtained.

Next we consider the problem of the linearization to develop the theory of sound waves. So far as assemblies of interacting particles are concerned, the physical meaning of the one particle aspect becomes obscure, hence we must describe the organized behavior of the assembly as a whole. It will be shown that the assembly in question is regarded as a fluid having the quantum mechanical pressure tensor $\boldsymbol{P}(x)$ and we consider the excitation of sound waves which is one of the most fundamental phenomena in quantal fluids as well as classical fluids. The theory of sound waves in question favors us to get insight into the medium-like behavior of the system and to find the approximate eigenvalue of the Hamiltonian.

Pines ond Bohm[3] have pointed out that the density fluctuation splits into two components. One component is associated with the random thermal motion of the individual particles. The other is associated with the organized oscillation of the system as a whole. For large wave-lengths the density fluctuations are primarily collective. Our theory of sound waves is aimed at making use of the simplicity of the collective behavior as a starting point for a tractable solution of the equations of motion. We consider an example of assemblies of neutral bosons interacting with short range forces. We shall add some remarks on the plasma oscillation caused by the long range coulomb interaction (§ 3). Further we investigate how an individual particle suffers the collective reaction from the assembly. The total hamiltonian involving the energy of the incident particle splits up into three parts: the hamiltonian of the particle, that of the assembly of N bosons and that of the interaction between the particle and the boson assembly which may be regarded as the assembly of sound quanta. Thus the problem is reduced to the study of the interaction of a particle and a vibrational field as much the same with the interaction between the electron and the electro-magnetic field. The reaction of the induced density fluctuation on the particle acts as a force in the inverse direction against the motion of the particle which loses energy emitting sound quanta in the wake. The energy loss is estimated by a classical treatment.

Such a reformulation regarding the assembly as a compressible fluid does not disguise the conventional quanrum theory; it will, however, be effective in affording us such new

aspects of the theory as the excitation of sound quanta and the collective interaction of particles.

Next we see how the linearized frequency is shifted in a self-consistent way, if the individual particle motion is appreciable. To pursue the behavior of the individual particle as well as the dynamical correlation among particles embedded in the assembly we apply the interaction representation of Tomonaga and Schwinger as well as the canonical transformation of Bloch and Nordsieck. On the dynamical correlation between particles the Hartree-Fock and the Thomas-Fermi method do not give any satisfactory account. As for many fermion problems, Tomonaga[4] applied Bloch's theory of sound waves and gave the most complete account of the one dimensional problem[5] and more recently Pines and Bohm have proposed an idea to describe the assembly as a whole collectively. Their method is quite classical and we applied it to boson assemblies. It has been shown by Macke[6] that the straight-forward perturbation calculus bears to find the correlation interaction. Our treatment given here does not deal with the direct interaction of particles, but the interaction of oscillation quanta with particles.

§ 1. Classical quantities

In the following we shall deal with average quantities with respect to Heisenberg wave packets. These quantities are essentially non-commutative with one another, but the commutation relations among them do not involve the δ–function. As demonstrated by Wigner the density matrix is connected with the phase space distribution function by a fourier transformation. This relationship has been more carefully examined by Husimi applying the Heisenberg wave packet. As a matter of fact, the expectation value of the Heisenberg wave packet will give rise to a complete approach to the classical particle distribution. Introducing the Heisenberg wave packet of the form

$$\Psi_{P,Q}(q)=\left(\frac{\gamma}{\pi}\right)^{3/4}\exp\left[-\frac{\gamma}{2}(q-Q)^2+\frac{i}{\hbar}Pq\right], \tag{1·1}$$

Husimi defined the classical density matrix in the form

$$\begin{aligned}\rho_{Cl}(P,Q)&=\iint\Psi^*_{P,Q}\left(x+\frac{r}{2}\right)\rho(x,r)\Psi_{P,Q}\left(x-\frac{r}{2}\right)dx\,dr\\&=\left(\frac{\gamma}{\pi}\right)^{3/2}\iint\rho(x,r)\exp\left[-\gamma(x-Q)^2-\frac{\gamma}{4}r^2-\frac{i}{\hbar}Pr\right]dx\,dr,\end{aligned} \tag{1·2}$$

which is essentially identical, aside from the normalization factor, with the classical phase space distribution function of the quantized theory which is defined by an average over the phase space:

$$f_{Cl}(P,Q)=\left(\frac{1}{\pi\hbar}\right)^3\iint f(p,q)\exp\left[-\gamma(q-Q)^2-\frac{1}{\gamma\hbar^2}(p-P)^2\right]dp\,dq$$

$$=\left(\frac{1}{2\pi\hbar}\right)^3 \rho_{\mathrm{Cl}}(P, Q). \tag{1·3}$$

Hence the classical density operator is expressed by

$$\langle \rho \rangle_Q = \int f_{\mathrm{Cl}}(P, Q)\, dp \tag{1·4}$$

$$=\left(\frac{\gamma}{\pi}\right)^{3/2} \int \rho(q) \exp[-\gamma(q-Q)^2]\, dq\,, \tag{1·5}$$

and the classical current operator takes the form

$$\langle J \rangle_Q = \int \frac{P}{m} f_{\mathrm{Cl}}(P, Q)\, dP = \left(\frac{\gamma}{\pi}\right)^{3/2} \left(\frac{1}{2\pi\hbar}\right)^3 \iiint\!\!\iint \left[\frac{\hbar}{mi} \frac{\partial \rho(x, r)}{\partial r} - \frac{\hbar\gamma}{2mi} r\rho(x, r)\right]$$

$$\times \exp\left[-\gamma(x-Q)^2 - \frac{\gamma}{4} r^2 - \frac{i}{\hbar} Pr\right] dx\, dr\, dP = \left(\frac{\gamma}{\pi}\right)^{3/2} \int J(x) \exp[-\gamma(x-Q)^2]\, dx$$

$$-\frac{\hbar^2\gamma}{2} \int \frac{\partial}{\partial P} f_{\mathrm{Cl}}(P, Q)\, dP\,, \tag{1·6}$$

in which the first term represents the average value of the quantized current operator and the second term vanishes.

On the other hand we introduce the classical velocity operator defined by

$$\langle v \rangle_Q = \left(\frac{\gamma}{\pi}\right)^{3/2} \int v(x) \exp[-\gamma(x-Q)^2]\, dx\,. \tag{1·7}$$

The commutation relation between the classical density operator and the classical velocity operator becomes

$$[\langle \rho \rangle_{Q'}, \langle v \rangle_Q] = 2i \left(\frac{\gamma}{\pi}\right)^3 \frac{\hbar}{m} \iint [\varepsilon(q) - 1] \gamma(q-Q') \exp[-\gamma(q-Q)^2 - \gamma(q-Q')^2]\, dq\,. \tag{1·8}$$

As already mentioned, we confine ourselves to such subspaces that the projection operator $\varepsilon(q)$ may be regarded as zero. Then the commutation relation turns out to be

$$[\langle \rho \rangle_{Q'}, \langle v \rangle_Q] = \frac{i\hbar}{m} \left(\frac{\gamma}{2\pi}\right)^{3/2} \gamma[Q'-Q] \exp[-\gamma(Q'-Q)^2]\,. \tag{1·9}$$

If γ is sufficiently large, it is permissible to set

$$\langle J \rangle_Q / \langle \rho \rangle_Q = \langle v \rangle_Q\,.$$

We introduce the average canonical variable $\langle \Phi \rangle$ in a similar manner and one finds the commutation relation of the form

$$[\langle \rho \rangle_{Q'}, \langle \Phi \rangle_Q] = i\hbar \left(\frac{\gamma}{2\pi}\right)^{3/2} \exp[-\gamma(Q-Q')^2/2]\,. \tag{1·10}$$

These relations are valid in those subspaces mentioned in I. From the commutation

relations of vector components of the velocity operator, which vanish in such subspaces, one can conclude that the velocity operator may be regarded as irrotational in those subspaces to the effect

$$v(x)=\frac{1}{m}\,\mathrm{grad}\,\Phi(x)\,, \tag{1·11}$$

or taking the average

$$\langle v\rangle_\varrho=\frac{1}{m}\,\mathrm{grad}\,\langle\Phi\rangle_\varrho\,. \tag{1·12}$$

Next we obtain the equations of these average quantities which involve non-average quantities. Such quantities are replaced by average quantities together with fluctuation terms which are very small if the parameter γ is sufficiently large; however, the magnitude of γ is to be bounded due to the following reasons.

Let the real and the imaginary parts of the wave function φ be q and p respectively, the c–density is expressed by

$$\langle\rho\rangle_x=\frac{1}{2}\{\langle p\rangle_x^2+\langle q\rangle_x^2\}-\left(\frac{\gamma}{2\pi}\right)^{3/2}+\langle\{R-\langle R\rangle_x\}^2\rangle_x\,, \tag{1·13}$$

in which

$$[p(x),\,q(x')]=\frac{1}{i}\delta(x-x')\,, \tag{1·14}$$

$$[\langle p\rangle_x,\,\langle q\rangle_{x'}]=\left(\frac{\gamma}{2\pi}\right)^{3/2}\exp[-\gamma(x-x')^2/2]\,, \tag{1·15}$$

and $\rho(x)=R^2(x)$. If and thus only if the parameter γ is sufficiently large and the expectation value of $\langle\rho\rangle_x$ is much larger at least than γ at the same time, both the fluctuation of $R(x)$ and the commutator (1·15) may be set equal to zero. In other words, the operators are regarded as commutative with one another and the fluctuation of the amplitude $R(x)$ is sufficiently small. Hence the classical theory is available if the expectation value of the c–density is large enough, regarding c–quantities as really c–number quantities.

§ 2. Equations of motion

Such transport equations as the equation of mass conservation, that of momentum conservation and that of energy conservation are derived from the equations of the density and the velocity potential or the velocity. We have only been concerned with such subspaces that the projection operator is set to be zero, and also the above-mentioned parameter γ is regarded as so large that fluctuation terms appearing are very small quantities.

First the equation of mass conservation law takes the form

$$\frac{\partial\langle\rho\rangle_x}{\partial t}+\mathrm{div}\langle\rho\rangle_x\langle v\rangle_x=0\,, \tag{2·1}$$

where

$$\langle J\rangle-\langle\rho\rangle\langle v\rangle$$

or

$$\langle J\rangle-\langle R\rangle\langle vR\rangle \text{ and } \langle vR\rangle-\langle v\rangle\langle R\rangle$$

are set to be zero. Regarding fluctuation terms as zero, the equation of the velocity potential operator takes the form

$$\frac{\partial\langle\Phi\rangle_x}{\partial t}+\frac{1}{2m}|\operatorname{grad}\langle\Phi\rangle_x|^2+\langle\Pi\rangle_x+\int G(x,x')\rho(x')dx'=0\,, \tag{2·2}$$

where

$$\langle\Pi\rangle_x=-\frac{\hbar^2}{8m}[|\operatorname{grad}\log\langle\rho\rangle_x|^2+2\Delta\log\langle\rho\rangle_x]=\frac{-\hbar^2}{2m}\frac{\Delta\langle R\rangle}{\langle R\rangle} \tag{2·3}$$

is called the quantum mechanical potential and $G(x, x')$ is the interaction energy. Taking the gradient of this equation and multiplied by the density $\langle\rho\rangle_x$, the equation of the velocity operator is expressed by

$$\langle\rho\rangle_x\frac{\partial\langle v\rangle_x}{\partial t}+\frac{1}{2}\langle\rho\rangle_x\operatorname{grad}\langle v\rangle_x^2+\operatorname{div}\langle\boldsymbol{P}\rangle_x+\frac{1}{m}\langle\rho\rangle_x\operatorname{grad}\int G(x,x')\langle\rho\rangle_x dx'=0\,, \tag{2·4}$$

where

$$m\langle\boldsymbol{P}\rangle_x=-\frac{\hbar^2}{4m}\langle\rho\rangle_x\operatorname{grad}\operatorname{grad}\log\langle\rho\rangle_x \tag{2·5}$$

is the pressure tensor. The equation of the total energy:

$$\langle E\rangle_x=\frac{m}{2}\langle\rho\rangle_x\langle v\rangle_x^2-\frac{\hbar^2}{8m}\langle\rho\rangle_x\Delta\log\langle\rho\rangle_x+\frac{1}{2}\langle\rho\rangle_x\int G(x,x')\langle\rho\rangle_{x'}dx' \tag{2·6}$$

is also obtained in the form

$$\frac{\partial\langle E\rangle_x}{\partial t}+\operatorname{div}\langle h\rangle_x+\frac{1}{2}\operatorname{div}\langle\rho\rangle_x\langle v\rangle_x\int G(x,x')\langle\rho\rangle_{x'}dx'$$
$$+\frac{1}{2}\langle\rho\rangle_x\int G(x,x')\operatorname{div}\langle\rho\rangle_{x'}\langle v\rangle_{x'}dx'=0\,, \tag{2·7}$$

where

$$\langle h\rangle_x=\left[\frac{m}{2}\langle\rho\rangle_x\langle v\rangle_x^2-\frac{\hbar^2}{8m}\langle\rho\rangle_x\Delta\log\langle\rho\rangle_x+m\langle\boldsymbol{P}\rangle_x\right]\langle v\rangle_x-\frac{\hbar^2}{8m}\langle\rho\rangle\operatorname{grad}\operatorname{div}\langle v\rangle_x \tag{2·8}$$

is the kinetic energy flow. Introducing the so-called substantial differentiation

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\langle v\rangle_x\cdot\operatorname{grad}\,, \tag{2·9}$$

the two fundamental equations turn out into the form

$$\frac{d}{dt}\langle\rho\rangle_x+\langle\rho\rangle_x\mathrm{div}\langle v\rangle_x=0\,, \tag{2·10}$$

$$\langle\rho\rangle_x\frac{d}{dt}\langle v\rangle_x+\mathrm{div}\langle \boldsymbol{P}\rangle_x+\frac{1}{m}\langle\rho\rangle_x\mathrm{grad}\int G(x,x')\langle\rho\rangle_{x'}dx'=0\,. \tag{2·11}$$

Further setting

$$\langle E\rangle_x=\langle\rho\rangle_x\langle U\rangle_x\,, \tag{2·12}$$

one finds the equation of the internal energy density $\langle U\rangle_x$ of the form

$$\langle\rho\rangle_x\frac{d\langle U\rangle_x}{dt}+\mathrm{div}\langle Q\rangle_x+m\langle \boldsymbol{P}\rangle_x\cdot\mathrm{grad}\langle v\rangle_x+\frac{1}{2}\langle\rho\rangle_x\int G(x,x')\langle\rho\rangle_{x'}\mathrm{div}\langle v\rangle_{x'}dx'=0\,, \tag{2·13}$$

where

$$\langle Q\rangle_x=-\frac{\hbar^2}{8m}\langle\rho\rangle_x\mathrm{grad}\,\mathrm{div}\langle v\rangle_x\,. \tag{2·14}$$

The fundamental equation (2·1) and (2·2) are summarized in the form

$$\frac{\partial W(x)}{\partial t}+\frac{1}{2m}|\mathrm{grad}\,W(x)|^2+\frac{\hbar}{2mi}\Delta W(x)+\int G(x,x')\langle\rho\rangle_{x'}dx'=0\,, \tag{2·15}$$

in which

$$W(x)=\langle\Phi\rangle_x+\frac{\hbar}{2i}\log\langle\rho\rangle_x\,. \tag{2·16}$$

The above function $W(x)$, corresponding to the principal function of Hamilton's partial differential equation, is the familiar one in the WKB approximation. Expanding $\langle\Phi\rangle$ and $\log\langle\rho\rangle$ in powers of $\hbar$:

$$\langle\Phi\rangle=\Phi_0+\hbar\Phi_1+\cdots\,, \tag{2·17}$$

$$\log\langle\rho\rangle=\log\rho_0+\hbar\log\rho_1+\cdots\,, \tag{2·18}$$

one obtains the following three equations for the first two terms of the expansion:

$$\frac{\partial\Phi_0(x)}{\partial t}+\frac{1}{2m}|\mathrm{grad}\,\Phi_0(x)|^2+\int G(x,x')\rho_0(x')dx'=0\,, \tag{2·19}$$

$$\frac{\partial\rho_0(x)}{\partial t}+\frac{1}{m}\mathrm{div}[\rho_0(x)\,\mathrm{grad}\,\Phi_0(x)]=0\,, \tag{2·20}$$

$$\frac{\partial\Phi_1(x)}{\partial t}+\frac{1}{m}\mathrm{grad}\,\Phi_0(x)\cdot\mathrm{grad}\,\Phi_1(x)=0\,. \tag{2·21}$$

Hereafter we restrict ourselves to the one dimensional problem. The last eqnation (2·21) shows that a curve represented by $\Phi_1=$const. in the (x,t)-plane may be regarded as the particle path along which particles move with the classical velocity $v=\mathrm{grad}\,\Phi_0/m$. We

expand the potential energy which is essentially non-local, in such a manner that the expression becomes local type :

$$\int G(x, x')\rho_0(x')dx' = \rho_0(x')\int G(r)dr + \frac{1}{2}\left|\frac{\partial\rho_0(x)}{\partial x}\right|^2\int r^2 G(r)dr + \cdots , \tag{2·22}$$

in which $\int r^{2n+1}G(r)dr$ is put to be zero, because G is an even function of r and considered as short range. In the one dimensional box of length L one may put

$$\int_{-L/2}^{L/2} G(r)dr = m\omega^2 L , \tag{2·23}$$

and the equations (2·19) and (2·20) turn out to be

$$\frac{\partial\rho_0(x)}{\partial t} + v_0(x)\frac{\partial\rho_0(x)}{\partial x} + \rho_0(x)\frac{\partial v_0(x)}{\partial x} = 0 , \tag{2·24}$$

$$\frac{\partial v_0(x)}{\partial t} + v_0(x)\frac{\partial v_0(x)}{\partial x} + \omega^2 L\frac{\partial\rho_0(x)}{\partial x} = 0 . \tag{2·25}$$

We have assumed that $\rho_1(x)$ is nearly equal to unity, for $\rho_0(x)$ should give the actual space density so far as our approximation method is available. The above equations represent the flow equation of a sort of polytropic gases[7] with the pressure

$$P(x) = \frac{\omega^2 L}{2}\rho_0(x)^2 , \tag{2·26}$$

and the characteristic curves in (x, t)–plane are given by

$$x_\alpha = t_\alpha(v_0 + c), \tag{2·27}$$

$$x_\beta = t_\beta(v_0 - c), \tag{2·28}$$

$$c^2 = \omega^2 L\rho_0(x). \tag{2·29}$$

Thus our problem may be regarded as that of the hyperbolic flow. There is another way to solve the equation (2·2). We regard the density operator involved in the interaction energy as a time-dependent parameter. Such a treatment resembles the adiabatic approximation. We shall apply this method in the following example.

Example

Coupled harmonic oscillators

We consider an assembly of coupled harmonic oscillators subjected to the following hamiltonian

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The stationary state problem has been considered by D. Bohm whose interpretation has recently been examined by Takabayasi, Prog. Theor. Phys. 8 (1952), 497.

$$H=-\frac{\hbar^2}{2m}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}+\frac{\varkappa}{2}\sum_{j,k=1}^{N}(x_j-x_k)^2+\frac{m\omega^2}{2}\sum_{k=1}^{N}x_k^2, \tag{E·1}$$

in which $\varkappa$ is the coupling constant which is so adjusted that the energy remains positive definite, therefore it depends on the total number of particles. The eigenvalue of the hamiltonian is obtained exactly by introducing the normal coordinates given by

$$\xi_\lambda=\sum_{\mu=1}^{N}O_{\lambda\mu}x_\mu, \qquad [O_{\lambda\mu}]: \text{the orthogonal matrix.} \tag{E·2}$$

These normal coordinates form the representative module of the irreducible regular representation of the symmetric group of order $N!$. One gets the eigenfrequency ω for the normal coordinate

$$\xi_1=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}x_j, \tag{E·3}$$

which corresponds to the center of gravity, and the eigenfrequency $[\omega^2+2(kN/m)]^{1/2}$ with the $(N-1)$ fold degeneracy.

Here we introduce the quantized wave functions φ and φ^*. The quantized hamiltonian takes the form

$$H=\frac{\hbar^2}{2m}\int\frac{\partial\varphi^*}{\partial x}\frac{\partial\varphi}{\partial x}\,dx+\left[\frac{m\omega^2}{2}+\varkappa N\right]\int x^2\varphi^*\varphi dx-\varkappa\left|\int x\varphi^*\varphi dx\right|^2, \tag{E·4}$$

and the associated classical equations are expressed by

$$\frac{\partial\log\langle\rho\rangle_x}{\partial t}+\frac{1}{m}\frac{\partial\langle\Phi\rangle_x}{\partial x}\frac{\partial\log\langle\rho\rangle_x}{\partial x}+\frac{1}{m}\frac{\partial^2\langle\Phi\rangle_x}{\partial x^2}=0, \tag{E·5}$$

$$\frac{\partial\langle\Phi\rangle_x}{\partial t}+\frac{1}{2m}\left|\frac{\partial\langle\Phi\rangle_x}{\partial x}\right|^2-\frac{\hbar^2}{8m}\frac{\partial^2\log\langle\rho\rangle_x}{\partial x^2}$$
$$+\left[\frac{m}{2}\omega^2+\frac{\varkappa N}{2}\right]x^2-\varkappa x\int x'\langle\rho\rangle_{x'}dx'+\frac{\varkappa}{2}\int x^2\langle\rho\rangle_{x'}dx'=0. \tag{E·6}$$

(a) Transformation function

Assuming the density $\langle\rho\rangle$ is independent of space coordinates, one gets the transformation function by a transformation of the form

$$x'=x-X(t),\ t'=t;\ \partial/\partial x=\partial/\partial x',\ \partial/\partial t=\partial/\partial t'-\dot{X}(t)\partial/\partial x', \tag{E·7}$$

$$\ddot{X}(t)+\Omega^2X(t)-\varkappa\int x\langle\rho\rangle_x\,dx=0, \tag{E·8}$$

$$\Omega^2=\omega^2+\varkappa N$$

and the following equation is obtained from (E·5) and (E·6)

$$\frac{\partial}{\partial t'}\left[\langle\Phi\rangle_{x'}+\frac{\hbar}{2i}\log\langle\rho\rangle_{x'}\right]+\left[\frac{1}{2m}\frac{\partial\langle\Phi\rangle_{x'}}{\partial x'}-\dot{X}(t)\right]\frac{\partial\langle\Phi\rangle_{x'}}{\partial x'}+\frac{m\Omega^2}{2}x'^2$$
$$+\frac{\hbar}{2mi}\frac{\partial\langle\Phi\rangle_{x'}}{\partial x'^2}+\frac{\varkappa}{2}\int x^2\langle\rho\rangle_x\,dx+\frac{m\Omega^2}{2}X(t)^2-\varkappa X(t)\int x\langle\rho\rangle_x\,dx=0. \tag{E·9}$$

If one sets[8]

$$\langle\Phi\rangle_{x'}+\frac{\hbar}{2i}\log\langle\rho\rangle_{x'}=\frac{1}{2}\left[a(t)x'^2+2b(t)x'+c(t)\right], \tag{E·10}$$

the transformation function in question is expressed by

$$U(x,y;t-t_0)=\left(\frac{m\Omega}{2\pi\hbar i\sin\Omega t}\right)^{1/2}\exp i\frac{m\Omega}{2\hbar}\Big\{[x-X(t)]^2\cot[\Omega(t-t_0)]$$
$$-2[x-X(t)]y\operatorname{cosec}[\Omega(t-t_0)]+\frac{2}{\Omega}[x-X(t)]\dot{X}(t)+y^2\cot[\Omega(t-t_0)]\Big\}\exp\frac{i}{\hbar}\int_{t_0}^{t}L(t')\,dt', \tag{E·11}$$

in which

$$L(t)=\frac{m}{2}\dot{X}(t)^2-\frac{m\Omega^2}{2}X(t)^2+\varkappa X(t)\int x\langle\rho\rangle_x\,dx-\frac{\varkappa}{2}\int x^2\langle\rho\rangle_x\,dx.$$

Let the wave function at the time T_0 be

$$\varphi(y, T_0)=\left(\frac{N^2m\Omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\Omega}{2\hbar}y^2\right], \tag{E·13}$$

and then the wave function at T becomes

$$\varphi(x, T)=\int U(x, y; T-T_0)\varphi(y, T_0)dy$$
$$=\left[\frac{N^2m\Omega}{\pi\hbar}\right]^{1/4}\exp\left[-\frac{i\Omega t}{2}+\frac{i}{\hbar}\int_{T_0}^{T}L(t')dt'+\frac{i}{\hbar}m\dot{X}(t)\{x-X(t)\}-\frac{m\Omega}{2\hbar}\{x-X(t)\}^2\right], \tag{E·14}$$

from which one observes that the unperturbed frequency Ω is shifted to the value $\Omega+(\varkappa/2m\Omega)$, which corresponds to the frequency of the adiabatic approximation.

The correct eigenvalue is obtained by the variation method. Assuming the wave function in the form

$$\langle R\rangle_x=\langle\rho\rangle_x^{1/2}=\left(\frac{\gamma}{\pi}\right)^{1/4}\exp\left[-\frac{\gamma}{2}\{x-X(t)\}^2\right], \tag{E·15}$$

and making use of the above transformation, and letting

$$\langle\Phi\rangle_{x'}=\langle\Phi\rangle'_{x'}+mx'\dot{X}(t)+\int_0^t L(t')dt',\ \langle\Phi\rangle'_{x'}=-E(\gamma)t+\text{const.} \tag{E·16}$$

in the equation (E·6), the postulation of the minimum energy:

$$\frac{\partial E(\gamma)}{\partial\gamma}=\frac{\hbar^2}{4m}-\frac{1}{4\gamma^2}[m\Omega^2+\varkappa N]=0, \tag{E·17}$$

gives rise to the correct value:

$$\Omega_0=\frac{\hbar\gamma}{m}=\left[\omega^2+\frac{2\varkappa N}{m}\right]^{1/2}. \tag{E·18}$$

(b) Energy increment with time

In our problem the field velocity is identical with that of the center of gravity X. From the expression of the energy density:

$$\langle E\rangle_x=\frac{m}{2}\langle\rho\rangle_x\langle v\rangle_x\langle v\rangle_x-\frac{\hbar^2}{8m}\langle\rho\rangle_x\frac{\partial^2\log\langle\rho\rangle_x}{\partial x^2}+\frac{m}{2}\omega^2x^2\langle\rho\rangle_x+\frac{\varkappa}{2}\int(x-x')^2\langle\rho\rangle_x\langle\rho\rangle_{x'}dx', \tag{E·19}$$

one gets

$$E=\frac{m}{2}N\dot{X}^2+\frac{m}{2}\omega^2NX^2-\frac{\hbar^2}{8m}\int\langle\rho\rangle_x\frac{\partial^2}{\partial x^2}\log\langle\rho\rangle_x dx+\frac{m\Omega_0^2}{2}\int\{x-X(t)\}^2\langle\rho\rangle_x dx. \tag{E·20}$$

Remembering that the ground state density is

$$\langle\rho\rangle_x=N\left(\frac{m\Omega_0}{\pi\hbar}\right)^{1/2}\exp\left[-\frac{m\Omega_0}{\hbar}\{x-X(t)\}^2\right], \tag{E·21}$$

the total energy at the time T is given by

$$E(T)=\left[\frac{m}{2}\dot{X}(T)^2+\frac{m\omega^2}{2}X(T)^2+\frac{1}{2}\hbar\Omega\right]N. \tag{E·22}$$

Thus the energy splits into two parts. One is the energy of the mass motion of the centre of gravity with the frequency ω. The other is the quantum-mechanical fluctuation of the system about the centre of gravity. In our approximation given above the motion of the centre of gravity is quite classical and subject to various initial conditions. One may suppose that this assembly behaves as if it were a molecule with internal structure. This example is too simple to grasp the collective behavior of interacting particles in general; it might, however, be possible to describe the motion of the assembly in such a similar manner that the quantal effect reveals itself as a fluctuation about the classical flow of quantal fluids.

§ 3. Propagation of sound waves

It has been shown that the assembly is regarded as a fluid having the quantum mechanical pressure tensor $\boldsymbol{P}(x)$, and we consider the excitation of sound waves. The theory of sound waves under consideration will favor us to get insight into the medium-like behavior of the system and to obtain the approximate eigenvalue of the hamiltonian.

We confine ourselves to those above-mentioned subspaces and develop the amplitude $R(x)$ in the power series of the density fluctuation $\rho'(x)$ as follows:

$$R(x)=[\rho_0+\rho'(x)]^{1/2}=\rho_0^{1/2}+\frac{1}{2}\frac{\rho'(x)}{\rho_0^{1/2}}-\frac{1}{8}\frac{\rho'(x)^2}{\rho_0^{3/2}}+\cdots, \tag{3·1}$$

and

$$\int_V \rho_0 dv=\rho_0 V=N, \tag{4·2}$$

in which ρ_0 is a constant of motion. Then the quadratic terms of the hamiltonian are expressed by

$$H=\frac{m}{2}N\sum_k v(k)v(-k)+\frac{\hbar^2}{8mN}\sum_k k^2\rho(k)\rho(-k)$$
$$+\frac{1}{2}\sum_{k\neq 0}G(k)\rho(k)\rho(-k)+\frac{1}{2}G(0)N^2-\frac{1}{2}G_0N-\frac{\hbar^2}{4m}\sum_k k^2. \tag{3·3}$$

Setting

$$v(k)=\frac{ik}{m}\Phi(-k), \tag{3·4}$$

one gets

$$H=\frac{N}{2m}\sum_k k^2\Phi(k)\Phi(-k)+\frac{\hbar^2}{8mN}\sum_k k^2\rho(k)\rho(-k)$$
$$+\frac{1}{2}\sum_{k\neq 0}G(k)\rho(k)\rho(-k)+\frac{1}{2}G(0)N^2-\frac{1}{2}G_0N-\frac{\hbar^2}{4m}\sum_k k^2. \tag{3·5}$$

Due to the commutation relation

$$[\rho(k),\ \Phi(k')]=i\hbar\delta_{k,k'}, \tag{3·6}$$

one finds the equations of motion

$$\frac{\partial\rho(k)}{\partial t}-\frac{N}{m}k^2\Phi(-k)=0, \tag{3·7}$$

$$\frac{\partial\Phi(k)}{\partial t}+\left[\frac{\hbar^2k^2}{4mN}+G(k)\right]\rho(-k)=0, \tag{3·8}$$

which leads to the equation

$$\frac{\partial^2\rho(k)}{\partial t^2}+k^2\left[\frac{\hbar^2k^2}{4m^2}+\frac{N}{m}G(k)\right]\rho(k)=0. \tag{3·9}$$

In the coordinate space this equation turns out to be

$$\frac{\partial^2\rho(x)}{\partial t^2}+\frac{\hbar^2}{4m^2}\Delta\Delta\rho(x)-\frac{\rho_0}{m}\int\Delta G(x,x')\rho(x')dx'=0\,. \tag{3·10}$$

Applying the expansion of the density given by (2·22) one gets again the equation of sound waves

$$\frac{\partial^2\rho(x)}{\partial t^2}=c^2\Delta\rho(x), \tag{3·11}$$

$$c^2=\frac{\rho_0}{m}\int G(x,x')dx', \tag{3·12}$$

neglecting the double Laplacian etc.

In the above formulation, we have assumed that the fluctuation of the density is sufficiently small. In the following we intend to examine whether the density fluctuations obtained are really small quantities or not.

The eigenfunction of the hamiltonian of sound waves is a function of fourier coefficients of the density. For the ground state we assume its form as

$$\Psi_0\{\rho_k\}=\prod_{k=-\infty}^{\infty}\varphi_0(\rho_k), \tag{3·13}$$

$$\varphi_0(\rho_k)=\exp\left[-\frac{1}{2}\gamma_k\rho(k)\rho(-k)\right], \tag{3·14}$$

$$\gamma_k=\frac{m}{\hbar N|k|}\left[\frac{\hbar^2k^2}{4m^2}+\frac{N}{m}G(k)\right]^{1/2}, \tag{3·15}$$

which is subject to the hamiltonian of the form

$$H_s=-\frac{\hbar^2N}{2m}\sum_k k^2\frac{\partial^2}{\partial\rho(k)\partial\rho(k)}+\frac{\hbar^2}{8mN}\sum_k\left[k^2+\frac{4mN}{\hbar^2}G(k)\right]\rho(k)\rho(k), \tag{3·17}$$

and

$$H_s\varphi_0(\rho_k)\varphi_0(\rho_{-k})=\hbar|k|c(k)\varphi_0(\rho_k)\varphi_0(\rho_{-k}), \tag{3·18}$$

$$c(k)=\left[\frac{\hbar^2k^2}{4m^2}+\frac{N}{m}G(k)\right]^{1/2}. \tag{3·19}$$*

Setting

$$\rho(k)=r_k\exp(i\varphi_k),\quad \rho(-k)=r_k\exp(-i\varphi_k), \tag{3·20}$$

the expectation value of $|\rho(k)|^2$ is given by

$$\langle|\rho(k)|^2\rangle_{AV}=\frac{\int r_k^2|\Psi_0|\prod_k r_k dr_k d\varphi_k}{\int|\Psi_0|^2\prod_k r_k dr_k d\varphi_k}=\frac{1}{2\gamma_k}\,. \tag{3·21}$$

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* This formula was once found by H. Nakano independently. (unpublished).

we estimate its magnitude for the following three cases:

(I) If $G(k)=0$, $\gamma_k=1/2N$ and $\langle r_k\rangle_{AV}=4N$.

(II) If $|k|$ is very small and $G|k|=4\pi e^2/k^2V$, $\gamma_k=(4\pi e^2mn)^{1/2}/\hbar|k|^2N$, $n=N/V$ and $\langle r_k^2\rangle_{AV}=\hbar k^2N/(16\pi e^2mn)^{1/2}$.

(III) If $|k|$ is very small and $G(k)=A(=\text{const.})/V$, $\gamma_k=[Amn]^{1/2}/\hbar|k|N$ and $\langle r_k^2\rangle_{AV}=\hbar|k|N/2[Amn]^{1/2}$.

Thus we see that these average values are sufficiently small compared with the square total number of particles: N^2, especially the values of the last two cases are almost negligible.

The second case is known as the problem of the plasma oscillation with the frequency $\omega_p=[4\pi e^2N/mV]^{1/2}$.

§ 4. One body problem aspect

In this section we study the collective response of the assembly to an individual particle moving through the system with velocity which is larger than the group velocity of the collective oscillation which measures the speed with which the disturbance is propagated. We show that the collective oscillation is excited in a wake tailing the particle. This wake resembles the Cerenkov radiation obtained when fast electrons go through a dielectric.

First the hamiltonian of $N+1$ particle system is considered. Letting the interaction energy of the projected particle with particles in the assembly be $g(x, x')$, the hamiltonian becomes

$$H(N+1)=\frac{P^2}{2M}+\int G(x, x')\rho(x')dx'+H(N), \tag{4·1}$$

in which M means the mass of the incident particle and

$$\rho(x)=\sum_{j=1}^{N}\delta(x-x_j), \tag{4·2}$$

and $H(N)$ is the hamiltonian of N particle system given by

$$H(N)=\sum_{j=1}^{N}\frac{P_j^2}{2m}+\frac{1}{2}\iint G(x, x')\rho(x')\{\rho(x)-\delta(x-x')\}dx\,dx'. \tag{4·3}$$

Here we rewrite the hamiltonian $H(N)$ in the quantized form. Then the hamiltonian $H(N+1)$ shows that a particle interacts with the assembly of N particles. As a first approximation $H(N)$ may be regarded as representing the hamiltonian of sound waves given in Section 3. From $H(N+1)$ one finds the equation of the density fluctuation in the form

$$\ddot{\rho}(k)+\omega(k)^2\rho(k)=-\frac{N}{m}g(k)k^2\exp[-ik\cdot v_0t], \tag{4·4}$$

using (4·5) and regarding the velocity v_0 of the incident particle as a parameter. Its particular solution representing the reaction of the individual particle motion on the density fluctuation, becomes

$$\rho(k)=-\left[\frac{N}{m}k^2g(k)\Big/\omega(k)^2-(k\cdot v_0)^2\right]\exp[-ik\cdot v_0t], \tag{4·5}$$

$$\omega(k)^2=\frac{\hbar^2k^4}{4m^2}+\frac{N}{m}G(k), \tag{4·6}$$

where we have assumed that the particle moves freely along the classical path with the constant velocity v_0 during the time interval t. In the coordinate space the variation of the density due to the motion of the particle becomes

$$\rho(x)=\frac{1}{V}\sum_k\rho(k)\exp[ikx]$$

$$=-\frac{N}{mV}\sum_k\frac{g(k)k^2}{\omega^2-(k\cdot v_0)^2}\exp[ikx-ikv_0t]. \tag{4·6}$$

As an example we assume the fourier coefficient of the potential in the form

$$G(k)=\frac{m}{N}c^2 \quad \text{for} \quad |k|\leqq k_0',\ G(k)=0 \quad \text{for} \quad |k|>k_0', \tag{4·7}$$

$$g(k)=\frac{m}{N}c_1^2 \quad \text{for} \quad |k|\leqq k_0<k_0',\ g(k)=0 \quad \text{for} \quad |k|>k_0, \tag{4·7'}$$

hence the frequency is $\omega=|k|c$. Then the density variation turns out to be

$$\rho(x)=-\frac{c_1^2}{V}\sum_k\frac{k^2}{\omega^2-(k\cdot v_0)^2}e^{ik\cdot(x-v_0t)}. \tag{4·8}$$

Choosing the direction of v_0 as z axis and setting

$$k_x'=ck_x,\ k_y'=ck_y,\ k_z'=[c^2-v_0^2]^{1/2}k_z;\ r_x'=r_x/c,$$

$$r_y'=r_y/c,\ r_z'=r_z/[c^2-v_0^2]^{1/2};\ r=x-v_0t, \tag{4·9}$$

(4·8) becomes

$$\rho(x)=\frac{(c_1^2/c^2)}{(2\pi)^3[c^3-v_0^2]^{1/2}}\Delta_r\int\frac{e^{ik'r'}}{|k'|^2}dk'=\frac{1}{[c^2-v_0^2]^{1/2}}\Delta_r\frac{1}{4\pi|r'|}, \tag{4·10}$$

where Δ_r means the Laplacian with respect to the variable r. Going back to the original coordinate system, one obtains

$$\rho(x)=\frac{\beta}{4\pi i}\Delta_r\frac{(c_1^2/c^2)}{[r_x^2+r_y^2-\beta^2r_z^2]^{1/2}}, \tag{4·11}$$

$$\beta^2=\frac{c^2}{v_0^2-c^2}. \tag{4·12}$$

From this expression one observes that $\rho(x)$ is merely real number within the cone $[r_x^2+r_y^2-\beta^2r_z^2]$ and sound quanta are excited in a wake tailing the particle which is moving at the vertex of the cone, if v_0 is greater than the sound speed c.

In the above formulation the integration limit with respect to the propagation number is infinite, but it must be smaller than k_0, hence the more careful presentation becomes

necessary. Considering this fact we estimate the energy loss of the particle in the wake. The force of the induced density fluctuation acts on the particle to pull it back. The z–component of the interaction force is given by

$$F_z^{\varrho}=-\frac{\partial}{\partial z}\int G(x,x')\rho(x')dx'=-\int G(x,x')\frac{\partial\rho(x')}{\partial z'}dx', \qquad (4\cdot13)$$

where the x–and y– components vanish in virtue of the symmetry and

$$\frac{\partial\rho(x)}{\partial z}=-\frac{ic_1^2}{V}\sum_{k}^{|k_0|}\frac{k_z k^2}{\omega^2-k_z^2v_0^2}\exp[ik\cdot(x-v_0t)]. \qquad (4\cdot14)$$

Choosing the counter in the complex k_z–plane so that $\rho(x)$ vanishes for $Z>V_0t$, (4·14) becomes

$$\frac{\partial\rho(x)}{\partial Z}=\frac{-i(c_1/c)^2}{(2\pi)^2(v_0^2-c^2)}\int_{\sqrt{k_x^2+k_y^2}=k_0(1-c^2/v_0^2)^{1/2}} k^2\cos[k_z^0(Z-v_0t)]\exp[ik_xx+ik_yy]dk_xdk_y, \qquad (4\cdot15)$$

$$k_Z^{\varrho}=c^2(k_x^2+k_y^2)/(v_0^2-c^2), \qquad (4\cdot16)$$

therefore the integration over the semi-sphere ($Z'\leqq v_0t$) gives

$$F_Z^{\varrho}=-\frac{V}{N}\frac{mc_1^2v_0^2}{4\pi(v_0^2-c^2)^2}\int_0^{k_0(1-c^2/v_0^2)^{1/2}}k^3dk=(-)\frac{mk_0^4c_1^2}{16\pi\rho_0}\left(\frac{c}{v_0}\right)^2. \qquad (4\cdot17)$$

Further the surface tension of the density fluctuation acts on the particle too. It comes from the quantum-mechanical pressure tensor which is primarily local type, and acts in the direction of the motion of the particle inversely to the classical force. If the density fluctuation is regarded as small, z–component of the quantum-mechanical force turns out into the form

$$F_Z^{\varrho}=-\frac{\partial}{\partial Z}\left[\frac{1}{2m}|\operatorname{grad}\Phi|^2-\frac{\hbar^2}{8m}\{|\operatorname{grad}\log\rho|^2+2\Delta\log\rho\}\right]. \qquad (4\cdot18)$$

The first term in the right-hand side vanishes and at the position of the particle, the head of the shock, it amounts to

$$F_Z^{\varrho}=\frac{\hbar^2k_0^6}{48m\pi\rho_0}\left(\frac{c_1}{v_0}\right)^2. \qquad (4\cdot19)$$

The quantum-mechanical force depends upon the magnitude of the density itself. The density fluctuation and the associated velocity potential vanish at the side ahead of the shock front and F_z is obtained by taking average of the two values at the front side and the back side. One observes that the contribution of the quantum-mechanical force is comparable with that of the interaction force and the rate of the energy loss in the z–direction is given by

$$\frac{\partial E}{\partial Z}=\bar{F}_z=\frac{1}{2}(F_Z^Q+F_Z^C)$$

$$=\frac{mc_1^2k_0^4}{96\pi\rho_0 v_0^2}\left[\left(\frac{\hbar k_0}{m}\right)^2-3C^2\right]. \tag{4·20}$$

The above process has a striking similarity to the Cerenkov radiation. Here we wish to give an analogous formulation with the radiation theory. In the coordinate space the energy density of sound waves is expressed by

$$H(x)=\frac{mc^2}{2\rho_0}|\rho(x)|^2+\frac{\rho_0}{2m}|\operatorname{grad}\Phi(x)|^2, \tag{4·21}$$

and

$$[\rho(x),\ \Phi(x')]=i\hbar\delta(x-x'). \tag{4·22}$$

From the equations of motion the rate of the energy changes with time in the region of volume V reads

$$\int_V\frac{\partial H(x)}{\partial t}dx=-\frac{c^2}{2}\int_S[\rho(x)\operatorname{grad}\Phi(x)+\operatorname{grad}\Phi(x)\rho(x)]\cdot nd\sigma=-c^2\int_S J(x)\cdot nd\sigma\,,$$

n: unit normal vector to the surface S, (4·23)

hence the energy flux per unit area is solely determined by the current operator $J(x)$. The equation of the density fluctuation turns out to be

$$-\frac{\partial^2\rho(x)}{\partial t^2}+c^2\Delta\rho(x)=S(x), \tag{4·24}$$

where $S(x)=(\rho_0/m)\sum k^2g(k)\exp[ik(x-v_0t)]$. The density fluctuation is given in the form

$$\rho(x)=\frac{1}{4\pi}\int\frac{S(x')}{|x-x'|}\exp[i(\omega/c)|x-x'|]dx'. \tag{4·25}$$

Since we are interested in the monochromatic wave of frequency ω, the conjugate quantity $\Phi(x)$ is given by

$$\Phi(x)=\frac{mc^2}{i\omega\rho_0}\rho(x). \tag{4·26}$$

§ 5. Self-consistent linearization and correlation interaction

In the above we have shown that the linearization of the hydrodynamical equation leads to the equation of sound waves; it, however, is valid merely for such cases that the velocity and the density fluctuation may be regarded as small quantities. It is our present task to observe the change of the vibrational motion when the motion of individual particles is appreciable.

If the interaction energy is absent there appears nothing collective. When the interaction is turned on, the free particle motion is transformed so as to excite the collective

oscillation associated with the density polarization. First we assume that the trial potential energy acting on the particle is given by such a density polarization, and then we find the density variation of the particle assembly arising from the individual particle motion. This reproduced density variation is identified to the above one applied to the trial potential so that the frequency of the collective oscillation is determined. We shall see that there appear N normal coordinates with the respective frequencies a· much the same with the vibrational branches of a crystal lattice. If the particles are at rest these various branches degenerate to the linearised frequency. The interaction of free particles with these normal coordinates gives rise to the self energy and the Møller interaction between particles.

First we consider the body problem that the individual particle interacts with oscillation quanta through the interaction energy given by (4·1). We shall treat this problem following the Tomonaga-Schwinger interaction representation[9] which reads

$$i\hbar\frac{\partial U}{\partial t}=H_{\text{INT.}}U\,, \tag{5·1}$$

$$H_{\text{INT.}}=\int G(x,x')\rho(x')dx'. \tag{5·2}$$

Introducing the annihilation and the creation operator of the oscillation as follows:

$$\rho(k)=|k|[\hbar N/2m\omega(k)]^{1/2}[b^*(-k)+b(k)]\,, \tag{5·3}$$

$$\Phi(k)=(1/i|k|)[\hbar m\omega(k)/2N]^{1/2}[-b^*(k)+b(-k)]\,, \tag{5·4}$$

$$[b(k),\,b^*(k')]=\delta_{k,k'}\,, \tag{5·5}$$

and taking the first order effect into consideration the solution of (5·1) becomes

$$U_1=1-\frac{i}{\hbar}\int_{-\infty}^{t=0}H_{\text{INT.}}(t')dt'=1-\frac{1}{\hbar}\sum_k F(k)[b^*(-k)\exp(ikx)/\{\omega(k)+(k\cdot p/m)+(\hbar k^2/2m)\}-b(k)\exp(ikx)/\{\omega(k)-(k\cdot p/m)-(\hbar k^2/2m)\}]\,, \tag{5·6}$$

$$F(k)=[\hbar N/2m\omega(k)]^{1/2}|k|G(k), \tag{5·7}$$

where the identity $\exp ik(x+pt/m)=\exp(ik\cdot x)\exp(ik\cdot pt/m)\exp(i\hbar k^2t/2m)$ is applied. The fourier coefficient of the density of N particles is expressed by

$$\rho_N(k)=\sum_{j=1}^{N}\exp(-ikx_j)\,, \tag{5·8}$$

hence the variation of the density becomes

$$\delta\rho_N(k)=\frac{k^2F(k)}{m}\left[\sum_{j=1}^{N}b^*(-k)/\{[\omega(k)+(k\cdot p_j/m)]^2-(\hbar^2k^4/4m^2)\}+\sum_{j=1}^{N}b(k)/\{[\omega(k)-(k\cdot p_j/m)]^2-(\hbar^2k^4/4m^2)\}\right], \tag{5·9}$$

since

$$\delta[\exp(-ikx)]=\frac{i}{\hbar}\left[\int_{-\infty}^{t=0}H_{\text{INT.}}(t')dt',\ \exp(-ikx)\right].$$

The density variation given here should be equal to $\rho(k)$ given by (5·3). Therefore one gets the dispersion relation in the form

$$1=\frac{k^2G(k)}{m}\sum_{j=}^{N}\frac{1}{[\omega(k)-(k\cdot p_j/m)]^2-[\hbar k^2/2m]^2}, \tag{5·10}$$

Now we consider the interaction of several free particles with oscillation quanta. To simplify the situation we assume that M particles which are identical with members of the assembly are projected with respective momenta and that there exists no direct interaction among these particles. Hence the total hamiltonian is given by

$$H=\sum_{j=1}^{M}\frac{p_j^2}{2m}+H_{\text{INT.}}+\sum_k\hbar\omega(k)b^*(k)b(k), \tag{5·11}$$

$$H_{\text{INT.}}=\sum_{j=1}^{M}\int G(x_j,x')\rho(x)dx', \tag{5·12}$$

The second order effect gives rise to the self-energy of particles and the Møller interaction between particles. One obtains (See Appendix II).

$$\Delta E=\left(\frac{-i}{\hbar}\right)^2\left\langle\int_{-\infty}^{t}H_{\text{INT.}}(1)d1\int_{-\infty}^{1}H_{\text{INT.}}(2)d2\right\rangle_{\text{AV.}}\Big/\left(\frac{-it}{\hbar}\right)$$

$$=\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\left\langle\left[\int_{-\infty}^{t}H_{\text{INT.}}(1)d1,\int_{-\infty}^{1}H_{\text{INT.}}(2)d2\right]\right\rangle_{\text{AV.}}\Big/\left(\frac{-it}{\hbar}\right)=H_{\text{S.E.}}+H_{\text{M.I.}}, \tag{5·13}$$

$$H_{\text{S.E.}}=-(N/2m)\sum_{j=1}^{M}\sum_k k^2G(k)^2/\{\omega(k)^2-[k\cdot v_j+\hbar k^2/2m]^2\}, \tag{5·14}$$

$$H_{\text{M.I.}}=-(N/2m)\sum_{j\neq j'}^{M}\sum_k k^2G(k)^2\exp ik(x_j'-x_j)/\{\omega(k)^2-[k\cdot v_j+\hbar k^2/2m]^2\}, \tag{5·15}$$

where

$$v_j=p_j/m.$$

The self-energy $H_{\text{S.E.}}$ is evaluated using the potential (4·8). If $\hbar k_0^2/2m\ll\omega(k_0)$, it becomes

$$H_{\text{S.E.}}=-\sum_{j=1}^{M}(L/2\pi)^3\int_0^{k_0}dk\int_0^{\pi}(mc^4/2N)2\pi k^3\sin\theta d\theta/(c^2-v_j^2\cos^2\theta). \tag{5·16}$$

If $v_j<c$, this expression turns out into

$$H_{\text{S.E.}}=-(L^3/(2\pi)^2)(mc^2k_0^3/6N)\sum_{j=1}^{M}(c/v_j)\log((c+v_j)/(c-v_j)), \tag{5·17}$$

and taking the principal value one gets the similar expression replacing $(c-v_j)$ by (v_j-c) for the case that $c<v_j$. For such a case, as shown in Section 4, the Cerenkov-like radiation takes place, while for the other case that $v_j<c$ the particles are surrounded by co-moving cloud of the polarized density. The last energy expression $H_{\text{M.I.}}$ gives the

Møller interaction between particles. Following the calculation of Section 4 one readily finds

$$H_{\text{M.I.}}=\frac{Vmc^2}{8\pi Ni}\sum_{j\neq j'}\sum \Delta_{r(j,j')}\beta_j/[r_x(j,j')^2+r_y(j,j')^2-\beta_j^2 r_z(j,j')^2]^{1/2}, \tag{5·21}$$

$$\beta_j=c/[v_j^2-c^2]^{1/2}, \quad |v_j|^2=|v_{j,z}|^2,$$

where the limit of the integration in the k-space is assumed to be infinity. For the case that $v_j<c$, the equi-potential surface is an ellipsoid, whose minor axis is parallel to the momentum of the particle. For the case that $c<v_j$ the interaction acts merely in the cones $(r_x^2+r_y^2-\beta^2 r_z^2)$. As the limit becomes larger the character of the interaction approaches to the above one.

We may conclude that the density fluctuation in the quiet assembly is represented by the linearized frequency, while for the assembly in which the individual particle motion is appreciable the frequency is replaced approximately by the doppler frequency $\omega(k)-k\cdot v$.

In the above formulation we may regard the M particles as members of the assembly themselves instead of external ones and further the density operator $\rho(x)$ involved in the interaction energy as an external parameter representing the coordinate of oscillator. Such an idea might be useful for practical purposes in place of the conventional self-consistent methods. Here we add a brief remark on the Bloch-Nordsieck transformation[10] in which the particle moves along the classical orbit. The kinetic energy of two particles is approximately given by

$$H_k=\frac{m}{2}(v_1^2+v_2^2)+(v_1\cdot p_1)+(v_2\cdot p_2), \quad v_1, v_2: \text{const.} \tag{5·22}$$

The momentum p_j and the creation operator b^* and the annihilation operator b are transformed acording to

$$p_j=p_j'-\hbar\sum_k k[b^*(k)'\exp[-ikx_j+i\omega(k)t]+b(k)'\exp[ikx_j-i\omega(k)t] + \sum_{j'}\sigma_{j'}(k)\exp ik(x_{j'}-x_j)]\sigma_j(k), \quad j, j'=1, 2., \tag{5·23}$$

$$b^*(k)=b^*(k)'+\sum_j \sigma_j(k)\exp[ikx_j-i\omega(k)t], \tag{5·24}$$

$$b(k)=b(k)'+\sum_j \sigma_j(k)\exp[-ikx_j+i\omega(k)t], \tag{5·25}$$

where

$$\sigma_j(k)=-F(k)/\hbar[\omega(k)-(k\cdot v_j)]. \tag{5·26}$$

Hence the transformed hamiltonian reads

$$H=H_k'+\hbar\sum_k \omega(k)b^*(k)'b(k)'+H_{\text{S.F.}}+H_{\text{M.I.}}, \tag{5·27}$$

$$H_{\text{S.F.}}=-(N/2m)2\sum_k k^2G(k)^2/\{\omega(k)^2-(k\cdot v)^2\}, \tag{5·28}$$

$$H_{\text{M.I.}}=-(N/m)\sum_k k^2G(k)^2\exp[ik(x_1-x_2)]/\{\omega(k)^2-(k\cdot v)^2\}, \tag{5·29}$$

where we have set $v_1=v_2$.

Thus again one finds the self-energy and the Møller interaction. The theory of forced linear harmonic oscillator mentioned in the example is applicable to this problem and it gives rise to the same result with the above one. Considering the one body problem, from (3·5) and (4·1) the total hamiltonian is given in the form

$$H=\frac{p^2}{2m}+\sum_{k>0}G(k)[\rho(k)\exp(ikx)+\rho(-k)\exp(-ikx)]$$
$$-\frac{N\hbar^2}{m}\sum_{k>0}k^2\frac{\partial^2}{\partial\rho(-k)\partial\rho(k)}+\frac{m}{N}\sum_{k>0}\frac{\omega(k)^2}{k^2}\rho(k)\rho(-k). \tag{5·30}$$

Applying a canonical transformation of the form

$$q_k=\rho(k)-Q_k,\ q_k^*=\rho(-k)-Q_k^*,\ t=t', \tag{5·31}$$

$$\ddot{Q}_k+\omega(k)^2Q_k+\frac{Nk^2G(k)}{m}\exp(-ikv_0t)=0, \tag{5·32}$$

$$\ddot{Q}_k^*+\omega(k)^2Q_k^*+\frac{Nk^2G(k)}{m}\exp(ikv_0t)=0, \tag{5·32'}$$

together with the transformation of the wave function:

$$\Psi=\prod_{k>0}\Psi_k=\prod_{k>0}\Psi_k'\exp\frac{i}{\hbar}\left(\frac{m}{Nk^2}\right)[q_k\dot{Q}_k^*+q_k^*\dot{Q}_k]\exp(iKx), \tag{5·33}$$

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi, \tag{5·34}$$

one gets the eigenfunction in the form

$$\Psi_k'=\Psi_k''\exp\frac{i}{\hbar}\int^t L_k(t')\,dt',\quad i\hbar\frac{\partial\Psi_k''}{\partial t}=\left[-\frac{N\hbar^2}{m}k^2\frac{\partial^2}{\partial q_k\partial q_k^*}+\frac{m\omega(k)^2}{Nk^2}q_kq_k^*\right]\Psi_k'', \tag{5·35}$$

where

$$L_k(t)=\frac{m}{Nk^2}\dot{Q}_k\dot{Q}_k^*-\frac{m}{N}k^2\omega(k)^2Q_kQ_k^*-G(k)[Q_k\exp(ikv_0t)+Q_k^*\exp[-ikv_0t)], \tag{5·36}$$

Since this Lagrangian does not depend on time explicitly, one finds the self-energy

$$E_S=-L=-\frac{N}{2m}\sum_k{}'\frac{k^2G(k)}{\omega(k)^2-(k\cdot v_0)^2}, \tag{5·37}$$

which is identical with (5·28).

§ 6. Concluding remark

In the scheme of the second quantization it has been shown that the hydrodynamical equation is obtained restricting ourselves to the above-mentioned subspace and applying the *WKB* approximation. Thus the behavior of the assembly as a whole is described by the

theory of the irrotational hyperbolic flow. The existence of the phonon state is confirmed, while no evidence of the "roton", which is predicted in Landau's theory of Helium II, is found. The vortex motion would appear in connection with a more macroscopic treatment after making use of some statistical averages. We have read off the zero density region in our approximation. The vortex motion does appear associated with such a region; it has, however, not yet been clarified what physical meaning it has. The starting point of our theory is the same with the others proposed by Born and Green as well as Irving and Zwanzig. They laid stress on the similarity of their formulation with the kinetic theory of gases to attack the irreversible processes. In the formulation given here, the many body problem is regarded as a problem of the hydrodynamics or that of the field of oscillation quanta. The behavior of the individual particle is described in terms of the interaction with such fields, rather than the interaction with each particle composing the assembly. So far as boson assemblies are concerned, a quantum mechanical extension of the classical description of Pines and Bohm has been achieved.

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Appendix I. Kinetic-theoretical stress tensor and energy flow

The relationship between the kinetic theoretical and the hydrodynamical quantities is shown applying the classical phase space distribution function as follows:

Kinetic stress tensor

$$\int \frac{p_j p_k}{m} f_{\mathrm{Cl}}(P, Q)\, dp = \frac{(-)\hbar^2}{m}\left(\frac{\gamma}{\pi}\right)^{3/2}\left(\frac{1}{2\pi\hbar}\right)^3 \iiint \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_k}\rho(x, r)$$
$$\times \exp\left[-\gamma(x-Q)^2 - \frac{\gamma}{4}r^2 - \frac{i}{\hbar}pr\right] \cdot dr\, dx\, dp$$
$$= (-)\frac{\hbar^2}{m}\left\langle \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_k}\rho(x, r)\right\rangle_H = \langle T_{jk}\rangle_H, \qquad (\mathrm{A}\cdot 1\cdot 1)$$

where $\langle\ \rangle_H$ means the average with respect to the Heisenberg wave packet and it is expressed by the classical density and the classical velocity operator in the form

$$\langle T_{jk}\rangle = m\langle\rho\rangle_Q\langle v_j\rangle_Q\langle v_k\rangle_Q - \frac{\hbar^2}{4m}\langle\rho\rangle_Q\frac{\partial}{\partial Q_j}\frac{\partial}{\partial Q_k}\log\langle\rho\rangle_Q. \qquad (\mathrm{A}\cdot 1\cdot 2)$$

Also the k-component of the energy-flow vector reads

$$\int p_k\frac{p^2}{2m^2}f_{\mathrm{Cl}}(P, Q)\, dp = \left(\frac{\hbar}{i}\right)^3\frac{1}{2m^2}\left\langle\frac{\partial}{\partial r_k}\frac{\partial^2}{\partial r^2}\rho(x, r)\right\rangle_H = \langle h_k\rangle_H$$
$$= \left[\frac{m}{2}\langle\rho\rangle_Q\langle v\rangle_Q^2 - \frac{\hbar^2}{8m}\langle\rho\rangle_Q\Delta\log\langle\rho\rangle_Q\right]\langle v_k\rangle_Q$$

$$+m[\langle \boldsymbol{P}\rangle_Q\cdot\langle v\rangle_Q]_k-\frac{\hbar^2}{8m}\langle\rho\rangle_Q\frac{\partial}{\partial Q_k}\operatorname{div}\langle v\rangle_Q\,, \tag{A·1·3}$$

where

$$\langle \boldsymbol{P}\rangle_Q=(-)\frac{\hbar^2}{4m}\langle\rho\rangle_Q\frac{\partial}{\partial Q}\frac{\partial}{\partial Q}\log\langle\rho\rangle_Q \tag{A·1·4}$$

is the kinetic pressure tensor and

$$\langle q\rangle_Q=(-)\frac{\hbar^2}{8m}\langle\rho\rangle_Q\operatorname{grad}\operatorname{div}\langle v\rangle_Q \tag{A·1·5}$$

corresponds to the heat flow of the classical kinetic theory.

Appendix II. The perturbation energy

We have obtained the second-order perturbation energy by the formula

$$(-i/\hbar)t\Delta E$$
$$=\left(\frac{-i}{\hbar}\right)^2\left\langle\int_{-\infty}^{t}H(1)d1\int_{-\infty}^{1}H(2)d2\right\rangle_{\text{AV.}}=\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\left\langle T\int_{-\infty}^{t}H(1)d1\int_{-\infty}^{t}H(2)d2\right\rangle_{\text{AV.}}, \tag{A·2·1}$$

where "T" is the so-called chronological operator and the average is taken with respect to the ground state, so to speak the vacuum state of the oscillation quanta. Here we assume that the interaction energy is adiabatically turned on from the infinite past and vanishes to the infinite future adiabatically as well. Then the above expression is transformed by virtue of the following relation

$$T\left(\int_{-\infty}^{\infty}H(1)d1\int_{-\infty}^{\infty}H(2)d2\right)=\int_{-\infty}^{\infty}H(1)d1\int_{-\infty}^{1}H(2)d2+\int_{1}^{\infty}H(2)d2\int_{-\infty}^{\infty}H(1)d1$$
$$=\left[\int_{-\infty}^{\infty}H(1)d1\right]^2+\left[\int_{-\infty}^{\infty}H(1)d1,\int_{-\infty}^{1}H(2)d2\right]. \tag{A·2·2}$$

Noting the fact that the adiabatic assumption leads to

$$\lim_{t\to\infty}\left\langle\left[\int_{-t}^{t}H(1)d1,\int_{-\infty}^{1}H(2)d2\right]\right\rangle_{\text{AV.}}=\lim_{t\to\infty}\left\langle 2t\left[H(1),\int_{-\infty}^{1}H(2)d2\right]\right\rangle_{\text{AV.}} \tag{A·2·3}$$

together with the identity

$$\left\langle\left|\int_{-\infty}^{\infty}H(1)d1\right|^2\right\rangle_{\text{AV.}}=0\,, \tag{A·2·4}$$

one gets the perturbation energy in the form

$$\Delta E=\frac{1}{2t}\left(\frac{-i}{\hbar}\right)\left\langle\left[\int_{-\infty}^{t}H(1)d1,\int_{-\infty}^{1}H(2)d2\right]\right\rangle_{\text{AV.}} \tag{A·2·5}$$

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Note added in proof.

1) Tomonaga's theory of many fermion problems is essentially a linear theory, while ours is a non-linear theory. The discussion of the phonon-phonon scattering caused by non-linear terms is left to another occasion. Recently H. Nakano applied Tomonaga's idea straight-forwardly to the many boson problems. The difference between his result and ours comes from non-linear terms.

2) Since the phonon state is obtained assuming the interaction energy in a special form, the application of our results seems to be very much restricted. It is our standpoint to start with a tractable solution such as the phonon state instead of dealing with the actual interaction energy directly and to proceed to the higher order calculation regarding the remainder of the interaction energy as perturbation terms.

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On the Relations between Beta-decay Nuclear Matrix Elements

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Ahrens and Feenberg[1] have found several relations between the first forbidden β-decay nuclear matrix elements. In this paper, they are extended to the general forbiddenness, and compared with experiments.

Ahrens and Feenberg[1] have found several relations between the first forbidden β-decay nuclear matrix elements with a rough but skilful method. In this paper, they are extended to the general forbiddenness. Comparing them with experiments, we shall expect some information about the β-decay interaction type.

According to Ahrens and Feenberg,[1] there is a relation between the nuclear matrix elements of any operator X and its commutator with the free Hamiltonian H_0. The relation is*

$$[(W_i-W_f)+0.48(e^2/R)(Z-1)-\{\delta/(2A)\}\{(-1)^N+(-1)^Z\}]\langle f|X|i\rangle \cong -\langle f|[H_0, X]|i\rangle. \quad (1)$$

Here, i and f denote 'initial' and 'final' respectively, W is the energy eigenvalue,[1] R is the nuclear radius, N and Z are the numbers of the neutrons and the protons of the product nucleus respectively, and δ is a quantity which has been used by Feenberg[2] in connection with the energy differences between even-even, odd-odd and even mass nuclei. We put

$$\{\Delta uZ/(2R)\}=(W_i-W_f)+0.48(e^2/R)(Z-1)-\{\delta/(2A)\}\{(-1)^N+(-1)^Z\}. \quad (2)$$

When Z is not small, the second term of the right-hand side of eq. (2) is apt to be considerably larger than the others, and A is near unity. Eq. (1) can be written as

$$\{\Delta uZ/(2R)\}\langle f|X|i\rangle \cong -\langle f|[H_0, X]|i\rangle. \quad (3)$$

The problem is to transform $\langle f|[H_0, X]|i\rangle$ into the familiar form of the β-decay theory.

Our objects are in Greuling's notation,[3] the following nuclear matrix elements: $Q_n(r, \boldsymbol{r}) \cong -Q_n(\beta r, \boldsymbol{r})$, $Q_n(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r}) \cong -Q_n(\beta\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})$, $Q_n(\alpha, \boldsymbol{r})$ and $Q_n(\beta\alpha, \boldsymbol{r})$, where $\cong$ denotes the equality in the non-relativistic approximation. The minus sign is the consequence of the representation used in the current β-decay theory, in which the large

* We use the units $\hbar=c=m=1$. (m is the electron rest mass.)

component belongs to the eigenvalue -1 of β. For the general cases, it is convenient to use the polarized solid harmonics $\mathcal{Y}_{LM}$, and it is sufficient to get a relation about the one component (e.g. $\mathcal{Y}_{LL}$) of them.[4] First, we consider $Q_n(\boldsymbol{r}, \boldsymbol{r})$, and we can take up $\sum_k \mathcal{Y}_{nn}(\boldsymbol{r}_k, \boldsymbol{r}_k) Q_k = \sum_k (r_{k1}+ir_{k2})^n Q_k$ as the corresponding polarized solid harmonics. (Normalization of $\mathcal{Y}_{LM}$ is unnecessary.) Here, k is the nucleon number, and Q is the displacement operator from neutron to proton. We substitute the above solid harmonics for the X of eq. (1) or (3). The free Hamiltonian is

$$H_0 = -\sum \alpha_k \cdot \boldsymbol{p}_k - M\sum \beta_k , \tag{4}$$

so that

$$[H_0, \sum (r_{k1}+ik_{k2})^n Q_k] = in\sum (\alpha_{k1}+i\alpha_{k2})(r_{k1}+ir_{k2})^{n-1} Q_k . \tag{5}$$

Now, $\sum (\alpha_{k1}+i\alpha_{k2})(r_{k1}+ir_{k2})^{n-1} Q_k = \sum \mathcal{Y}_{nn}(\boldsymbol{\alpha}_k, \boldsymbol{r}_k) Q_k$ is the operator for the $Q_n(\boldsymbol{\alpha}, \boldsymbol{r})$. Therefore,

$$\Lambda aZ/(2R) \cdot Q_n(\boldsymbol{r}, \boldsymbol{r}) = -\Lambda aZ/(2R) \cdot Q_n(\beta\boldsymbol{r}, \boldsymbol{r}) = -inQ_n(\boldsymbol{\alpha}, \boldsymbol{r}) . \tag{6}$$

A little preparation is necessary before we deal with $Q_n(\boldsymbol{\sigma} \times \boldsymbol{r}, \boldsymbol{r})$. First, the method to express a matrix element in the non-relativistic form is shown.[5] To avoid the needless complexity, we write the total Hamiltonian of a nucleus as the sum of the Hamiltonians for the individual nucleons,

$$H = \sum H_{0k} + \sum H_{1k} .$$

Here, H_{1k}'s represent the interaction between nucleons. Of course, it is wrong to write as this, but the results are correct because H_{1k}'s are neglected afterwards. The operator in question is the sum over all individual nucleons, so if we know the non-relativistic form of a matrix element for a nucleon, the complete matrix element is obtained as the sum of them. In the case of one nucleon problem, the small component φ and the large component χ are made from a relativistic wave function ψ as follows :

$$\varphi = (1/2)(1+\beta)\psi , \quad \chi = (1/2)(1-\beta)\psi , \tag{7}$$

$$\varphi + \chi = \psi , \quad \beta\varphi = \varphi , \quad \beta\chi = -\chi . \tag{8}$$

We define the following operator $X_{\pm\pm}$ from X,

$$X_{\pm\pm} = (1/4)(1 \pm \beta) X (1 \pm \beta), \tag{9}$$

$$X = X_{++} + X_{+-} + X_{-+} + X_{--} . \tag{10}$$

The Schrödinger equation becomes

$$-(2M + W + H_{1++})\varphi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + H_{1+-})\chi , \tag{11}$$

$$-(W + H_{1--})\chi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + H_{-+})\varphi . \tag{12}$$

In the non-relativistic approximation we get from eq. (11)

$$\varphi \cong -\{1/(2M)\}(\boldsymbol{\alpha} \cdot \boldsymbol{p} + H_{1+-})\chi . \tag{13}$$

Therefore,

$$\langle\psi_f|X|\psi_i\rangle\cong\langle\chi_f|X_{--}|\chi_i\rangle$$
$$-\{1/(2M)\}\langle\chi_f|X_{-+}(\boldsymbol{\alpha}\cdot\boldsymbol{p}+H_{1+-})+(\boldsymbol{\alpha}\cdot\boldsymbol{p}+H_{1-+})X_{+-}|\chi_i\rangle. \quad (14)$$

In the right-hand side of eq. (14) the first term is the largest, but this vanishes in many cases. In the second term, the part which contains the $\boldsymbol{p}$ will be larger than the one which contains the H_1, because momentum is larger than energy in the non-relativistic case. When either the first term or the part of the second term which includes the $\boldsymbol{p}$ remains, the H_1 can be neglected, and eq. (14) becomes

$$\langle\psi_f|X|\psi_i\rangle\cong\langle\chi_f|X_{--}|\chi_i\rangle-\{1/(2M)\}\langle\chi_f|(X\boldsymbol{\alpha}\cdot\boldsymbol{p}+\boldsymbol{\alpha}\cdot\boldsymbol{p}X)_{--}|\chi_i\rangle$$
$$=\langle\chi_f|X|\chi_i\rangle-\{1/(2M)\}\langle\chi_f|X\boldsymbol{\alpha}\cdot\boldsymbol{p}+\boldsymbol{\alpha}\cdot\boldsymbol{p}X|\chi_i\rangle. \quad (15)$$

In the same way,

$$\langle\psi_f|[H_0, X]|\psi_i\rangle\cong\{1/(2M)\}\langle\chi_f|[p^2, X]_{--}|\chi_i\rangle$$
$$=\{1/(2M)\}\langle\chi_f|[p^2, X]|\chi_i\rangle. \quad (16)$$

When this formula is derived, the part which includes the $\boldsymbol{p}$ vanishes in the largest term. However, if X commutes with β, the part which includes the H_1 vanishes too, and eq. (16) is valid. When X includes $\boldsymbol{\alpha}$ or γ_5, eq. (16) is invalid, but eq. (15) is valid except for $\beta\gamma_5$. Let us consider the X which is the product of the Dirac matrices, $\boldsymbol{p}$ and $\boldsymbol{r}$. $\boldsymbol{r}$ is the only one which does not commute with p^2. Using

$$p^2 r_i=r_i p^2-2ip_i,$$

the right-hand side of eq. (16) is $-i/M$ times as large as the sum of the terms which are constructed by substituting p_i for each r_i in the X. This will be connected with the fact that $[H, X]$ is $-i$ times the time-derivative of X.

Now, let us consider the relation between $Q_n(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})$ and $Q_n(\beta\boldsymbol{\alpha}, \boldsymbol{r})$. We substitute $\mathcal{Y}_{nn}(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})=\{(\boldsymbol{\sigma}\times\boldsymbol{r})_1+i(\boldsymbol{\sigma}\times\boldsymbol{r})_2\}(r_1+ir_2)^{n-1}$, corresponding to $Q_n(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})$, for X in eq. (16). By the convenient relation

$$(r_1+ir_2)(p_1+ip_2)=(p_1+ip_2)(r_1+ir_2), \quad (17)$$

eq. (16) becomes

$$\langle\psi_f|[H_0, \mathcal{Y}_{nn}(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})]|\psi_i\rangle\cong-(i/M)\langle\chi_f|\{(\boldsymbol{\sigma}\times\boldsymbol{p})_1+i(\boldsymbol{\sigma}\times\boldsymbol{p})_2\}(r_1+ir_2)^{n-1}$$
$$+(n-1)\{(\boldsymbol{\sigma}\times\boldsymbol{r})_1+i(\boldsymbol{\sigma}\times\boldsymbol{r})_2\}(p_1+ip_2)(r_1+ir_2)^{n-2}|\chi_i\rangle. \quad (18)$$

On the other hand, from eq. (15),

$$\langle\psi_f|\mathcal{Y}_{nn}(\beta\boldsymbol{\alpha}, \boldsymbol{r})|\psi_i\rangle=(1/M)\langle\chi_f|\sigma_3(p_1+ip_2)(r_1+ir_2)^{n-1}$$
$$-(\sigma_1+i\sigma_2)p_3(r_1+ir_2)^{n-1}|\chi_i\rangle. \quad (19)$$

By eqs. (18) and (19),

$$\langle\psi_f|[H_0, \mathcal{Y}_{nn}(\boldsymbol{\sigma}\times\boldsymbol{r}, \boldsymbol{r})]|\psi_i\rangle-n\langle\psi_f|\mathcal{Y}_{nn}(\beta\boldsymbol{\alpha}, \boldsymbol{r})|\psi_i\rangle$$
$$=-(i/M)(n-1)\langle\chi_f|(\sigma_1+i\sigma_2)\{(\boldsymbol{p}\times\boldsymbol{r})_1+i(\boldsymbol{p}\times\boldsymbol{r})_2\}(r_1+ir_2)^{n-2}|\chi_i\rangle. \quad (20)$$

The right-hand side of eq. (20) vanishes in the approximation of Ahrens and Feenberg.[1] It is known as follows. We regard newly the operator in the right-hand side of eq. (20) as the X in eq. (3), then by the above method we get

$$\begin{aligned}&\langle\chi_f|(\sigma_1+i\sigma_2)[(\boldsymbol{p}\times\boldsymbol{r})_1+i(\boldsymbol{p}\times\boldsymbol{r})_2](r_1+ir_2)^{n-2}|\chi_i\rangle\\&\quad\simeq-\{2R/(\Lambda\alpha Z)\}(-i/M)(n-2)\langle\chi_f|(\sigma_1+i\sigma_2)\{(\boldsymbol{p}\times\boldsymbol{r})_1\\&\quad+i(\boldsymbol{p}\times\boldsymbol{r})_2\}(p_1+ip_2)(r_1+ir_2)^{n-3}|\chi_i\rangle.\end{aligned}$$

This procedure is repeated until eq. (20) becomes

$$\begin{aligned}&\langle\psi_f|[H_0,\ \mathcal{Y}_{nn}(\sigma\times\boldsymbol{r},\ \boldsymbol{r})]|\psi_i\rangle-n\langle\psi_f|\mathcal{Y}_{nn}(\beta\alpha,\ \boldsymbol{r})|\psi_i\rangle\\&\quad\simeq\{-2R/(\Lambda\alpha Z)\}^{n-2}(-i/M)^{n-1}(n-1)!\langle\psi_f|(\sigma_1+i\sigma_2)\{(\boldsymbol{p}\times\boldsymbol{p})_1\\&\quad+i(\boldsymbol{p}\times\boldsymbol{p})_2\}(p_1+ip_2)^{n-2}|\chi_i\rangle=0.\end{aligned}\tag{21}$$

Except for $|\Lambda|\ll1$, eq. (21) is valid for the usual n values. The obtained relation is

$$\{\Lambda\alpha Z/(2R)\}Q_n(\sigma\times\boldsymbol{r},\ \boldsymbol{r})=-\{\Lambda\alpha Z/(2R)\}Q_n(\beta\sigma\times\boldsymbol{r},\ \boldsymbol{r})=-nQ_n(\beta\alpha,\ \boldsymbol{r}).\tag{22}$$

This resembles eq. (6) very much.

Finally we compare these results with experiments. There are five examples, Cl^{36}, Rb^{87}, Tc^{99}, Sb^{124} and Tm^{170} which have been explained by the mixture of $Q_n(\beta\sigma\times\boldsymbol{r},\ \boldsymbol{r})$ and $Q_n(\beta\alpha,\ \boldsymbol{r})$ in the tensor type interaction. Among them Cl^{36}, Rb^{87} and Tc^{99} are the most reliable, but Cl^{36} will be too light to apply the above theory, so it is omitted. There are many explanations for Sb^{124}, and we write two of them.[6][7] For Tm^{170} the explanation of Nakamura *et al.*[7] is doubtful.[8] For the four examples, the theoretical values of $nQ_n(\beta\alpha,\ \boldsymbol{r})/Q_n(\beta\sigma\times\boldsymbol{r},\ \boldsymbol{r})$, namely $\Lambda\alpha Z/(2R)$, and the (semi)-empirical values of them are given in Table I. Besides, the values of Λ are given for reference. We take as the nuclear radius

$$R=\frac{1.4\times10^{-13}\text{cm}\,A^{1/3}}{\hbar/(mc)}.$$

Table I

	n	$\left[\dfrac{nQ_n(\beta\alpha,\ \boldsymbol{r})}{Q_n(\beta\sigma\times\boldsymbol{r},\ \boldsymbol{r})}\right]_{\text{theor.}}$	$\left[\dfrac{nQ_n(\beta\alpha,\ \boldsymbol{r})}{Q_n(\beta\sigma\times\boldsymbol{r},\ \boldsymbol{r})}\right]_{\text{exp.}}$	Λ
Rb^{87}	3	7.12	12.6[9]	0.828
Tc^{99}	2	8.07	13.4[10]	0.845
Sb^{124}	2[6] (or 1[7])	10.64	26[6] (or 0∼12[7])	1.014
Tm^{170}	1	10.82	10.3	0.853

Observing Table I, the orders of the theoretical and the experimental values are in agreement, but the experimental values are a little larger than the theoretical ones. In particular, for the reliable Rb^{87} and Tc^{99} the experimental values are about 70% larger than the theoretical ones. From these results only, it is indistinct whether this difference

is due to the roughness of this theory or the mixture of other interaction type. It is desirable to evaluate the relative magnitude of $Q_n(\sigma \times \nu, \nu)$ and $Q_n(\nu, \nu)$. Lastly, it should be noted that almost the same results are obtained for $-inQ_n(\alpha, \nu)/Q_n(\nu, \nu)$ in the vector type interaction.

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Comments on an Article of Takabayasi concerning the Formulation of Quantum Mechanics with Classical Pictures

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Certain of Takabayasi's criticisms of a causal re-interpretation of the quantum theory are answered in detail. It is indicated how this interpretation can be extended to the Dirac equation, and the problem of the identity of equivalent particles is discussed, for the case of particles that satisfy the Pauli principle. Possible extensions of theory are indicated. Finally, criticisms at the philosophical level are discussed, especially the question of what it means to call a theory "metaphysical" and the reasons why the concept of causality is indispensable in scientific theories.

§ 1. Introduction

In a recent article[1], Takabayasi has criticized a causal re-interpretation of the quantum theory suggested by the present author[2].

The following is a summary of some of his more important criticisms:

(1) The proposed re-interpretation of the quantum theory cannot be extended to include spin, nor can it represent the electron as a quantum field.

(2) The theory can be developed only in the space-time representation of the wave-function, and therefore does not reflect the invariance of the theory to a unitary transformation.

(3) The ψ field is so different from other objectively real fields, such as the electromagnetic, that it does not provide a satisfactory model for quantum phenomena.

(4) The hidden particle variables are "metaphysical", since "they have no connection with observation in the end".

(5) To proceed along classical lines is not "along any probable lines of development" of the quantum theory.

Before answering these criticisms in detail, the author would like to call attention to a new paper[3], in which the relationship of probability to the function is clarified. In this paper, it is assumed that the basic meaning of the ψ function is that of a new and objectively real field of force, which determines the quantum potential through the relation, $U=-\hbar^2\nabla^2 R/2mR$, where $\psi=Re^{iS/\hbar}$. If the particle velocities satisfy the subsidiary condition, $v=\nabla S/m$, and if ψ satisfies Schrödinger's equation, then, as shown in the paper, an arbitrary initial probability distribution, $P(\boldsymbol{X})$ will ultimately approach $|\psi(\boldsymbol{X})|^2$ after

a large number of "random" collisions with other systems. Thus, the analogy between the ψ field and other objectively real fields is strengthened, because we no longer need to assume any *intrinsic* relation between ψ and the probability. For all quantum mechanical systems now available in practice have had billions of years in which to "come to equilibrium" through this process of collision; so that no matter what the probability distribution may once have been, it is now sure to be equal to $|\psi(\boldsymbol{X})|^2$. This result is the direct analogue of Boltzmann's H theorem in classical statistical mechanics.

We now take up the criticisms of Takabayasi in detail.

§ 2. Extension of theory to include spin, and the effects of the exclusion principle

The theory can be extended to include spin through a causal re-interpretation of the Dirac equation. In order to accomplish this, one introduces an objectively real spinor field, $\psi_i(\boldsymbol{X})$, having four complex components; but one does not introduce the subsidiary condition, $\dot{\boldsymbol{x}}=\boldsymbol{p}S/m$ for the particle velocity. Instead, one introduces the condition

$$\dot{\boldsymbol{x}}=\frac{\psi^*\boldsymbol{\alpha}\psi}{\psi^*\psi}. \tag{1}$$

From the well-known conservation equation, it follows that

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\dot{\boldsymbol{x}})=0, \quad \text{where } \rho=\psi^*\psi. \tag{2}$$

This means that it is consistent to assume a probability distribution of particles $P(\boldsymbol{X})=\rho(\boldsymbol{X})=\psi^*\psi$; since if this assumption is satisfied at any given time, say $t=0$, it will, because of eq. (2), be satisfied for all time. But with the aid of the same methods that are developed in Paper III, it can easily be shown that even if $P(\boldsymbol{X})$ were not initially equal to $\psi^*\psi$, it will approach $\psi^*\psi$ after the particle has suffered many collisions with other systems. Thus, the probability distribution giving the mean particle density will be exactly the same as it is in the usual interpretion. Moreover, it can also be shown that the relation, $\dot{\boldsymbol{x}}=\psi^*\boldsymbol{\alpha}\psi/\psi^*\psi$ is a consistent subsidiary condition, if we assume that the particle is acted on by a quantum vector-potential, $B^\nu(X^\alpha)$ and a quantum-scalar potential, $\phi(X^\alpha)$ according to the equation

$$\frac{dv^\mu}{ds}=v^\nu\left(\frac{\partial B^\mu}{\partial X^\nu}-\frac{\partial B^\nu}{\partial X^\mu}\right)+g^{\mu\nu}\frac{\partial\phi}{\partial X^\nu}$$

where $v^\mu=dx^\mu/ds=\psi^+\gamma^\mu\psi/\psi^+\psi$, and ds is the element of proper time. $B^\mu(X^\alpha)$ and $\phi(X^\alpha)$ are functions of the ψ field, given by

$$B^\mu=\psi^+\gamma^\mu\psi/\psi^+\psi; \quad \phi=(\psi^+\gamma^\mu\psi)(\psi^+\gamma_\mu\psi)/(\psi^+\psi)^2 \text{ and } \psi^+=\psi^*\beta.$$

That this interpretation leads to all of the results of the usual interpretation will be demonstrated in detail in a future article, which will also show the relationship of the ψ

field to the spin of the particle[4]. That this theory approaches the non-relativistic causal interpretation in the limit of small v/c follows from the fact that in this limit, $\dot{x} \rightarrow \nabla S/m$ where $S/\hbar$ is the mean phase, of the four components of the Dirac ψ function[5], which is also approximately equal to the phase appearing in Schrödinger's equation.

As for the representation of the electron as a quantum field, let us note that the quantum field theory of electrons, with anti-commutation relations, is completely equivalent in all of its results to that of a particle theory, in which we restrict ourselves to antisymmetric wave functions, and assume that all negative energy states are occupied in the vacuum. To explain the *experimental facts* to the same degree of accuracy as is possible with the quantum field formalism for the electrons, it is therefore adequate to interpret causally the many-particle theory with antisymmetric wave functions. As has already been shown in papers I and II, this interpretation leads to precisely the same results as does the usual interpretation in this problem. In this connection, it may be helpful to remark that causal interpretation is obliged only to reproduce those experimental predictions of the usual interpretation which are known to be correct. It is not obliged to use the same mathematical apparatus in every problem, nor is it required to give the same significance to intermediate mathematical quantities which appear in the calculation.

At this point, it is appropriate to make some comments on the problem of indistinguishability of equivalent particles. As is well known, one can identify equivalent particles in classical theory by their trajectories, which are continuous. In the usual form of quantum theory, this method does not work; because as the wave function is restricted to being totally anti-symmetric (or totally symmetric), the system cannot be put into a quantum state in which one of the particles could be identified as something following a definite trajectory, or possessing any other definite property which could be used to distinguish it from the other particles[6]. Thus, it follows both in the usual interpretation of the quantum theory and in the causal interpretation, that in those domains in which the present general form of the quantum theory is valid, equivalent particles cannot be distinguished from each other experimentally. But the usual interpretation goes farther, for it not only implies that the particles cannot be distinguished from each other, but also that they really do not have separate identities. On the other hand, the causal interpretation implies that the particles do have separate identities, but that we simply are not yet able to distinguish them, as long as we are restricted to doing experiments at a level where the usual form of the quantum theory is valid. But if there is a level at which the present form of the quantum theory fails (perhaps at 10^{-13} cm), then it would be quite possible to distinguish the particles experimentally.

To demonstrate this possibility, we recall that the reason that we cannot identify a particle by its continuous trajectory is basically that when we observe its position, we impart to it an uncontrollable momentum, in accordance with the well-known Uncertainty Principle of Heisenberg. But as shown in Paper II (Sec. 6) according to the causal interpretation of the quantum theory, this uncontrollable transfer of momentum arises in the circumstance that in the interaction of the measuring apparatus with the particle under observation there appears not only the classical potential, $V(x)$, acting on the particle,

but also a change of the wave function introduced by the appearance of $V(x)$ in the wave equation. As shown in paper II, this change in the wave function produces violent and chaotic alterations in the particle orbit even when $V(x)$ is small. Thus, we cannot avoid imparting an uncontrollable change of momentum to the particle which is responsible for the "uncertainties" appearing in Heisenberg's principle. But in the causal interpretation, it is not necessary that *all* interactions must necessarily involve $V(x)$ in just this way. In fact, it is quite possible that at levels where the usual form of the quantum theory breaks down, there could exist interactions, $V(x)$, between the particles, without the corresponding terms appearing in the wave equation. As shown in paper II, Sec. 6, such an interaction would permit measurements of the particle position, x, without appreciable disturbances, either of the particle momentum, or of the wave function. We should then be able, in principle, to follow the trajectory of the particle and thus to identify it, as in classical physics. Moreover, such an identification would not interfere with the validity of any of the conclusions of the quantum theory (including the exclusion principle and the Fermi-Dirac statistics) which hold at the level where the usual form of the quantum theory is a good approximation. For all of these conclusions follow from: (a) Schrödinger's (or Dirac's) equation for the ψ field, (b) The subsidiary condition, $v=\nabla S/m$, (c) The antisymmetry of the wave function, (d) The relation, $P=|\psi|^2$. But an observation of the trajectory of an individual particle need not change any of those things. Thus, in the hypothetical process of observation described above, the particle velocity is not altered*), the wave function is not altered (and therefore remains antisymmetric), while after the measurement is over, collisions will soon re-establish the validity of the relationship, $P=|\psi|^2$.

The conclusion that equivalent particles have no identity follows, not from the quantum theory as such, but rather from the speculative assumption that the present general form of the quantum theory must hold at all levels that will ever be investigated. For this assumption implies that it will *never* be possible to identify equivalent particles, so that the hypothesis that they have an identity can *never* have any physical consequences. Thus, a proof that the particles have no identity could be obtained only by proving the unlimited validity of the present general form of the quantum theory. Since such a proof is obviously impossible even in principle, there is no reason not to assume that the particles actually do have an identity, which may disclose itself to us later in connection with processes in which the present general form of the quantum theory breaks down**).

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*) Even if it were altered, then as shown in Paper I, Sec. 9, we could assume modifications in Schrödinger's equation, important only at the level of 10^{-13}cm, which would re-establish the relation, $v=\nabla S/m$, at the atomic level.

**) This assumption is analogous to one made in the 19th century that individual atoms existed (each presumably having an identity), but that their individuality would disclose itself only in connection with the breakdown of large-scale laws, appropriate to the level at which matter acts approximately as if it were continuous. Such breakdowns were found later, for example, in connection with Brownian motion, the oil drop experiment, tracks in a cloud chamber, etc.

§ 3. Rule of canonical invariance in causal interpretation

In the causal interpretation of the quantum theory, canonical invariance plays a much less important role than it does in the usual interpretation. In this connection, however, it is appropriate to repeat that the causal interpretation is required to reproduce only the *experimental* predictions of the quantum theory, and is not necessarily required to give the same significance to all of the mathematical quantities that may happen to be employed in obtaining these predictions. The following are the essential *experimentally verified* properties of matter that are most characteristic of the quantum level:

(a) The existence of discrete energy levels of matter.
(b) The quantization of the electromagnetic energy.
(c) The appearance of a full quantum of energy in the photo-electric effect even after the electromagnetic wave has spread out over a long distance.
(d) The appearance of interference phenomena even when photons are sent into the system *separately and independently*.
(e) The parallel phenomena with electrons—the transfer of a full quantum of energy to a hydrogen atom in the Franck-Hertz experiment by an exciting particle which may pass far away, so that its force of interaction is very weak and the appearance of electron diffraction even when electrons are sent *separately* and *independently* into a diffracting system.

All of these phenomena are inexplicable on the basis of classical theory. In fact, it was just because of this that physicists were led to accept the idea that simultaneous space-time description and causal explanation must be given up at the quantum level. Yet, everyone of these phenomena can be given a simultaneous space-time description and causal explanation in terms of the causal interpretation of the quantum theory, as can every other phenomenon that can successfully be treated in terms of the present form of the quantum theory. On the other hand, in the causal point of view, the transformation theory is regarded largely as a mathematical superstructure, which is useful in the expansion of the wave function as a series of eigenfunctions of convenient operators, but which has a comparatively limited physical significance.

To clarify our position, we first note that even within the framework of the usual interpretation of quantum mechanics, the transformation theory rests on very weak experimental grounds. For its basic hypothesis; viz, that physical operations exist that would permit the observation of an arbitrary Hermitean operator, has not yet obtained adequate experimental support. In fact, the only observables known so far are position, momentum, angular momentum, a few functions of position, such as dipole and quadripole moment, energy, and spin. But all of these can now be included within the causal interpretation. At present, it would therefore appear that the causal interpretation provides a more plausible treatment of this phase of the problem than does the usual interpretation. For the fundamental quantities which appear in the causal interpretation correspond to just the quantities that have been found to be necessary in experiment; whereas in order to provide a physical basis for the transformation theory, which plays so basic a role in the usual

interpretation, a whole host of additional "observables" must be postulated, the existence of which remains to be demonstrated.*)

As we have already stated, the transformation theory plays no very fundamental physical role in the causal interpretation of the quantum theory, but is relegated to the position of a mathematical technique for expanding the wave function. This technique will, however, be useful only when the equations governing the underlying wave function are linear. The essentially mathematical character of the transformation theory may be emphasized further by pointing out that all *linear* equations can conveniently be treated by mathematical techniques resembling those of the quantum-mechanical transformation theory. For example, Maxwell's equations could, if we wished, be put in "matrix" form, and transformed at will in the associated Hilbert space. Similarly, in the theory of linear electrical circuits, one often introduces an "impedance matrix". Something very much resembling transformation theory may be applied in solving for the free oscillations of such a circuit. Examples of linear equations to which transformations in Hilbert space could conveniently be applied may be multiplied almost without limit. Why, then, should this possibility be singled out for so much special attention in the quantum theory?

We may also ask "Why should there always be an underlying linear theory?" In all other linear equations ever studied (elasticity, sound waves, electrodynamics, gravitation, etc.), linear equations have always turned out to be the linear approximation to non-linear equations. Why should quantum mechanics be different? But as soon as the underlying equations become non-linear, the entire transformation theory becomes useless. It is inevitable that when this happens a certain "representation" should be favored. But it is only natural that, as suggested by the causal interpretation, this favored representation will be that of space and time. In all other fields, space and time stand out as the natural means of describing the progress of physical phenomena. Common experience suggests that *absolute* canonical invariance is most implausible. The author regards it as an advantage of the theory that it shows why space and time are to be favored over arbitrary canonical variables.**)

In sum, then, from the point of view of the causal interpretation, one is led to the conclusion that the significance of the transformation theory and the associated matrices has been exagerated, in the sense that elements entering into useful *mathematical techniques* have been raised to the level of fundamental concepts entering into the *physical theory*.

*) Such a postulate is logically admissible without a doubt. In fact, it stands on much the same footing as does the assumption of the causal interpretation that future experiments may permit the verification of the assumed hidden particle variables. It would therefore seem inconsistent to accept the postulate that all Hermitean operators are observable, in the hope that future experiments will prove it, while objecting to the postulate of hidden particle variables, based on a similar hope.

**) Some of the various criticisms of the quantum theory outlined here have already been suggested individually by various authors like Pauli, Feynman, and others. But the causal interpretation brings out the interconnection of these criticisms and suggests how they might be met in terms of a new point of view.

§ 4. The physical interpretation of the ψ field

Takabayasi has the following objections to the author's suggestion that the ψ function be regarded as a mathematical representation of an objectively real field satisfying Schrödinger's equation.

a) The field is independent of the particle, and is conserved, its absolute value having no significance.

b) In a many body problem, it must be defined in configuration space, so that in ordinary space, we obtain non-local interactions.

c) He regards the subsidiary condition restricting the velocity to $v=\nabla S/m$ as unnatural*).

To all of these objections the author would first like to remark that none of them involve any logical inconsistencies in the theory**). Essentially, they amount to the statement that the ψ field does not act completely like other fields to which we are accustomed, such as the electromagnetic. But it must be remembered that the phenomena that we are called upon to explain causally (e. g., quantization of energy, photoelectric effect, Franck-Hertz experiment, interference effects, etc.) are also completely unlike those that are familiar at the normal classical level. It would be surprising indeed, if we could explain such phenomena without postulating new types of entities, acting in new ways.

Secondly, the author would like to state that he does not regard his proposed interpretation of quantum mechanics as a final theory. His principal objective in proposing this interpretation has been to prove that at least one logically self-consistent causal interpretation of the quantum theory is possible, which is also consistent with all of the experimental facts that can be understood in terms of the usual interpretation. This result shows the falsity of the hitherto prevalent impression that no such interpretation is possible, on the basis of which had arisen the notion that it is necessary to give up causality at the quantum level. Naturally, as is quite normal in causal theories, more than one causal explanation of the same basic experimental facts may be possible. In this case, a choice between the various suggested explanations would have to await an extension of the experimental data, as well as a further working out of the consequences of the various theories; although in the absence of sufficient experimental facts, one would tend to prefer the theory having the minimum number of primitive assumptions.

Finally, the author would like to point out that there are almost unlimited possibilities for modifying the theory, in such a way as to remove some of the aspects of the ψ field which Takabayasi regards as objectionable. In paper I, Sec. 9, the author has made

*) He also asserts that the ψ field must contract in a measurement. However, as has been shown in paper II, this is not necessary; and Takabayasi has kindly admitted in a private letter to the author that this statement was an oversight. Some of the remaining objections to the ψ field given in Takabayasi's paper have been answered in paper III, where it is shown that there is no necessary relationship between the probability and the ψ field.

**) At first sight the non local interactions might be thought to contradict the theory of relativity, but as shown in paper II, Sec. 8, such contradictions are not necessarily implied.

some proposals of this type which are, however, not intended as definitive theories, but rather as examples demonstrating only that such changes are possible. Recently, Vigier[7] has suggested a program which seems quite promising. He proposes that one starts only with a non-linear field theory, resembling perhaps the Born-Infeld theory[8] or some of the new unified theories of Einstein[9] or Schrödinger[10]. Because of the nonlinearity, particle-like solutions should be possible, which consist of local regions in which the field "bunches up". His program is to seek those field equations whose particle-like singularities move on the trajectories, $\dot{x} = \frac{\psi'^* \alpha \psi'}{\psi'^* \psi'}$, where ψ' is a solution of the Dirac equation, which is related to the non-singular part of the solution of the basic field equations. Vigier states that preliminary work on his part indicates the possibility of finding such solutions*. Such a program, if successful, would remove most of the arbitrary elements from the theory; for it would be necessary only to assume the ψ' field, from which would follow the particles and their interaction with the ψ' field, *including the subsidiary condition mentioned in objection (c) of Takabayasi.* Moreover, all of the results would follow only in the limit in which the change of the general "back ground field" over the region of the size of the particle (presumably of the order of 10^{-13}cm) could be neglected. In the domain of the size of the particle, totally new behavior would be obtained, which could not be described in terms of the usual interpretation of the quantum theory.

The author quotes the above suggestions only to give a concrete example of how further changes in the causal interpretation are possible, which may lead to new results, as well as perhaps to a more connected formulation of the theory as a whole.

§ 5. On the question of whether the hidden particle variables are "metaphysical"

The objection has been raised, not only by Takabayasi, but also by many other physicists**) that the hidden particle variables in the causal interpretation of the quantum theory are "metaphysical", because they cannot yet be verified experimentally. Now, the author would agree that any hypotheses that *as a matter of principle* can *never* be verified by experiment is "metaphysical" in the sense that it could *never* have any physical consequences, so that it could *never* make any practical difference whether we included it in the theory or not. In terms of the above definition, however, the hidden particle variables are not "metaphysical", because there is nothing in the quantum theory, *causally interpreted*, which could lead one to the conclusion that it is *in principle* impossible to verify the existence of these variables. It just happens that at the level at which the usual form of the quantum theory is valid, they cannot be observed in a precise way; but as we have already seen, it is possible to conceive of modifications, consistent with the

*) Private communication.
**) Private communications.

causal interpretation, and with all known experimental facts, which would permit them to be observed, at least in principle, with unlimited precision.

Proponents of the usual interpretation of the quantum theory often argue that a hypothesis is "metaphysical" unless it is *already* possible to furnish a prescription showing in principle how this hypothesis could be verified *without the need for an extension of existing theories*. In other words, it is not admissible to introduce entities, the proof of the existence of which is based on experimental and theoretical results which we hope will be obtained in the future. This requirement, however, cannot be satisfied even within the framework of the usual interpretation, because the latter needs to introduce the postulate that all Hermitean operators are observable; and as we have seen in Sec. 3, the theory contains no prescriptions telling how one might go about trying to verify this postulate. Moreover, such a requirement is based on an implicit hypothesis; namely, that the world is so constituted that nothing could be gained by assuming the existence of entities before one can give a prescription for how their existence is to be verified. At best, we might regard this hypotheses as a general conclusion drawn from experience with scientific theories in the past. Past experience in a wide variety of fields would, however, suggest the falsity of such a conclusion; for it has very often turned out to be very fruitful indeed to postulate the existence of things before any one knew, even in principle, how to go about trying to verify their existence. In fact, the means of verification are very often discovered precisely because of researches springing from the hypothesis themselves. To require that the means of verification already be known when a new hypothesis is suggested would therefore often make almost impossible the suggestion of new concepts.

The atomic theory is an excellent case in point. For when the atomic theory was first suggested, it was certainly accompanied by no prescriptions telling how individual atoms might be observed. Nor did it have any new experimental consequences not already contained in large-scale laws, such as the law of definite proportions, the law of multiple proportions, the gas laws, etc. In fact, it was for just these reasons that 19th century positivists like Mach criticized the atomic hypothesis as "metaphysical" and refused to believe in the existence of atoms. Nevertheless, the advantage of the atomic theory at that time was already that it provided a causal explanation of the inter-connection of various fields of phenomena, the relationships of which would have to be regarded as arbitrary and co-incidental in the absence of such an explanation. This causal explanation moreover led in time to various new types of experimental research, which had as their aim the discovery of manifestations of the individual atoms; and as is well known, these experiments were eventually crowned with success (but not until long after the atomic hypothesis was first proposed).

The causal interpretation of the quantum theory presents a close analogy. For just as the atomic theory introduced new but "hidden" degrees of freedom in the form of the coordinates and momenta of the various atoms, the causal interpretation introduces the precisely definable particle variables and the objectively real ψ fields as new but at present "hidden" degrees of freedom. In his article, Takabayasi criticizes this analogy between the new atomic degrees of freedom and the hidden particle variables, asserting that the latter

are not really new degrees of freedom, because they merely "alter the concept of motion" in connection with existing degrees of freedom (e. g,, the electrons, etc. as treated by the usual interpretation), and because no new constants are introduced (analogous to the mass, size, etc. of the atom). We disagree with Takabayasi's conclusion. First of all, in the usual interpretation of the quantum theory there is no concept of motion to be altered. It is specifically required in the usual interpretation to renounce all possibility of conceiving, for example, of how an individual electron moves from one stationary state to another. Such things are simply assumed to happen, but in a way the precise details of which could have no meaning. It therefore remains only for us to accept the large-scale experimental results, without aiming for a conception that would explain precisely how they have come about. The hidden particle variables of the causal interpretation (plus the objectively real ψ' fields) therefore permit us to introduce a hitherto non-existent precise concept of motion at the quantum level. Moreover, new constants are implied. For example, in the discussion of Vigier's suggestions in Sec. 3, we showed, in a concrete example, how existing theories may be expected to fail in the domain of the size of an "elementary" particle. We do not know, for certain, what this size is, but when the atomic theory was first introduced, the mass of the atom was likewise unknown.

At the present level, the main advantage of the hidden particle variables is, as in the case of the atomic theory, that they permit a less arbitrary and more connected description of the relationship between various experimental phenomena. For example, in the case of the α-decay of a nucleus, the usual interpretation requires us to assume that the precise time at which a particular nucleus decays is completely arbitrary (See Paper III, Sec. 3); while only the probability of decay in a statistical ensemble is related to the previous state of the system (through the wave function). Nevertheless, such an individual decay process clearly has physical significance, since, for example, it could activate a Geiger counter. Moreover, the wave function spreads out over a wide region of space; and yet the track of something at least resembling a particle appears in a cloud chamber, in a comparatively small region of space. In the usual interpretation, one is required to assume that this particle-like manifestation appears, almost as if by magic, in a way the precise details of which are necessarily forever beyond human comprehension. A similar arbitrariness and lack of connection appear in relation to other processes at the quantum level, such as transitions between stationary states, interference in diffraction problems, etc. However, by postulating the existence of the hidden precisely definable particle variables (together with the objectively real ψ' fields), the causal interpretation explains the appearance of a definite track in the cloud chamber in a natural way, by simply assuming that there really is a particle which, under the action of its own field, is able to cross the potential barrier and thereafter to move until it reaches the cloud chamber, where it manifests itself (See Paper I, Sec. 8). Moreover, the precise time of decay of an individual nucleus is no longer arbitrary, but is in principle determined by the precise initial locations of the particles in the nucleus, which, however, we do not yet know how to observe without seriously disturbing the system. Nevertheless, we can already predict that as a result of chaotic collisions with other systems, a statistical ensemble of nuclei will show a distribution

of decay times, which have a probability that can be calculated from the relation, $P=\psi^*\psi$. On the other hand, the usual interpretation must assume this probability distribution *ad-hoc*. Finally, we are able to suggest that if we wish to definitely prove the existence of these particles, we should make investigations in the domain in which their size cannot be neglected ; and as we have pointed out, there is already evidence suggesting that this domain may be connected with the breakdown of existing theories in the region of 10^{-13}cm.

§ 6. On the question of whether causality is along the probable lines of development of the theory

It is a common opinion among theoretical physicists that the probable lines of development of the quantum theory do not lie in the direction of a return to causality, but rather in that of a further renunciation of causality in connection, for example, with the " fundamental length " suggested originally by Wataghin and later developed by Heisenberg. This opinion is based on the prevailing impression that it was just such a renunciation of causality that was responsible for the great progress originally made with the aid of the quantum theory at the atomic level. It must be remembered, however, that the successful predictions of the present quantum theory come, not from the abandonment of causality, but rather from certain much more specific elements in the theory. These elements are : The Schrödinger equation, the Dirac equation, and the statistical interpretation of Born. But these very same predictions could equally well have come if the wave equation had intially received a causal interpretation. Indeed, the general lines of such a causal interpretation were already laid down by de Broglie even before the current a-causal interprtation had been fully developed[11] ; but de Broglie gave up this interpretation, because in its early form, it did meet certain objections[12]. With additional work, however, these objections could have been answered, as is demonstrated by the work of the present author (See especially Paper II). And since the causal interpretation leads at the atomic level to exactly the same probability distribution as that adopted by Born, the quantum theory causally interpreted would have been at least as successful in predicting experimental results as it has actually been with the aid of the a-causal interpretation, and perhaps more so, since there would have existed an impetus to attack problems, which according to the usual interpretation are meaningless, but which might really be quite significant. In any case, it hardly seems justifiable to attribute the success of the current quantum theory to its a-causal philosophy, since this philosophy is not needed for predicting or understanding any of the experimental results treated by the theory. Thus, the conclusion that in the future, progress will probably be made by a further renunciation of causality carries little weight, since it is based on the false assumption that it was a renunciation of causality that has been responsible for the success of the quantum theory in the past.

The very possibility of a causal interpretation of the quantum theory demonstrates that there is no *need* to give up causality at the quantum level. Moreover, we have seen in the previous paragraph that no real advantage has actually come from giving up causality. We wish now to show that there are very strong reasons for not giving up

causality in the absence of positive proof that the assumption of causality is inconsistent with the experimental facts.

In order to demonstrate this contention, it would be helpful to begin with a brief definition of causality. A necessary condition for causality is that a given event or state, which we call the effect, shall always be preceded by another event or state, which we call the cause (mose generally by a set which we call the set of causes). This however is not sufficient to define causality, since an event that always precedes another may not actually be relevant; it may just be something that generally happens to be associated with the true cause or causes. The true causes can be distinguished as those conditions or events whose *reproduction* is both necessary and sufficient for the later *reproduction* of the effect (or effects). To be sure of the true cause, we must therefore demonstrate experimentally that the effects are reproduced every time that the causes are reproduced, and at no other times. Strictly speaking, no effect can be completely independent of anything in the universe, because everything is really interconnected to some extent with everything else. Nevertheless, one can generally isolate what we may call the "controlling causes" the reproduction of which is adequate to determine, for all practical purposes, the reproduction of the effects; and for the limited purposes of predicting or controlling the effects in question, all other causes may be neglected.

It can easily be seen from the above definition that the domain of causality defines the domain of science itself. For science is concerned primarily with the reproducible relationships between objects, phenomena, or events. For example, in the quantum theory, we can study only those properties of matter that can be reproduced, such as energy levels, dipole moments, various constants of matter such as $\hbar$, etc. Since we do not yet understand the factors that might control the reproduction of certain properties of the individual system at the quantum level, (such as the precise time of decay of a nucleus), we are at the present restricted to studying the statistical averages of such properties, which are approximately reproducible. By assuming that there are no causal factors that might control the reproduction of such properties of the individual system at the quantum level, however, the usual interpretation relegates real physical phenomena to a domain that is by definition forever beyond the possibility of scientific investigation. It does this, not on the basis of having proved that such phenomena have no causes (indeed, the causal interpretation of the quantum theory demonstrates that they could have causes) but rather just for the philosophical reason of avoiding the postulation of entities that cannot be observed in terms of existing theories. The untenable character of such philosophical reasons has already been demonstrated in Sec. 5. Here, we need only add that when we face a new and as yet unexplained phenomena, the most fruitful attitude is always to assume they have a cause, which we must discover. Even if there really were no cause, no error could come from the assumption that there was one. All that would happen would be that our efforts to find the cause would not be successful. But if, as is much more likely, there really is a cause, and we assume there is not, then we may be led to overlook important new factors that are needed in the theory. Moreover, it is not enough to wait until experiments by themselves disclose these "hidden" causal factors, since the theory must guide

the experiments, especially in problems as complicated as are those faced by physics in the domain of 10^{-13}cm.

Now, many physicists who would agree with the need for returning *some* form of causality would nevertheless object to the causal interpretation proposed by the author, because it represents a return to the excessively simple form of causality appearing in classical mechanics; namely, "Newtonian mechanism".*) The author would agree that the *unlimited* extension of causal laws of the type appearing in classical mechanics would lead to most implausible results. For example, if we imagine, with Laplace, that the entire universe is made of particles obeying the laws of classical mechanics, then we are led to conclude that once the universe has been set in motion in a given way, its future motion is determined rigidly for all time. Moreover, it could equally well be said that the future determines the past, as that the past determines the future. Thus, one of the principal aspects of causality, its irreversible direction of action in time is lost. Indeed, causality itself loses all meaning, since each cause is rigidly determined by preceding causes, and so on ad infinitum. Thus, the unlimited extension of this simple form of causality leads to a theory of mechanical determinism, in which real causality has no place.

From this paradox, and from other difficulties into which the unlimited extension of Newtonian mechanics leads, some physicists have drawn the conclusion that Newtonian mechanics is basically inadequate, and should be replaced as rapidly as possible by a superior form of causality, if not by the abandonment of causality itself. On the other hand, it is not necessary to take such radical steps in order to solve these problems. For the difficulties of the type cited above come, not from Newtonian mechanics as such, but ratheer from the completely unjustified assumption of the *unlimited* validity of this form of mechanics. As soon as we admit that Newtonian mechanics is an approximation valid only in a limited domain, we get into no such difficulties. For example, the complete determinism of Laplace can no longer be deduced from the theory; the most that can be deduced is that there will be determinism in a limited domain. And to this conclusion there can be no objections, since it is already known that the deterministic scheme of Newtonian mechanics can correctly and consistently be applied in at least one domain; namely the classical domain. The only remaining question is this: In what additional domains are laws resembling Newton's laws of motion valid? Only a detailed investigation of any particular domain can tell us to what extent a simple theory resembling Newton's laws of motion may be valid there. But the very fact that a causal interpretation of the quantum theory is possible along Newtonian lines suggests that such theories may be at least *partially* adequate in the quantum domain, and perhaps in the domain of 10^{-13}cm. Thus, a "return to classical types of laws" *in a limited domain* would not be excluded *a-priori* on the basis of philosophical considerations, however plausible. For there is nothing intrinsically wrong with classical types of laws, as long as we do not try to

*) Apparently Takabayasi shares this point of view. His article does not make clear exactly what are his objections to causality, but in a private communication, he implies that it is just the classical mechanical form of causality to which he objects.

extrapolate them unjustifiably by imagining (with Laplace) that they furnish a final theory, or at least a final general framework, within which the details only remain to be filled in.

In order to clarify our position on this point, we note that whenever throughout the history of the development of science, it was thought that the final truth (or at least its general outlines) had been reached, new levels were later discoverd, which completely overturned many of the conclusions, drawn from the unjustified extrapolation of laws appropriate to the previous levels. (Conclusions drawn *only* within the limited domain of the previous laws were however never overturned). The fact that toward the end of the 19th century, it was felt that physics was only a matter of "filling in the next decimal points" is a vivid reminder of the danger of supposing that we already have the general outlines of a final theory. Indeed, it seems most unlikely that we shall ever be able to obtain experimental proof that our general conceptual framework is a final one, that is valid at *all* levels. Thus, it is necessary to formulate our theories in such a way that we *explicitly* recognize the possibility of an inexhaustible number of new levels, in which entirely new types of laws may be needed*). If we do this, then even if we discover that simple Newtonian types of laws do hold in the domain of 10^{-13}cm, we know that the final course of the world is not necessarily determined "mechanically" by such laws. For there is a continual interaction between all levels; and the more complex laws that may be appropriate to the unlimited number of new levels (which we have hardly even begun to scratch) could easily invalidate the conclusions coming from the unfounded extrapolation of Newtonian laws to *all* levels. Thus, the unsatisfactory aspects of Newtonian types of laws are not present in a theory that limits itself to a finite domain, in which it might hope to verify such laws; but are present only when we try to fit all possible future human knowledge into the limited conceptual framework of these laws**). The real problem then is to find out which types of laws are valid in any given domain; and the only way of doing this is to try every type that can be thought of, and to see which type works best.

In conclusion, the author would like to comment on another phase of Takabayasi's paper; namely his use of the hydrodynamical model of the quantum theory, with the suggestion of the addition of vortex flow. This work seems promising. However, the introduction of the classical hydrodynamical "pictures" followed by their "quantization" seems to be, at the very least, a methodological contradiction. For only existing justification for introducing the "pictures" is to explain quantization. It therefore seems

*) For example, below the level of 10^{-13}cm probably lies still another level, etc. ad infinitum. Why should the possibility of analysis necessarily stop just where we happen to be? Also, we must not restrict ourselves only to levels of size, as there may be new levels associated with very high density, with very high energy, unusual complexity of organization, etc. etc.

**) There is a close analogy between the assumption that the general framework of all future laws was to be expressible in terms of interacting particles, and the implicit assumption underlying the usual interpretation of the quantum theory, that this general framework will always be expressible in terms of linear operators, Hilbert spaces, the uncertainty principle, etc. The former unverifiable assumption leads to mechanistic determinism, and the latter equally unverifiable assumption leads to the conclusion that causality must be given up.

unjustifiable to finish up by mechanically "quantizing" the model. A procedure that seems more reasonable to the author would be to try to relate the vortex motion of the fluid to electron spin. That this may be possible is already indicated by the causal interpretation of Dirac's equation, for which the corresponding hydrodynamical analogue already implies vortex motion (See eq. (1)). Preliminary calculations by the author suggest that such a relation may in fact be found.

Finally, the author would like to thank Dr. Takabayasi for sending him a prepublication copy of his manuscript, and also for valuable correspondence.

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On the Lattice Vibration and the Effective Mass of Conduction Electrons

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The effective mass of electrons interacting with lattice vibrations is calculated by means of the second-order perturbation theory. The result is that the magnitude of the effective mass of electrons near the Fermi surface is of the order of $10^{-3}\sim10^{-4}m$, irrespective as to whether the distribution of electrons is normal or superconductive. The effect of small effective mass on the diamagnetic susceptibility of electrons is discussed.

§ 1. Introduction

Theories of superconductivity based on the interaction between electrons and lattice vibrations have been proposed by Fröhlich[1] and Bardeen[2]. Bardeen has found that electrons with energies near the Fermi surface have small effective mass, $m_{\text{eff}}\sim10^{-3}m$, and he has indicated that a sufficiently large diamagnetism for superconductivity will occur if there is a sufficient number of electrons of small effective mass. However, Bardeen's result is obtained after a questionable approximation for the average energy of the intermediate states. Drell[3] has calculated the energy of electrons interacting with the lattice vibrations by means of Bloch-Nordsieck-type transformation. His result is contrary to the conclusion that electrons have small effective mass. The approximation involved in the Bloch-Nordsieck transformation is the neglect of electron recoil, hence it is not a good one for lattice waves of short wavelength.

As the smallness of the effective mass is an interesting property, it is desirable to clarify whether the interaction between electrons and lattice vibrations actually yields small effective mass. In the present paper we shall calculate the effective mass of conduction electrons at the absolute zero of temperature by means of the second-order perturbation theory.

§ 2. Calculation

If we denote the mean value of the effective mass of an electron with wave vector $\boldsymbol{k}$ by m_{eff}, we have, by definition that

$$\frac{1}{m_{\text{eff}}}=\frac{1}{3\hbar^2}\mathrm{Tr}\left(\frac{\partial^2 E}{\partial k_s \partial k_j}\right), \tag{1}$$

where Tr means trace. Considering that E is a function of $|\boldsymbol{k}|=k$, we can rewrite Eq. (1) as

$$\frac{1}{m_{\text{eff}}}=\frac{1}{3\hbar^2}\,\frac{1}{k}\,\frac{d^2}{dk^2}(kE). \tag{1'}$$

Using the second-order perturbation calculation, Fröhlich has obtained E as the sum of three terms

$$E_0(k)=\frac{\hbar^2k^2}{2m}, \tag{2,a}$$

$$E_1(k)=-\frac{8F\zeta\sigma_0}{3n}\,\frac{1}{(2\pi)^3}\int\frac{w}{q^2-k^2+\sigma_0 w}\,d\boldsymbol{w}, \tag{2,b}$$

$$E_2(k)=\frac{8F\zeta\sigma_0}{3n}\,\frac{1}{(2\pi)^3}\int\frac{wf(q)}{q^2-k^2+\sigma_0 w}\,d\boldsymbol{w}. \tag{2,c}$$

Here

$$\boldsymbol{q}=\boldsymbol{k}-\boldsymbol{w},\quad \zeta=\frac{\hbar^2k_0^2}{2m},\quad \sigma_0=\frac{2ms}{\hbar},\quad F=c^2/3\zeta Ms^{2*},$$

and n is the number of electrons per unit volume, s and k_0 being the velocity of sound and the radius of the Fermi sphere respectively. Performing the angular integration in Eq. (2,b) using Cauchy's principal value, we get

$$E_1(k)=-\frac{8F\zeta\sigma_0}{3n(2\pi)^2}\,\frac{1}{k}\,\frac{J(k)}{2}, \tag{3}$$

with

$$J(k)=\int_0^{w_0} w^2\log\frac{w+2k+\sigma_0}{|\,2k-w-\sigma_0|}\,dw. \tag{4}$$

Case 1. $(4\nu)^{1/3}F<1$.**) In this case the distribution of electrons in momentum space in the ground state is the Fermi distribution. The allowed region of integration for $E_2(k)$ is the shaded one shown in Fig. 1. A coordinate system appropriate for the evaluation of the integral is also shown in Fig. 1. Using the relation

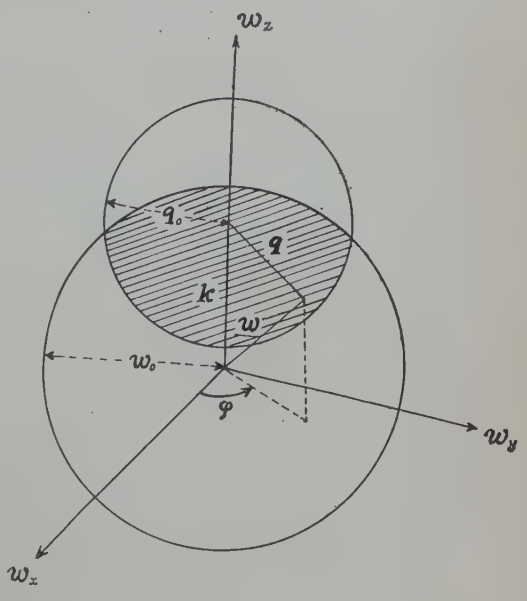

Fig. 1.

*) C is half the value of Sommerfeld and Bethe's C.

**) ν is the number of electrons per atom.

$$dw = (wq/k)\, dw\, dq\, d\varphi,$$

we find for not very small k***) that

$$E_2(k) = \frac{8F\zeta\sigma_0}{3n(2\pi)^2}\,\frac{1}{k}\,G(k,k_0), \tag{5}$$

where

$$G(k,k_0) = \int_0^{k_0-k} w^2 dw \int_{|k-w|}^{k+w} \frac{q\,dq}{q^2-k^2+\sigma_0 w} + \int_{k_0-k}^{k} w^2 dw \int_{|k-w|}^{k_0} \frac{q\,dq}{q^2-k^2+\sigma_0 w} + \int_{k}^{w_0} w^2 dw \int_{|k-w|}^{k_0} \frac{q\,dq}{q^2-k^2+\sigma_0 w}. \tag{5'}$$

These integrals are divergent, but this divergence is an apparent one because the divergence of $E_2(k)$ is cancelled by that of $E_1(k)$ and the self energy itself is finite. Carrying out the integrations by the use of principal values, we obtain

$$G(k,k_0) = \frac{1}{2}\int_0^{k_0-k} w^2 \log\frac{2k+w+\sigma_0}{|2k-w-\sigma_0|}\, dw + \frac{1}{2}\int_{k_0-k}^{w_0} w^2 \log\frac{k_0^2-k^2+\sigma_0 w}{w(2k-w-\sigma_0)}\, dw.$$

Eq. (1′) can thus be written as

$$\frac{1}{m_{\text{eff}}} = \frac{8F\zeta\sigma_0}{9n\hbar^2(2\pi)^2}\,\frac{\partial^2}{\partial k^2}\left\{-\frac{J(k)}{2} + G(k,k_0)\right\} + \frac{1}{m}. \tag{6}$$

Now we shall calculate the effective mass of electrons near the Fermi surface. If $k \sim k_0$ and $4\nu > 1$, then $2k-w-\sigma_0 > 0$ and so we are allowed to interchange the order of integration and differentiation. Differentiating first, we get

$$\frac{\partial^2}{\partial k^2} G(k,k_0) = -2k^2 \int_{k_0-k}^{w_0} \frac{w^2 dw}{(k_0^2-k^2+\sigma_0 w)^2} + \text{smaller terms.}$$

Integrating over w, we obtain

$$\frac{\partial^2}{\partial k^2} G(k,k_0) = \frac{2w_0 k_0^2}{\sigma_0}\left\{\frac{2}{\alpha}\log(1+\alpha) - \frac{1}{1+\alpha} - 1\right\} + O(k_0^2/\sigma_0) \tag{7}$$

with

$$\frac{1}{\alpha} = \frac{\delta}{\sigma},\quad \delta = k_0 - k \quad \text{and} \quad \sigma = (1/4\nu)^{1/3}\sigma_0.$$

As a^2J/dk^2 is found to be negligible as compared with d^2G/dk^2, we finally obtain

$$\frac{1}{m_{\text{eff}}} \approx \frac{4F}{3}\left(\frac{\nu}{2}\right)^{2/3}\frac{k_0}{\sigma_0}\left\{\frac{2}{\alpha}\log(1+\alpha) - \frac{1}{1+\alpha} - 1\right\}\frac{1}{m}$$

$$= \frac{4F}{3}\left(\frac{\nu}{2}\right)^{2/3}\frac{k_0}{\sigma_0}\psi_1(\alpha)\frac{1}{m}. \tag{8}$$

***) If k is very small and one of the sphere is wholly inside the other the integration limits are different from Eq. (5′). We also impose the condition that $4\nu > 1$.

Since the magnitude of k_0/σ_0 is of the order of 10^3, m_{eff} is $\sim 10^{-3}m$. $\psi_1(\alpha)$ is shown by the curve in Fig. 2. It is to be noted that the sign of m_{eff} is opposite to that of m. As the value of k decreases, the contribution of the self energy to the effective mass becomes smaller and it becomes negligibly small when $k \ll k_0$. Therefore the sign of m_{eff} near the Fermi surface is opposite to that of m_{eff} near the center of the Fermi sphere and at some wave number k', $1/m_{\text{eff}}$ vanishes. Accordingly $m_{\text{eff}}(k'+0) = +\infty$ and $m_{\text{eff}}(k'-0) = -\infty\ (m > 0)$.

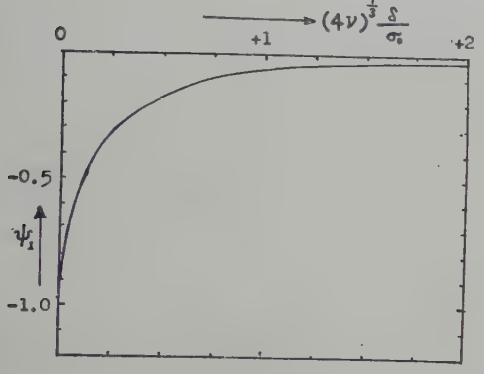

Fig. 2.

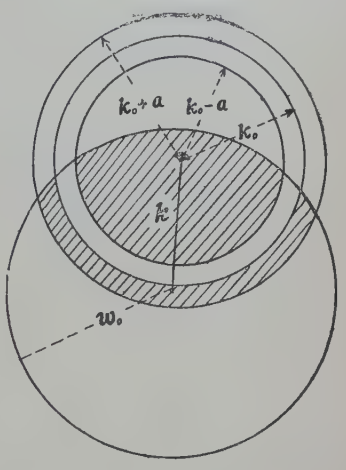

Fig. 3.

Case 2. $(4\nu)^{1/3}F > 1$. The distribution is different from the Fermi distribution and consists of an occupied sphere, a gap and an occupied shell. This distribution is identified with the superconducting state by Fröhlich[1]. In this case the integration domain is the shaded region shown in Fig.3. Using the variables (q, w, φ) as before, we get for electrons with the wave number $k_0 < k < k_0 + a$, i. e. in the shell,

$$E_2(k) = \frac{8F\zeta\sigma_0}{3n(2\pi)^2}\,\frac{1}{k}\,[G(k, k_0+a) - H(k, k_0) + H(k, k_0-a)], \qquad (9)$$

where the three terms in the square bracket are the integrals over the lense-shaped regions. We find

$$H(k, k_0) = \int_{k-k_0}^{k} w^2 dw \int_{|k-w|}^{k_0} \frac{q\,dq}{q^2-k^2+\sigma_0 w} + \int_{k}^{w_0} w^2 dw \int_{|k-w|}^{k_0} \frac{q\,dq}{q^2-k^2+\sigma_0 w}$$

$$= \frac{1}{2}\int_{k-k_0}^{w_0} w^2 \log\left|\frac{k_0^2-k^2+\sigma_0 w}{w(2k-w-\sigma_0)}\right| dw.$$

Integrations are carried out using principal values. Contrary to Case 1, the self energy of electrons is divergent in this case. This is due to the degeneracy of the unperturbed state. Performing the integration of $H(k, k_0)$ first and then differentiating with respect to k, we get

$$\frac{\partial^2}{\partial k^2} H(k, k_0) \approx 2w_0 \frac{k_0^2}{\sigma_0^2} \left\{ -\frac{2}{\alpha} \log(1-\alpha) - \frac{1}{1-\alpha} - 1 \right\}, \tag{10}$$

with

$$\frac{1}{\alpha} = \frac{\delta}{\sigma}, \quad \delta = k - k_0.$$

Similarly we obtain

$$\frac{\partial^2}{\partial k^2} H(k, k_0 - a) \approx 2w_0 \frac{k_0^2}{\sigma_0^2} \left\{ -\frac{2}{\beta} \log(\beta-1) - \frac{1}{1-\beta} - 1 \right\}, \tag{11}$$

with

$$\frac{1}{\beta} = \frac{\delta + a}{\sigma}.$$

From Eq. (7) we get

$$\frac{\partial^2}{\partial k^2} G(k, k_0 + a) = -2w_0 \frac{k_0^2}{\sigma_0^2} \left\{ -\frac{2}{\alpha} \log(1-\gamma) - \frac{1}{1-\gamma} - 1 \right\}, \tag{12}$$

where

$$\frac{1}{\gamma} = \frac{\delta - a}{\sigma}.$$

Substituting (10), (11) and (12) into (9) and using the relation (1′), we finally obtain

$$\frac{1}{m_{\text{eff}}} = \frac{4F}{3} \left(\frac{\nu}{2} \right)^{\frac{2}{3}} \cdot \frac{k_0}{\sigma_0} \frac{1}{m} \left[\frac{2}{\alpha} \log(1-\alpha) - \frac{2}{\beta} \log(\beta-1) - \frac{2}{\gamma} \log(1-\gamma) - \frac{1}{1-\alpha} + \frac{1}{\beta-1} - \frac{1}{1-\gamma} - 1 \right]. \tag{13}$$

The effective mass of electrons near the surface of the inner sphere is given by the following equation

$$\frac{1}{m_{\text{eff}}} = \frac{1}{m} + \frac{8F\zeta\sigma_0}{9n\hbar^2(2\pi)^2} \frac{1}{k} \times \frac{\partial^2}{\partial k^2} \left[\frac{J(k)}{2} + H(k, k_0 + a) - H(k, k_0) + H(k, k_0 - a) \right], \tag{14}$$

which is found to be identical with Eq. (13). Thus, for electrons in the shell and near the surface of the inner sphere we can use the common expression (13), which we may put as

$$\frac{1}{m_{\text{eff}}} = \frac{4F}{3} \left(\frac{\nu}{2} \right)^{2/3} \frac{k_0}{\sigma_0} \frac{1}{m} \psi_2.$$

ψ_2 is illustrated by Fig. 4. Fig. 4 shows that if the width of the shell, a, is smaller than 0.2σ, the sign of the effective mass of electrons in the shell and near the surface

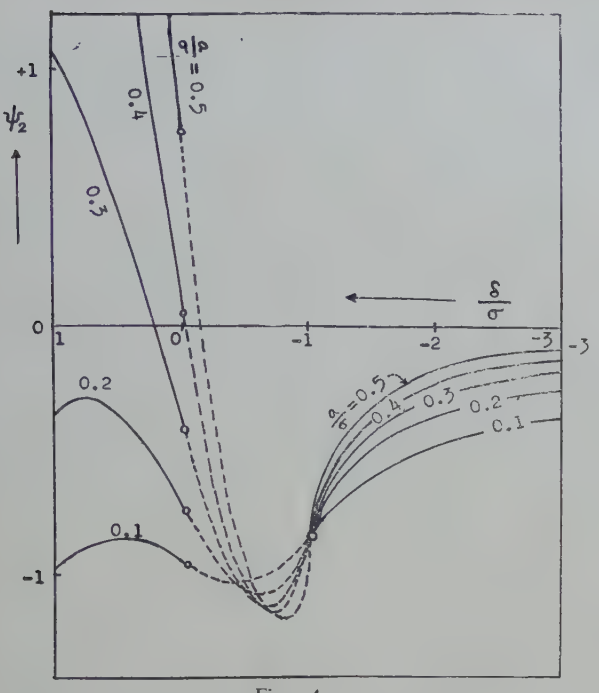

Fig. 4.

of the inner sphere is opposite to m and the order of magnitude of m_{eff} is $10^{-3}m$, and that if a/σ is about 0.4, the sign of m_{eff} in the shell is equal to m and its magnitude is of the order of 10^{-4} m near the outer surface of the shell. If a/σ is about 1.3, $1/m_{\text{eff}}$ changes its sign in the shell and so a small number of electrons in the shell have very large effective mass of both signs.

The component of the effective mass tensor m is given by

$$\frac{1}{m_{ij}}=\frac{1}{\hbar^2}\frac{\partial^2 E}{\partial k_i \partial k_j},$$

which is rewritten as

$$\frac{1}{m_{ij}}=\frac{3k_ik_j}{k^2}\frac{1}{m_{\text{eff}}}+\frac{1}{\hbar^2}\frac{dE}{dk}\left(\frac{\delta_{ij}}{k}-\frac{k_ik_j}{k^3}\right).$$

The second term is smaller by the factor σ_0/k_0 than the first term for electrons with $k\sim k_0$. Therefore we get

$$m_{ij}=\frac{k^2}{3k_ik_j}m_{\text{eff}}. \tag{15}$$

§ 3. Discussion

We have found that the effective mass of electrons near the Fermi surface is very small, irrespective as to whether the distribution of electrons is normal or superconductive.

The expression for the interaction energies obtained by the Bloch-Nordsieck transformation are identical with Eqs. (2,a) and (2,b) except that in the integrands the denominator $(k-zv)^2-k^2+\sigma_0 zv$ is replaced by $-2(\boldsymbol{k}\cdot\boldsymbol{w})+\sigma_0 zv$. Small effective mass, however, is not obtained, if $q^2-k^2+\sigma_0 zv$ is replaced by $q^2-k^2+\sigma_0 zv-zv^2$ in (2,a) and similar calculations as § 2 is carried out. Accordingly the approximation of neglecting electron recoil is not a good one for our problem.

Peierls' formula[4] for the diamagnetic susceptibility of electrons in a periodic crystalline field consists of three terms, of which we are concerned with the following term

$$\chi=\frac{e^2}{12\hbar^2c^2}\sum_k\left[\frac{\partial^2\varepsilon_0}{\partial k_x^2}\frac{\partial^2\varepsilon_0}{\partial k_y^2}-\left(\frac{\partial^2\varepsilon_0}{\partial k_x\partial k_y}\right)^2\right]f''\left[E_0+\varepsilon_0(\boldsymbol{k})\right],$$

where $f(E) = -kT\log(1+\exp(E_0-E)/kT)$ and ε_0 is the perturbation energy of the periodic field. If the self energy of electrons is formally added to ε_0, the order of magnitude of the susceptibility will not be changed much, because according to (15)

$$\frac{1}{\hbar^2}\left\{\frac{\partial^2(E_1+E_2)}{\partial k_x^2}\cdot\frac{\partial^2(E_1+E_2)}{\partial k_y^2}-\left[\frac{\partial^2(E_1+E_2)}{\partial k_x\partial k_y}\right]^2\right\}\approx O\left(\frac{1}{m}\right).$$

The estimation of the order of magnitude of the susceptibility may be made by the following consideration. In quantum electrodynamics the self energy of electron is interpreted as the contribution to the mass. The self energy of electrons interacting with lattice vibrations may be interpreted analogously, then we can write as

$$E_0(k)+E_1(k)+E_2(k)=\hbar^2k^2/2m^*, \quad \text{with } m^*=m+\Delta m.$$

If we write as $E_1(k)+E_2(k)=\hbar^2k^2/2m'$, we easily find that for electrons near the Fermi surface $m'=O(mk_0/\sigma_0)$. Since $1/m^*=1/m+1/m'$, we get

$$\Delta m=O(m\sigma_0/k_0).$$

We may take the Hamiltonian of a conduction electron in a magnetic field as

$$\mathcal{H}=\frac{1}{2m^*}\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)^2,$$

in which the interaction of electrons with lattice vibrations is taken into account as the mass renormalization. Since m^* is of the order of m, the diamagnetic susceptibility will be of the order of Landau diamagnetism of free electrons. The above result is, of course, not conclusive. It is imperative to consider the interaction of electrons in a magnetic field with lattice vibrations.

Note added in proof: Recently Fröhlich has shown that the same eqs. (2a), (2b) and (2c) can also be derived by means of a canonical transformation, which leads to a renormalization of the velocity of sound and of the interaction parameter F. Therefore s and F in this paper are to be considered as the renormalized sound velocity and the renormalized interaction parameter respectively.

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On the Gauge Invariance of the Neutral Vector Meson Theory*

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We discuss the neutral vector meson theory in a way which, by exploiting its similarity to quantum electrodynamics, simplifies the transition to vanishing meson rest mass. In particular the gauge invariance of electrodynamics is shown to be a special case of an invariance holding more generally in the meson theory.

It has been known for some time that the analogy between quantum electrodynamics and the neutral vector meson theory is considerably stronger than the usual meson theoretical formalism suggests. In the following note we shall examine this correspondence in detail, and show in particular the way in which the two theories coalesce in the limit of vanishing meson mass. Many of the similarities of the formalisms are due to a type of gauge invariance which the neutral vector meson theory may be shown to possess in common with electrodynamics.

In the familiar formulation[1] of the neutral vector meson theory the uncoupled field components $\varphi_\nu(x)$ obey the equations of motion

$$(\Box-\mu^2)\varphi_\nu(x)=0 \tag{1}$$

$$\frac{\partial}{\partial x_\nu}\varphi_\nu(x)=0\,, \tag{2}$$

in which μ is the meson mass***. These are accompanied by the commutation relation

$$[\varphi_\lambda(x),\varphi_\nu(x')]=i\left(\delta_{\lambda\nu}-\frac{1}{\mu^2}\frac{\partial}{\partial x_\lambda\partial x_\nu}\right)\Delta(x-x')\,, \tag{3}$$

where the function $\Delta(x)$ is defined by Schwinger[2]. The interaction Hamiltonian for vector coupling to spinor fields is

$$\mathcal{H}(x)=j_\nu(x)\varphi_\nu(x)+\frac{1}{2\mu^2}(n_\nu j_\nu)^2, \tag{4}$$

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* This note is based on material in the author's Ph. D. thesis submitted to Harvard University in June 1949, and further developed during a stay in Zurich the following year.

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*** We use units in which $\hbar=1$, $c=1$.

in which $j_\nu(x)$ is the usual Dirac current vector with a mesonic charge, and the n_ν are components of the normals to a space-like surface through x. The latter term, which is necessary for the Schrödinger equation to be integrable cannot, of course, contribute to physical effects since the choice of space-like surfaces through x is arbitrary. Both this term, and the second term of the commutator (3) owe their presence to the equation of motion (2), which says, in effect, that of the four quantities $\varphi_\nu(x)$ only three are independent. Rather than destroy the symmetry of the treatment by explicitly eliminating one component, we use a reformulation of the theory due to Stueckelberg[3]. The equations of the reformulation are required to have the same physical content as those above, while retaining four-dimensional symmetry.

Following Stueckelberg, we employ five field variables, the components of a four-vector $A_\nu(x)$ and a scalar $B(x)$. The Lagrangian without coupling may be taken to be

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial A_\nu}{\partial x_\lambda}\frac{\partial A_\nu}{\partial x_\lambda}+\mu^2 A_\nu A_\nu\right)+\frac{1}{2}\left(\frac{\partial B}{\partial x_\lambda}\frac{\partial B}{\partial x_\lambda}+\mu^2 B^2\right), \tag{5}$$

from which it follows that A_ν and B obey the Klein-Gordon equation

$$(\Box-\mu^2)A_\nu=0, \qquad (\Box-\mu^2)B=0 \tag{6}$$

and may be taken to satisfy the simple commutation rules

$$[A_\lambda(x), A_\nu(x')]=i\delta_{\lambda\nu}\Delta(x-x'), \qquad [B(x), B(x')]=i\Delta(x-x'), \tag{7}$$
$$[A_\lambda(x), B(x')]=0.$$

Stueckelberg has shown[3] that the expectation value of the Hamiltonian which follows from (5) will always be positive if the fields obey the self-consistent supplementary conditions

$$(\partial A_\nu(x)/\partial x_\nu+\mu B(x))\Psi=0, \tag{8}$$

where Ψ is the state vector. These conditions effectively reduce to four the number of independent field variables.

Only three independent variables are required, of course, to specify the state of a vector field. The fourth independent variable still present, while preserving the symmetry of the theory, must be required, in effect, to be an ignorable coordinate. The group of transformations under which the theory is correspondingly required to remain invariant may be taken as the gauge transformations,

$$A_\nu(x)\to A_\nu(x)+(1/\mu)\partial\Lambda(x)/\partial x_\nu, \qquad B(x)\to B(x)-\Lambda(x). \tag{9}$$

These already leave the integrated Lagrangian, the equations of motion, and the supplementary conditions unchanged, provided $\Lambda(x)$ is any function which satisfies the Klein-Gordon equation. In particular the choice $\Lambda(x)=B(x)$ eliminates the scalar field, leaving a four-vector field which may then be identified with the vector $\varphi_\nu(x)$ of the usual formulation[4]

$$\varphi_\nu(x)=A_\nu(x)+(1/\mu)\partial B(x)/\partial x_\nu. \tag{10}$$

To confirm the identification we observe that (10) implies the commutation relation (3), and that, owing to (8), the equation of motion (2) is replaced by an analogous supplementary condition,

$$(\partial\varphi_\nu(x)/\partial x_\nu)\Psi=0. \tag{11}$$

Since the relation (10) is itself gauge-invariant, direct use may be made of it to generalize the familiar form of the theory. In particular the interaction Hamiltonian (4) may be rewritten in the form

$$\mathcal{H}(x)=j_\nu A_\nu+(1/\mu)\,j_\nu(\partial B/\partial x_\nu)+(1/2\mu^2)\,(n_\nu j_\nu)^2, \tag{12}$$

an expression whose advantage has also been discussed by Matthews[5].

The last two terms of (12) may be recognized as the interaction Hamiltonian for the vector coupling of the scalar field B. Such couplings, as Dyson has indicated[6] lead to no dynamical effects, and may be eliminated by an elementary redefinition of the state vector. It suffices to define the new state vector $\Psi'(\sigma)$ by the relation

$$\Psi(\sigma)=e^{-iS(\sigma)}\Psi'(\sigma) \tag{13}$$

with

$$S(\sigma)=\frac{1}{\mu}\int_\sigma j_\nu(x')B(x')\,d\sigma', \tag{14}$$

the integration being carried out over the contemporary surface σ. ($d\sigma_\nu$ is a directed surface element parallel to n_ν.) It is then easily verified that $\Psi'(\sigma)$ exactly satisfies the Schrödinger equation with the simple interaction Hamiltonian

$$\mathcal{H}'(x)=j_\nu(x)A_\nu(x), \tag{15}$$

which is the direct analogue of the electrodynamical interaction.

The supplementary condition (8) is transformed by writing it in the form

$$e^{iS(\sigma)}(\partial A_\nu(x)/\partial x_\nu+\mu B(x))e^{-iS(\sigma)}\Psi'=0 \tag{16}$$

where, for generality, x need not lie on σ. The exponentials commute with the first term, and the multiple commutator expansion may be used in evaluating the second one, yielding the transformed condition:

$$\left\{\frac{\partial A_\nu(x)}{\partial x_\nu}+\mu B(x)+\int\Delta(x-x')j_\nu(x')\,d\sigma'_\nu\right\}\Psi'=0. \tag{17}$$

The latter relation, in the limit of vanishing μ, is just the electrodynamical supplementary condition given by Schwinger[2].

Calculations with the interaction (15) proceed almost exactly as in electrodynamics. The field $B(x)$ has no further bearing upon the interaction, and, as a consequence, the supplementary condition (17) may be regarded as a definition of $B(x)$ in terms of the four variables $A_\nu(x)$. The condition then no longer restricts the values A_ν may assume, and may, for most purposes, be ignored.

The invariance under the gauge transformations (9) assumes a more familiar form in the modified theory. The substitutions

$$A_\nu \to A_\nu - \partial\Lambda/\partial x_\nu, \qquad \Psi' = e^{-iG}\Psi'', \tag{18}$$

where

$$G = \int j_\nu(x')\Lambda(x')dx'_\nu \tag{19}$$

leave the Schrödinger equation unaltered, a property which is directly analogous to the gauge invariance of quantum electrodynamics.

Gauge transformations allow, in effect, the elimination of one of the four independent field components A_ν, the remaining ones representing the three orientations of a particle of unit spin. For vanishing meson mass, the field $B(x)$ no longer enters the supplementary condition, which then resumes its importance as a restriction on the initial values of $A_\nu(x)$. This corresponds physically to the fact that a photon has only two independent states of polarization.

In calculating radiative corrections to the quantum mass, the situation encountered differs somewhat from that of electrodynamics. Correction to the meson mass may be separated into a non-gauge-invariant, and a gauge-invariant part, both of which are formally divergent. Here, as in electrodynamics, the non-gauge-invariant integrals must be presumed to vanish. (In second order the integral is identical to the electrodynamical one.)

The remaining gauge-invariant addition to the meson mass is a real divergence of the theory. It may however, be removed along with similar difficulties for the mass and charge of the nucleon, by the procedure of renormalization. For this purpose it must be noted that the field $B(x)$, although eliminated from the interaction is still present in the free-field Hamiltonian. Gauge-invariant corrections to the energy density arising from mass renormalizations must be shown to have the form $\delta(\mu^2)\{A_\nu A_\nu + B^2\}$, which may be done by using the supplementary condition to reintroduce B at the last step. (Its reintroduction is otherwise unnecessary.) The meson mass correction is logarithmically divergent and vanishes as μ goes to zero :

$$\delta_\mu = -\mu\left(\frac{g^2}{4\pi}\right)\left\{\frac{1}{12\pi}\int_0^\infty \frac{\cos w}{w}dw + \text{finite terms, } O(\mu^2)\right\}. \tag{20}$$

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On the Transition Amplitude in Quantum Electrodynamics

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A set of orthonormal states is constructed by operating (free) photon-, electron- and positron-creation operators one after another to the (free) vacuum state Ψ_0. Any state, say $\Psi[\sigma]$, can be expanded in terms of these orthonormal states. An equation is derived in a covariant form which is satisfied by a coefficient of the expansion which is a functional of σ and whose absolute square may be interpreted as the transition probability.

The procedure may be regarded as the transition to the Fock space. In the last section, relation to the S-matrix is also discussed.

§ 1. Introduction

As we stated previously[1], the convergence of the S-matrix series may be attributed to that of the coefficients of the expansion of the S-matrix into the normal constituents. As explicit calculations of the coefficients are very difficult, it seems desirable that other strategies will be prepared (the more, the better). This amounts to search other methods to evaluate the transition probability.

Now, a set of orthonormal states can be constructed by operating (free) photon-, electron- and positron-creation operators one after another to the free vacuum state Ψ_0. We assume the set complete. When any state, say $\Psi[\sigma]$, is expanded in terms of the orthonormal states, the coefficients may be regarded as wave functions in the Fock space and their absolute squares give the transition probabilities. The idea is not a new one and have already been used by several authors, for example, W. Thirring and B. Touschek[2], G. Lüders[3], etc. Their results, however, are yet unsatisfactory to apply to our case. Transition to the Fock space in the interaction representation has been done by I. Sato[4] in a slightly different way.

The transition amplitude is essentially non-covariant, but in § 3 a covariant function associated with it is defined and the covariant equations satisfied by the function are derived. In § 4, connection with the S-matrix is discussed.

§ 2. A set of orthonormal states

We will follow the Schwinger's notations[5], unless we give some remarks, and we introduce for the sake of convenience the following notations:

$$p^+ = (\boldsymbol{p}, i\varepsilon(\boldsymbol{p})), \qquad p^- = (\boldsymbol{p}, -i\varepsilon(\boldsymbol{p})). \tag{1}$$

for electrons and positrons, where

$$\varepsilon(\boldsymbol{p})=\sqrt{\boldsymbol{p}^2+\varkappa^2}, \tag{2}$$

and

$$p^+=(\boldsymbol{p}, i|\boldsymbol{p}|), \qquad p^-=(\boldsymbol{p}, -i|\boldsymbol{p}|) \tag{3}$$

for photons.

a) *The electromagnetic field*

The four-dimensional vector potential $A_\mu(x)$ satisfies the commutation relation,

$$[A_\mu(x), A_\nu(x')]=i\hbar c\,\delta_{\mu\nu}D(x-x'), \tag{4}$$

and can be expanded as

$$A_\mu(x)=\int \delta(k^2)\exp(ikx)A_\mu(k)\,(dk). \tag{5}$$

Then we define

$$\begin{aligned} a_\mu(\boldsymbol{k})&=\sqrt{(2\pi)^3/2\hbar c|\boldsymbol{k}|}\;A_\mu(k^+),\\ a_\mu^*(-\boldsymbol{k})&=\sqrt{(2\pi)^3/2\hbar c|\boldsymbol{k}|}\;A_\mu(k^-), \end{aligned} \tag{6}$$

which follow the commutation relations:

$$\begin{aligned} [a_\mu(\boldsymbol{k}), a_\nu(\boldsymbol{k}')]&=[a_\mu^*(\boldsymbol{k}), a_\nu^*(\boldsymbol{k}')]=0,\\ [a_\mu(\boldsymbol{k}), a_\nu^*(\boldsymbol{k}')]&=\delta_{\mu\nu}\delta(\boldsymbol{k}-\boldsymbol{k}'). \end{aligned} \tag{7}$$

Here $a_4^*(\boldsymbol{k})$ is not hermitian conjugate of $a_4(\boldsymbol{k})$ but antihermitian conjugate. We have, however, denoted it by an asterisk for simplicity sake and we will use Φ^* in the same meaning as $\Phi^\dagger=\Phi^*\eta$ of S. N. Gupta[6] and K. Bleuler[7]. Then we need not seriously take the supplementary condition.

b) *The electron positron field*

$\psi(x)$ which satisfies the free Dirac equation can expanded as

$$\psi_\alpha(x)=\int (i\gamma q-\varkappa)_{\alpha\beta}\delta(q^2+\varkappa^2)\psi_\beta(q)e^{iqx}(dq), \tag{8}$$

and follows the commutation relation:

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}=-iS_{\alpha\beta}(x-x'). \tag{9}$$

As $\psi(q)$ in (8) has superfluous freedom, we may impose upon it the restrictions:

$$\gamma_4\psi(q^+)=\psi(q^+), \quad \gamma_4\psi(q^-)=-\psi(q^-), \tag{10}$$

or

$$\psi_\rho(q^+)=\psi_\sigma(q^-)=0, \quad (\rho=3, 4\,;\,\sigma=1, 2)$$

Then we define $c_\rho(\boldsymbol{q})$ and $d_\sigma(\boldsymbol{q})$ by

$$\begin{aligned} \psi_\rho(q^+)&=\sqrt{2\varepsilon(\boldsymbol{q})/(2\pi)^3(\varepsilon(\boldsymbol{q})+\varkappa)}\;c_\rho(\boldsymbol{q}), \quad (\rho=1, 2)\\ \psi_\sigma(q)&=\sqrt{2\varepsilon(\boldsymbol{q})/(2\pi)^3(\varepsilon(\boldsymbol{q})+\varkappa)}\;d_\sigma^*(\boldsymbol{q}), \quad (\sigma=3, 4) \end{aligned} \tag{11}$$

which satisfy the commutation relations (See, Appendix I) :

$$\begin{aligned}&\{c_\rho(\boldsymbol{k}),\, c^*_{\rho'}(\boldsymbol{k}')\}=\delta_{\rho\rho'}\delta(\boldsymbol{k}-\boldsymbol{k}') \quad (\rho,\,\rho'=1,\,2)\\&\{d_\sigma(\boldsymbol{k}),\, d^*_{\sigma'}(\boldsymbol{k}')\}=\delta_{\sigma\sigma'}\delta(\boldsymbol{k}-\boldsymbol{k}'), \quad (\sigma,\,\sigma'=3,\,4)\\&\text{other anticommutators}=0.\end{aligned} \tag{12}$$

c) *The orthonormal states*

The (free) vacuum Ψ_0 is defined by

$$a_\mu(\boldsymbol{k})\Psi_0=c_\rho(\boldsymbol{k})\Psi_0=d_\sigma(\boldsymbol{k})\Psi_0=0, \tag{13}$$

and assumed to be normalized as,

$$\Psi_0{}^*\Psi_0=1. \tag{14}$$

Now we define

$$\begin{aligned}&\Phi_{\mu_1\cdots\mu_l\,\rho_1\cdots\rho_m\,\sigma_1\cdots\sigma_n}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_l\,;\,\boldsymbol{p}_1,\ldots,\boldsymbol{p}_m\,;\,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)\\&=(1/\sqrt{l!\,m!\,n!})\,a^*_{\mu_1}(\boldsymbol{k}_1)\ldots a^*_{\mu_l}(\boldsymbol{k}_l)c^*_{\rho_1}(\boldsymbol{p}_1)\ldots c^*_{\rho_m}(\boldsymbol{p}_m)d^*_{\sigma_1}(\boldsymbol{q}_1)\ldots d^*_{\sigma_n}(\boldsymbol{q}_n)\Psi_0.\end{aligned} \tag{15}$$

These states are orthonormal in the following meaning : unless $l=l'$, $m=n'$, $n=n'$,

$$\begin{aligned}&\Phi^*([\boldsymbol{k}_1']\ldots[\boldsymbol{k}_{l'}'];\,\{\boldsymbol{p}_1'\}\ldots\{\boldsymbol{p}_{m'}'\};\,\{\boldsymbol{q}_1'\}\ldots\{\boldsymbol{q}_{n'}'\})\times\\&\quad\Phi([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})=0,\end{aligned} \tag{16}$$

where we ceased writing vector and spinor indices explicitly and denoted it by square brackets and braces as,

$$\mu_i,\,\boldsymbol{k}_i=[\boldsymbol{k}_i], \quad \rho_i,\,\boldsymbol{p}_i=\{\boldsymbol{p}_i\}, \quad \sigma_i,\,\boldsymbol{q}_i=\{\boldsymbol{q}_i\}, \tag{17}$$

and in the case of $l=l'$, $m=m'$, $n=n'$,

$$\begin{aligned}&\Phi^*([\boldsymbol{k}_1']\ldots[\boldsymbol{k}_l'];\,\{\boldsymbol{p}_1'\}\ldots\{\boldsymbol{p}_m'\};\,\{\boldsymbol{q}_1'\}\ldots\{\boldsymbol{q}_n'\})\times\\&\quad\Phi([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l],\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\},\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\\&=(1/l!)\sum_P\delta([\boldsymbol{k}_1]-[\boldsymbol{k}'_{p(1)}])\ldots\delta([\boldsymbol{k}_l]-[\boldsymbol{k}'_{p(l)}])\\&\times(1/m!)\sum_Q\delta_Q\,\delta(\{\boldsymbol{p}_1\}-\{\boldsymbol{p}'_{q(1)}\})\ldots\delta(\{\boldsymbol{p}_m\}-\{\boldsymbol{p}_{q(m)}\})\\&\times(1/n!)\sum_R\delta_R\,\delta(\{\boldsymbol{q}_1\}-\{\boldsymbol{q}_{r(1)}\})\ldots\delta(\{\boldsymbol{q}_n\}-\{\boldsymbol{q}_{r(n)}\}),\end{aligned} \tag{18}$$

where

$$\delta([\boldsymbol{k}_i]-[\boldsymbol{k}'_j])=\delta_{\mu_i\mu_j}\delta(\boldsymbol{k}_i-\boldsymbol{k}'_j), \quad \text{etc.}$$

and the summation with respect to P, Q, R should be taken over all permutation :

$$P=\begin{pmatrix}1, & 2, & \ldots, & l\\ p(1), & p(2), & \ldots, & p(l)\end{pmatrix},$$

$$Q=\begin{pmatrix}1, & 2, & \ldots, & m\\ q(1), & q(2), & \ldots, & q(m)\end{pmatrix},$$

$$R=\begin{pmatrix}1, & 2, & \ldots, & n\\ r(1), & r(2), & \ldots, & r(n)\end{pmatrix},$$

and δ_Q and δ_R are ± 1 according as the corresponding permutation is even or odd. The above Eqs. (16) and (18) result directly from (15), (7), (12) and (13). The right hand side of (18) reduces simply to

$$\begin{aligned}&\delta([\boldsymbol{k}_1]-[\boldsymbol{k}_1'])\ldots\delta([\boldsymbol{k}_l]-[\boldsymbol{k}_l'])\times\\&\delta(\{\boldsymbol{p}_1\}-\{\boldsymbol{p}_1'\})\ldots\delta(\{\boldsymbol{p}_m\}-\{\boldsymbol{p}_m'\})\times\\&\delta(\{\boldsymbol{q}_1\}-\{\boldsymbol{q}_1'\})\ldots\delta(\{\boldsymbol{q}_n\}-\{\boldsymbol{q}_n'\}),\end{aligned}\tag{19}$$

if we multiply it by a function, symmetric with respect to $[\boldsymbol{k}_1'],\ldots,[\boldsymbol{k}_l']$ and antisymmetric with respect to $\{\boldsymbol{p}_1'\},\ldots,\{\boldsymbol{p}_m'\}$ and $\{\boldsymbol{q}_1'\},\ldots,\{\boldsymbol{q}_n'\}$.

If we assume the set of the orthonormal states (15) is complete, any state Ψ can be expanded as

$$\begin{aligned}\Psi=\sum_{l,m,n=0}^{\infty}\int &d\boldsymbol{k}_1\ldots d\boldsymbol{k}_l\,d\boldsymbol{p}_1\ldots d\boldsymbol{p}_m d\boldsymbol{q}_1\ldots d\boldsymbol{q}_n\times\\&u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\times\\&\Phi([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}).\end{aligned}\tag{20}$$

Taking into account the definition of Φ (15) and the commutation relations (7) and (12), we see that u must have the very property mentioned just below the Eq. (19). For a given Ψ, u can be determined from

$$\begin{aligned}&u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\\&\quad=\Phi^*([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\cdot\Psi.\end{aligned}\tag{21}$$

d) *The normalization*

i) When Ψ is normalized as

$$\Psi^*\Psi=1,$$

u must satisfy

$$\begin{aligned}\sum_{l,m,n=0}^{\infty}\int &d\boldsymbol{k}_1\ldots d\boldsymbol{k}_l\,d\boldsymbol{p}_1\ldots d\boldsymbol{p}_m d\boldsymbol{q}_1\ldots d\boldsymbol{q}_n\times\\&u^*([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\times\\&u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})=1.\end{aligned}\tag{22}$$

ii) In scattering problems, another normalization is often available such as

$$\Psi^*\Psi=\infty$$

so that the probability that a particle exists in some unit volume $=1$. With this normalization, for instance, in the case that l' free photons with momenta (and polarization), $[\boldsymbol{k}_1^0],\ldots,[\boldsymbol{k}_{l'}^0]$, m' free electrons and n' free positrons with momenta (and spin) $\{\boldsymbol{p}_1^0\},\ldots,\{\boldsymbol{p}_{m'}^0\}$ and $\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\}$ respectively, we have,

$$u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_{l'}]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_{m'}\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_{n'}\})$$
$$= \Pi\,\delta([\boldsymbol{k}_i]-[\boldsymbol{k}_i^0])\;\Pi\,\delta(\{\boldsymbol{p}_j\}-\{\boldsymbol{p}_j^0\})\;\Pi\,\delta(\{\boldsymbol{q}_k\}-\{\boldsymbol{q}_k^0\}). \tag{23}$$

§ 3. Covariant formulation and the equation of motion

It seems desirable in many cases to perform calculations in covariant forms. For the purpose we expand Ψ as

$$\Psi = \sum_{l,m,n=0}^{\infty}(1/\sqrt{l!\,m!\,n!})\int(dx_1)\ldots(dx_l)\,(dy_1)\ldots(dy_m)\,(dz_1)\ldots(dz_n)\times$$
$$v([x_1]\ldots[x_l]\,;\,\{y_1\}\ldots\{y_m\}\,;\,\{z_1\}\ldots\{z_n\})\,A^{(-)}([x_1])\ldots A^{(-)}([x_l])\times$$
$$\bar{\psi}^{(-)}(\{y_1\})\ldots\bar{\psi}^{(-)}(\{y_m\})\,\psi^{(-)}(\{z_1\})\ldots\psi^{(-)}(\{z_n\})\,\Psi_0, \tag{24}$$

instead of (20). We obtain in the momentum space*),

$$\Psi = \sum_{l,m,n=0}^{\infty}(1/\sqrt{l!\,m!\,n!})\int(dk_1)\ldots(dk_l)\,(dp_1)\ldots(dp_m)\,(dq_1)\ldots(dq_n)\times$$
$$v([k_1]\ldots[k_l]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})\prod_{i=1}^{l}A^{(-)}([k_i])\,\delta(k_i^2)\times$$
$$\prod_{j=1}^{m}\bar{\psi}^{(-)}(\{p_j\})\,(i\gamma p_j-\varkappa)\,\delta(p_j^2+\varkappa^2)\prod_{k=1}^{n}(i\gamma q_k-\varkappa)\,\delta(q_k^2+\varkappa^2)\,\psi^{(-)}(\{q_k\})\cdot\Psi_0, \tag{25}$$

where

$$v([k_1]\ldots[k_l]\,;\,\{p_1\}\ldots\{p_m\}\,;\,\{q_1\}\ldots\{q_n\})$$
$$=\int(dx_1)\ldots(dx_l)\,(dy_1)\ldots(dy_m)\,(dz_1)\ldots(dz_n)\times$$
$$v([x_1]\ldots[x_l]\,;\,\{y_1\}\ldots\{y_m\}\,;\,\{z_1\}\ldots\{z_n\})\times$$
$$\exp\left[i\{k_1x_1+\ldots+k_lx_l-p_1y_1-\ldots-p_my_m+q_1z_1+\ldots+q_nz_n\}\right], \tag{26}$$

and

$$A_\mu^{(-)}(k)=A_\mu(k)\theta(-k),\quad \psi^{(-)}(q)=\psi(q)\theta(-q),$$
$$\bar{\psi}^{(-)}(p)=\overline{\psi^{(+)}}(p)=\bar{\psi}(p)\theta(p).$$

Performing the integrations with respect to k_{i0}, p_{j0}, q_{k0}, we have

$$\Psi = \sum_{l,m,n=0}^{\infty}(1/\sqrt{l!\,m!\,n!})\int d\boldsymbol{k}_1\ldots d\boldsymbol{k}_l\,d\boldsymbol{p}_1\ldots d\boldsymbol{p}_m d\boldsymbol{q}_1\ldots d\boldsymbol{q}_n\times$$
$$v([k_1^-]\ldots[k_l^-]\,;\,\{p_1^+\}\ldots\{p_m^+\}\,;\,\{q_1^-\}\ldots\{q_n^-\})\times$$
$$\prod_{i=1}^{l}(1/2|k_i|)A([k_i^-])\prod_{j=1}^{m}(1/2\varepsilon(\boldsymbol{p}_j^+))\bar{\psi}(\{p_j^+\})\,(i\gamma p_j^+-\varkappa)\times$$
$$\prod_{k=1}^{n}(1/2\varepsilon(\boldsymbol{q}_k))\,(i\gamma q_k^--\varkappa)\psi(\{q_k^-\})\,\Psi_0. \tag{27}$$

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* The multiplicands must be placed in increasing orders of the indices.

Since we have from (10)

$$\bar{\psi}(p_j^+) = \psi^*(p_j^+),$$

considering (6), (11), (15) and (20), a relation between u and v is derived:

$$u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})$$
$$= (\hbar c)^{\frac{l}{2}}(2\pi)^{-\frac{3}{2}(l+m+n)} \prod_{i=1}^{l} (1/\sqrt{2|\boldsymbol{k}_i|}) \prod_{j=1}^{m} (1/\sqrt{2\varepsilon(\boldsymbol{p}_j)(\varepsilon(\boldsymbol{p}_j)+\varkappa)})(i\gamma p_j^+ - \varkappa) \times$$
$$v([k_1^-]\ldots[k_l^-],\{p_1^+\}\ldots\{p_m^+\},\{q_1^-\}\ldots\{q_n^-\}) \times$$
$$\prod_{k=1}^{m} (1/\sqrt{2\varepsilon(\boldsymbol{q}_k)(\varepsilon(\boldsymbol{q}_k)+\varkappa)})(i\gamma q_k^- - \varkappa). \tag{28}$$

Factors multiplied into v in (28) show that u is not Lorentz-invariant. A covariant function U associated with u can be defined as follows:

$$U([x_1]\ldots[x_l];\{y_1\}\ldots\{y_m\};\{z_1\}\ldots\{z_n\}) = (-)^m(\hbar c)^{-\frac{l}{2}} i^{l+m+n} \times$$
$$(2\pi)^{-\frac{3}{2}(l+m+n)} \int d\boldsymbol{k}_1\ldots d\boldsymbol{k}_l\, d\boldsymbol{p}_1\ldots d\boldsymbol{p}_m d\boldsymbol{q}_1\ldots d\boldsymbol{q}_n \prod_{i=1}^{l} (1/\sqrt{2|\boldsymbol{k}_i|})\, e^{-ik_i^- x_i} \times$$
$$\prod_{j=1}^{m} \sqrt{(\varepsilon(\boldsymbol{p}_j)+\varkappa)/2\varepsilon(\boldsymbol{p}_j)}\, e^{ip_j^+ y_j} \prod_{k=1}^{n} \sqrt{(\varepsilon(\boldsymbol{q}_k)+\varkappa)/2\varepsilon(\boldsymbol{q}_k)}\, e^{-iq_k^- z_k} \times$$
$$u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l],\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}). \tag{29}$$

U can be written also in terms of v,

$$U([x_1]\ldots[x_l];\{y_1\}\ldots\{y_m\};\{z_1\}\ldots\{z_n\})$$
$$= \int (dx'_1)\ldots(dx'_l)(dy'_1)\ldots(dy'_m)(dz'_1)\ldots(dz'_n) \times$$
$$D^{(-)}([x_1']-[x_1])\ldots D^{(-)}([x_l']-[x_l])\, S^{(+)}(\{y_1\}-\{y_1'\})\ldots S^{(+)}(\{y_m\}-\{y_m'\}) \times$$
$$S^{(-)}(\{z_1'\}-\{z_1\})\ldots S^{(-)}(\{z_n'\}-\{z_n\})\, v([x_1']\ldots[x_l'],\{y_1'\}\ldots\{y_m'\},\{z_1'\}\ldots\{z_n'\}). \tag{30}$$

Insert the expansion (23) regarding v as a functional of σ into the Tomonaga-Schwinger's equation of motion for $\Psi[\sigma]$:

$$i\hbar c(\delta/\delta\sigma(x))\Psi[\sigma] = H(x)\Psi[\sigma],$$
$$H(x) = -(1/c) j_\mu(x) A_\mu(x),$$
$$j_\mu(x) = (iec/2)[\bar{\psi}(x)\gamma_\mu, \psi(x)], \tag{31}$$

and operate

$$\Psi_0{}^*\bar{\psi}^{(+)}(\{z_n\})\ldots\bar{\psi}^{(+)}(\{z_1\})\psi^{(+)}(\{y_m\})\ldots\psi^{(+)}(\{y_1\})A^{(+)}([x_l])\ldots A^{(+)}([x_1]),$$

from the left and take into account the commutation relations:

$$[A_\mu^{(+)}(x), A_\nu^{(-)}(x')] = -i\hbar c\,\delta_{\mu\nu} D^{(-)}(x'-x),$$
$$\{\psi_\alpha^{(+)}(x), \bar{\psi}_\beta^{(-)}(x')\} = -iS_{\alpha\beta}^{(+)}(x-x'),$$
$$\{\bar{\psi}_\beta^{(+)}(x), \psi_\alpha^{(-)}(x')\} = -iS_{\alpha\beta}^{(-)}(x'-x), \tag{32}$$

and the definition of the vacuum

$$A_\mu^{(+)}\Psi_0 = \psi_\alpha^{(+)}\Psi_0 = \bar{\psi}_\alpha^{(+)}\Psi_0 = 0,$$
$$\Psi_0^* A_\mu^{(-)} = \Psi_0^*\bar{\psi}^{(-)} = \Psi_0^*\psi^{(-)} = 0. \tag{33}$$

Then we have the equation of motion for U:

$$\partial/\partial\sigma(x)\cdot U([x_1]\dots[x_l];\ \{y_1\}\dots\{y_m\};\ \{z_1\}\dots\{z_n\};\ \sigma)$$
$$= -ie\gamma_{\alpha\beta}^{\mu}[(i/\hbar c)\sqrt{(m+1)(n+1)/l}\ \sum_{i=1}^{l} D^{(-)}([x]-[x_i])\times$$
$$U([x_1]\dots[x_{i-1}][x_{i+1}]\dots[x_l];\ \{y_1\}\dots\{y_m\}\{x\};\ \{\bar{x}\}\{z_1\}\dots\{z_n\};\ \sigma)$$
$$+\sqrt{(l+1)(m+1)(n+1)}\,U([x][x_1]\dots[x_l];\ \{y_1\}\dots\{y_m\}\{x\};\ \{\bar{x}\}\{z_1\}\dots\{z_n\};\ \sigma)$$
$$+i\sqrt{l+1}\sum_{j=1}^{m} S^{(+)}(\{y_j\}-\{\bar{x}\})U([x][x_1]\dots[x_l];\ \{y_1\}\dots\{y_{j-1}\}\{x\}\{y_{j+1}\}\dots\{y_m\};$$
$$\{z_1\}\dots\{z_n\};\ \sigma)$$
$$-i\sqrt{l+1}\sum_{k=1}^{n} S^{(-)}(\{x\}-\{z_k\})U([x][x_1]\dots[x_l];\ \{y_1\}\dots\{y_m\};\ \{z_1\}\dots\{z_{k-1}\}\{\bar{x}\}$$
$$\{z_{k+1}\}\dots\{z_n\};\ \sigma)$$
$$-(1/\hbar c)(1/\sqrt{l})\sum_{j=1}^{l} D^{(-)}([x]-[x_j])\sum_{j=1}^{m} S^{(+)}(\{y_j\}-\{\bar{x}\})\times$$
$$U([x_1]\dots[x_{i-1}][x_{i+1}]\dots[x_l];\ \{y_1\}\dots\{y_{j-1}\}\{x\}\{y_{j+1}\}\dots\{y_m\};$$
$$\{z_1\}\dots\{z_n\};\ \sigma)$$
$$+\sqrt{l/mn}\sum_{j=1}^{m}(-1)^{m-j} S^{(+)}(\{y_j\}-\{x\})\sum_{k=1}^{n}(-1)^k S^{(-)}(\{\bar{x}\}-\{z_k\})\times$$
$$U([x][x_1]\dots[x_l],\ \{y_1\}\dots\{y_{j-1}\}\{y_{j+1}\}\dots\{y_m\},\ \{z_1\}\dots\{z_{k-1}\}\{z_{k+1}\}\dots\{z_n\};\ \sigma)$$
$$+(i/\hbar c)\sqrt{1/lmn}\sum_{i=1}^{l} D^{(-)}([x]-[x_i])\sum_{j=1}^{m}(-1)^{m-j} S^{(+)}(\{y_j\}-\{x\})\sum_{k=1}^{n}(-1)^k$$
$$S^{(-)}(\{\bar{x}\}-\{z_k\})\times$$
$$U([x_1]\dots[x_{i-1}][x_{i+1}]\dots[x_l];\ \{y_1\}\dots\{y_{j-1}\}\{y_{j+1}\}\dots\{y_m\};$$
$$\{z_1\}\dots\{z_{k-1}\}\{z_{k+1}\}\dots\{z_n\};\ \sigma), \tag{34}$$

where $[x]=\mu, x$; $\{x\}=\beta, x$; $\{\bar{x}\}=\alpha, x$ and U has been defined by (29) or (30).

If we give an initial condition:

$$u^0([\boldsymbol{k}_1^0]\dots[\boldsymbol{k}_l^0];\ \{\boldsymbol{p}_1^0\}\dots\{\boldsymbol{p}_m^0\};\ \{\boldsymbol{q}_1^0\}\dots\{\boldsymbol{q}_n^0\})$$
$$= \Phi^*([\boldsymbol{k}_1^0]\dots[\boldsymbol{k}_{l'}^0];\ \{\boldsymbol{p}_1^0\}\dots\{\boldsymbol{p}_{m'}^0\};\ \{\boldsymbol{q}_1^0\}\dots\{\boldsymbol{q}_{n'}^0\})\cdot\Psi[\sigma_0], \tag{21'}$$

and solve (34) and obtain,

$$u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}),$$

then the transition probability is given by its absolute square. Accordingly, by this procedure the transition probability can be computed, in principle, without using the assumption of the adiabatic switching-on. However in practice we do not know how to choose the actual initial state. If we adopt the state where l' free photons, m' free electrons and n' free positrons are present (normalization ii) :

$$\Phi(\,[\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0]\,;\,[\boldsymbol{p}_1^0]\ldots[\boldsymbol{p}_{m'}^0]\,;\,[\boldsymbol{q}_1^0]\ldots[\boldsymbol{q}_{n'}^0]), \tag{35}$$

as the initial state, it is equivalent to the adiabatic procedure. This is related to the fundamental difficulty of the present field theory. Since we adopted as the basis of the theory the free fields which have not any real correspondence in the nature, it is not surprising that we are obliged to take the visionary free particle state as the initial state.

I believe that this point should be corrected at the future stage of development of the field theory.

§ 4. The relation to the S-matrix

We consider the transformation function $S[\sigma,\sigma_0]$ instead of the S-matrix. $S[\sigma,\sigma_0]$ connects $\Psi[\sigma]$ at σ with $\Psi[\sigma_0]$ at σ_0, by the relation,

$$\Psi[\sigma]=S[\sigma,\sigma_0]\,\Psi[\sigma_0]. \tag{36}$$

Now we expand $\Psi[\sigma_0]$ as

$$\begin{aligned}\Psi[\sigma_0]=\sum_{l',m',n'=0}^{\infty}\int d\boldsymbol{k}_1^0\ldots d\boldsymbol{k}_{l'}^0\,d\boldsymbol{p}_1^0\ldots d\boldsymbol{p}_{m'}^0\,d\boldsymbol{q}_1^0\ldots d\boldsymbol{q}_{n'}^0\times\\ u^0([\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0]\,;\,\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\}\,;\,\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\})\times\\ \Phi([\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0]\,;\,\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\}\,;\,\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\}),\end{aligned} \tag{20'}$$

and operate

$$\Phi^*([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})$$

from the left and we have

$$\begin{aligned}&u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}\,;\,\sigma)\\ &=\sum_{l',m',n'=0}^{\infty}\int d\boldsymbol{k}_1^0\ldots d\boldsymbol{k}_{l'}^0\,d\boldsymbol{p}_1^0\ldots d\boldsymbol{p}_{m'}^0\,d\boldsymbol{q}_1^0\ldots d\boldsymbol{q}_{n'}^0\times\\ &([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l]\,;\,\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\}\,;\,\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}\,|\,S[\sigma_1\sigma_0]\,|\,[\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0]\,;\,\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\}\,;\\ &\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\})\times u^0([\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0]\,;\,\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\}\,;\,\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\}),\end{aligned} \tag{37}$$

where*

---

* $(\boldsymbol{k}\,;\,\boldsymbol{p}\,;\,\boldsymbol{q}\,|\,S\,|\,\boldsymbol{k}^0\,;\,\boldsymbol{p}^0\,;\,\boldsymbol{q}^0)$ is often referred as Green-function of quantized field.

$$(\ldots|S[\sigma,\sigma_0]|\ldots)=$$

$$\boldsymbol{\Phi}^*([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})S[\sigma,\sigma_0]\times$$

$$\boldsymbol{\Phi}([\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0];\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\};\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\}). \tag{38}$$

If we take (35) as the initial state, we have

$$u([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\})=$$

$$([\boldsymbol{k}_1]\ldots[\boldsymbol{k}_l];\{\boldsymbol{p}_1\}\ldots\{\boldsymbol{p}_m\};\{\boldsymbol{q}_1\}\ldots\{\boldsymbol{q}_n\}|S|[\boldsymbol{k}_1^0]\ldots[\boldsymbol{k}_{l'}^0];\{\boldsymbol{p}_1^0\}\ldots\{\boldsymbol{p}_{m'}^0\};\{\boldsymbol{q}_1^0\}\ldots\{\boldsymbol{q}_{n'}^0\}). \tag{39}$$

When the S-matrix has been calculated as

$$S=\sum_{L,N=0}^{\infty}\int(dx_1)\ldots(dx_L)(dy_1)\ldots(dy_N)(dz_1)\ldots(dz_N)\times$$

$$M([x_1]\ldots[x_L];\{y_1\}\ldots\{y_N\};\{z_1\}\ldots\{z_N\})\times$$

$$:A([x_1])\ldots A([x_L])\bar{\psi}(\{y_1\})\ldots\bar{\psi}(\{y_N\})\psi(\{z_1\})\ldots\psi(\{z_N\}): \tag{40}$$

it may be convenient to obtain the relation between u and M.

Introducing

$$M([K_1]\ldots[K_L];\{P_1\}\ldots\{P_N\};\{Q_1\}\ldots\{Q_N\})$$

$$=\int(dx_1)\ldots(dx_L)(dy_1)\ldots(dy_N)(dz_1)\ldots(dz_N)\times$$

$$M([x_1]\ldots[x_L];\{y_1\}\ldots\{y_N\};\{z_1\}\ldots\{z_N\})\times$$

$$\exp[i\{K_1x_1+\ldots+K_Lx_L-P_1y_1-\ldots-P_Ny_N+Q_1z_1+\ldots+Q_Nz_N\}],$$

and following the similar procedure as the transition from (24) to (27), we obtain,

$$S=\sum_{L,N=0}^{\infty}\sum_{L_1}\binom{L}{L_1}\sum_{N_1}\binom{N}{N_1}\sum_{N_2}\binom{N}{N_2}(-1)^{(N-N_1)(N_2+1)}\prod_{i=1}^{L}\sqrt{\hbar c/(2\pi)^3}|\bar{\boldsymbol{K}}_i|\times$$

$$\prod_{j=1}^{N}\sqrt{1/(2\pi)^3 2\varepsilon(\boldsymbol{P}_j)}\,\overline{(\varepsilon(\boldsymbol{P}_j)+\varkappa)}\prod_{k=1}^{N}\sqrt{1/(2\pi)^3 2\varepsilon(\boldsymbol{Q}_k)}\,\overline{(\varepsilon(\boldsymbol{Q}_k)+\varkappa)}\times$$

$$\int d\boldsymbol{K}_1\ldots d\boldsymbol{K}_L\,d\boldsymbol{P}_1\ldots d\boldsymbol{P}_N\,d\boldsymbol{Q}_1\ldots d\boldsymbol{Q}_N\prod_{j=1}^{N_1}(i\gamma P_j^+-\varkappa)\prod_{j=N_1+1}^{N}(i\gamma P_j^--\varkappa)\times$$

$$M([K_1^-]\ldots[K_{L_1}^-][K_{L_1+1}^+]\ldots[K_L^+];\{P_1^+\}\ldots\{P_{N_1}^+\}\{P_{N_1+1}^-\}\ldots\{P_N^-\},$$

$$\{Q_1^-\}\ldots\{Q_{N_2}^-\}\{Q_{N_2+1}^+\}\ldots\{Q_N^+\})\prod_{k=1}^{N_2}(i\gamma Q_k^--\varkappa)\prod_{k=N_2+1}^{N}(i\gamma Q_k^+-\varkappa)\cdot$$

$$a^*([-\boldsymbol{K}_1])\ldots a^*([-\boldsymbol{K}_{L_1}])\ldots a(\boldsymbol{K}_{L_1+1})\ldots(\boldsymbol{K}_L)c^*(\{\boldsymbol{P}_1\})\ldots c^*(\{\boldsymbol{P}_{N_1}\})\times$$

$$d^*(\{\boldsymbol{Q}_1\})\ldots d^*(\{\boldsymbol{Q}_{N_2}\})d(\{\boldsymbol{P}_{N_1+1}\})\ldots d(\{\boldsymbol{P}_N\})c(\{\boldsymbol{Q}_{N_2+1}\})\ldots c(\{\boldsymbol{Q}_N\}). \tag{41}$$

Thus we have immediately,

$$(\ldots|S|\ldots)=\sum_{f=0}\sum_{g=0}\sum_{h=0}\frac{(-1)^{(n-n')(g+h-n')}}{f!\,g!\,h!\sqrt{l!\,m!\,n!\,l'!\,m'!\,n'!}}\binom{l+l'-2f}{l-f}\binom{m+n'-g-h}{m-g}\times$$

$$\binom{n+m'-g-h}{n-h}\sum_{P}\sum_{P'}\sum_{Q}\delta_{Q}\sum_{Q'}\delta_{Q'}\sum_{R}\delta_{R}\sum_{R'}\delta_{R'}\times$$

$$\delta([\boldsymbol{k}_{p(l-f+1)}]-[\boldsymbol{k}^{0}_{p'(l'-f+1)}])\ldots\delta([\boldsymbol{k}_{p(l)}]-[\boldsymbol{k}^{0}_{p'(l')}])\times$$

$$\delta(\{\boldsymbol{p}_{q(m-g+1)}\}-\{\boldsymbol{p}^{0}_{q'(m'-g+1)}\})\ldots\delta(\{\boldsymbol{p}_{q(m)}\}-\{\boldsymbol{p}^{0}_{q'(m')}\})\times$$

$$\delta(\{\boldsymbol{q}_{r(n-h+1)}\}-\{\boldsymbol{q}^{0}_{r'(n'-h+1)}\})\ldots\delta(\{\boldsymbol{q}_{r(n)}\}-\{\boldsymbol{q}^{0}_{r'(n')}\})\times$$

$$\prod_{i=1}^{l-f}\sqrt{\hbar c/(2\pi)^{3}2|\boldsymbol{k}_{p(i)}|}\prod_{i=1}^{l'-f}\sqrt{\hbar c/(2\pi)^{3}2|\boldsymbol{k}^{0}_{p'(i)}|}\times$$

$$\prod_{j=1}^{m-g}\sqrt{1/(2\pi)^{3}2\varepsilon(\boldsymbol{p}_{q(j)})(\varepsilon(\boldsymbol{p}_{q(j)})+\varkappa)}\prod_{k=1}^{n'-k}\sqrt{1/(2\pi)^{3}2\varepsilon(\boldsymbol{q}^{0}_{r'(k)})(2\varepsilon(\boldsymbol{q}^{0}_{r'(k)})+\varkappa)}\times$$

$$\prod_{k=1}^{n-h}\sqrt{1/(2\pi)^{3}2\varepsilon(\boldsymbol{q}_{r(k)})(\varepsilon(\boldsymbol{q}_{r(k)})+\varkappa)}\prod_{j=1}^{m'-g}\sqrt{1/(2\pi)^{3}2\varepsilon(\boldsymbol{p}^{0}_{q'(j)})(\varepsilon(\boldsymbol{p}^{0}_{q'(j)})+\varkappa)}\times$$

$$\prod_{j=1}^{m-g}(i\gamma p^{+}_{q(j)}-\varkappa)\prod_{k=1}^{n'-h}(i\gamma q^{0-}_{r'(k)}-\varkappa)\,M([-k^{+}_{p(1)}]\ldots[-k^{+}_{p(l-f)}];[k^{0+}_{p'(l'-f)}]\ldots[k^{0+}_{p'(1)}];$$

$$\{p^{+}_{q(1)}\}\ldots\{p^{+}_{q(m-g)}\}\{q^{0-}_{r'(n'-h)}\}\ldots\{q^{0-}_{r'(1)}\};\{q^{-}_{r(1)}\}\ldots\{q^{-}_{r(n-h)}\}\{p^{0+}_{q'(m'-g)}\}\ldots\{p^{0+}_{q'(1)}\})$$

$$\prod_{k=1}^{n-h}(i\gamma q^{-}_{r(k)}-\varkappa)\prod_{j=1}^{m'-g}(i\gamma p^{0+}_{q'(j)}-\varkappa),\qquad(42)$$

where the upper limits of the summations with respect to f, g and h are respectively $\min(l, l')$, $\min(m, m')$ and $\min(n, n')$. If we write the spinor indices explicitly, we have

$$([\boldsymbol{k}];\rho\boldsymbol{p};\sigma\boldsymbol{q}|S|[\boldsymbol{k}^{0}];\rho^{0}\boldsymbol{p}^{0};\sigma^{0}\boldsymbol{q}^{0})=\sum\sum\sum(\text{factors})(\delta\text{-functions})\times$$

$$\Pi(i\gamma p^{+}-\varkappa)_{\rho\rho'}\,\Pi(i\gamma q^{0-}-\varkappa)_{\sigma\sigma''}\,M([-k^{+}],[k^{0+}];\{\rho'p^{+}\},\{\sigma''q^{0-}\};$$

$$\{\sigma'q^{-}\},\{\rho''p^{0+}\})\,\Pi(i\gamma q^{-}-\varkappa)_{\sigma'\sigma}\,\Pi(i\gamma p^{0+}-\varkappa)_{\rho''\rho_{0}}.$$

In general $([\boldsymbol{k}];\{\boldsymbol{p}\};\{\boldsymbol{q}\}|S|[\boldsymbol{k}^{0}];\{\boldsymbol{p}^{0}\};\{\boldsymbol{q}^{0}\})$ contains a four-dimensional δ-function $\delta(P-P^{0})$:

$$([\boldsymbol{k}];\{\boldsymbol{p}\};\{\boldsymbol{q}\}|S|[\boldsymbol{k}^{0}];\{\boldsymbol{p}^{0}\};\{\boldsymbol{q}^{0}\})$$

$$=(2\pi)^{4}\delta(P-P^{0})([\boldsymbol{k}];\{\boldsymbol{p}\};\{\boldsymbol{q}\}|\boldsymbol{S}|[\boldsymbol{k}^{0}];\{\boldsymbol{p}^{0}\};\{\boldsymbol{q}^{0}\}),\qquad(43)$$

where

$$\boldsymbol{P}=\sum\boldsymbol{k}_{i}+\sum\boldsymbol{p}_{j}-\sum\boldsymbol{q}_{k},\qquad\boldsymbol{P}^{0}=\sum\boldsymbol{k}^{0}_{i}+\sum\boldsymbol{p}^{0}_{j}-\sum\boldsymbol{q}^{0}_{k},$$

$$P_{0}=\sum|\boldsymbol{k}_{i}|+\sum\varepsilon(\boldsymbol{p}_{j})+\sum\varepsilon(\boldsymbol{q}_{k}),\quad P^{0}_{0}=\sum|\boldsymbol{k}^{0}_{i}|+\sum\varepsilon(\boldsymbol{p}^{0}_{j})+\sum\varepsilon(\boldsymbol{q}^{0}_{k}).\qquad(44)$$

Then the scattering probability per unit volume and per unit time is

$$w\,d\boldsymbol{k}_{1}\ldots d\boldsymbol{p}_{1}\ldots d\boldsymbol{q}_{1}\ldots=$$

$$c|([\boldsymbol{k}];\{\boldsymbol{p}\};\{\boldsymbol{q}\}|\boldsymbol{S}|[\boldsymbol{k}^{0}];\{\boldsymbol{p}^{0}\};\{\boldsymbol{q}^{0}\})|^{2}(2\pi)^{4}\delta(P-P^{0})\,d\boldsymbol{k}_{1}\ldots d\boldsymbol{p}_{1}\ldots d\boldsymbol{q}_{1}\ldots,\qquad(45)$$

(See Appendix II).

Multiplying into (45) the normalization volume of the initial state $(2\pi)^{3(l'+m'+n')}$ and dividing it by the flux of the incident particles φ, we obtain the differential cross-section:

$$\sigma=(2\pi)^{3(l'+m'+n')}(c/\varphi)\,|\,([\boldsymbol{k}],\{\boldsymbol{p}\},\{\boldsymbol{q}\}\,|\boldsymbol{S}|\,[\boldsymbol{k}^0],\{\boldsymbol{p}^0\},\{\boldsymbol{q}^0\})\,|^2\times$$
$$(2\pi)^4\delta(P\quad P^0)\,d\boldsymbol{k}_1\ldots d\boldsymbol{p}_1\ldots d\boldsymbol{q}_1\ldots\ . \tag{46}$$

For the essential scattering we need only retain the term $f=0$, $h=0$, $g=0$. If we write $M=(2\pi)^4\delta(P-P^0)K$, we obtain finally by averaging over the initial spin states and summing over the final spin-states*

$$\bar{\sigma}=l!\,m!\,n!\,l'!\,m'!\,n'!\left[\binom{l+l'}{l}\binom{m+n'}{m}\binom{n+m'}{n}\right]^2(c/\varphi)\times$$
$$\prod_{i=1}^{l'}(\hbar c/|2\boldsymbol{k}_i^0|)\prod_{j=1}^{m'}((\varepsilon(\boldsymbol{p}_j^0)+\varkappa)/4\varepsilon(\boldsymbol{p}_j^0))\prod_{k=1}^{n'}((\varepsilon(\boldsymbol{q}_k^0)+\varkappa)/4\varepsilon(\boldsymbol{q}_k^0))\times$$
$$\sum_{\rho,\rho'=1,2}\ \sum_{\sigma,\sigma'=3,4}|K([-k^+][k^{0+}];\{\rho p^+\}\{\sigma' q^0\};\{\sigma q^-\}\{\rho' p^{0+}\})\,|^2\times$$
$$\prod_{i=1}^{l}(\hbar c/(2\pi)^3 2|\boldsymbol{k}_i|)\,d\boldsymbol{k}_i\prod_{j=1}^{m}((\varepsilon(\boldsymbol{p}_j)+\varkappa)/(2\pi)^3 2\varepsilon(\boldsymbol{p}_j))\,d\boldsymbol{p}_j\prod_{k=1}^{n}((\varepsilon(\boldsymbol{q}_k)+\varkappa)/$$
$$(2\pi)^3 2\varepsilon(\boldsymbol{q}_k))\,d\boldsymbol{q}_k\,. \tag{47}$$

Generalization in the case where external field is present will be straight-forward.

The author is much indebted to Mr. Oneda for his helpful discussions.

Appendix I

Performing the integration with respect to q_0 in the eq. (8), we have

$$\psi_\alpha(x)=\int\frac{d\boldsymbol{q}}{2\varepsilon(\boldsymbol{q})}\{(i\gamma q^+-\varkappa)_{\alpha\beta}\psi_\beta(q^+)e^{iq^+x}+(i\gamma q^--\varkappa)_{\alpha\beta}\psi_\beta(q^-)e^{iq^-x}\}\,. \tag{A·1}$$

Now we define $u_\alpha^{(i)}$, $(i=1,2,3,4)$ as follows,

$$u_\alpha^{(1)}=N(i\gamma p^+-\varkappa)_{\alpha 1}\,,\qquad u_\alpha^{(2)}=N(i\gamma p^+-\varkappa)_{\alpha 2}\,,$$
$$u_\alpha^{(3)}=N(i\gamma p^--\varkappa)_{\alpha 3}\,,\qquad u_\alpha^{(4)}=N(i\gamma p^--\varkappa)_{\alpha 4}\,, \tag{A·2}$$

and we have accordingly

$$u_\alpha^{(i)}{}^*=N[(i\gamma p^+-\varkappa)\gamma_4]_{i\alpha}\,,\quad(i=1,2)$$
$$u_\alpha^{(i)}{}^*=-N[(i\gamma p^--\varkappa)\gamma_4]_{i\alpha}\,,\quad(i=3,4)\,. \tag{A·3}$$

Then we can easily verify the relations,

$$u_\alpha^{(i)}{}^*u_\alpha^{(j)}=N^2 2\varepsilon(\boldsymbol{p})\,(\varepsilon(\boldsymbol{p})+\varkappa)\delta_{ij}, \tag{A·4}$$

$$\bar{u}_\alpha^{(i)}\,u_\alpha^{(j)}=\pm N^2\,2\varkappa(\varepsilon(\boldsymbol{p})+\varkappa)\delta_{ij},$$
$$+\text{ for }\ i=1,2\,;\quad-\text{ for }\ i=3,4. \tag{A·5}$$

Thus we see from (A4) that the functions $u_\alpha^{(i)}$ are orthogonal and can be normalized, if we choose,

$$N=1/\sqrt{2\varepsilon(\boldsymbol{p})\,(\varepsilon(\boldsymbol{p})+\varkappa)}. \tag{A·6}$$

Taking into account the restrictions (10), we have

$$\psi_s(\boldsymbol{p},t)=(2\pi)^{-\frac{3}{2}}\int d\boldsymbol{q}e^{i\boldsymbol{q}\boldsymbol{r}}\Big(\sum_{\rho=1,2}u_\alpha^{(\rho)}c_\rho(\boldsymbol{q})e^{-i\varepsilon(q)t}+\sum_{\sigma=3,4}u_\alpha^{(\sigma)}d_\sigma^*(\boldsymbol{q})e^{i\varepsilon(q)t}\Big), \tag{A·7}$$

which gives the commutation relations (12), when inserted into the eq. (6). Another choice of $N=1/\sqrt{2\varkappa(\varepsilon(\boldsymbol{p})+\varkappa)}$ is also often used, so that

$$\begin{aligned} &u_\alpha^{(i)*}u_\alpha^{(j)}=(\varepsilon(\boldsymbol{p})/\varkappa)\delta_{ij}\,,\\ &\bar{u}_\alpha^{(i)}u_\alpha^{(j)}=\pm\delta_{ij}(=(\gamma_4)_{ij}). \end{aligned} \tag{A·9}$$

Appendix II.

From (39) and (43), u can be written as,

$$u(\boldsymbol{k};\boldsymbol{p};\boldsymbol{q})=\int e^{i(P-P_0)x}dx\cdot(\boldsymbol{k};\boldsymbol{p};\boldsymbol{q}|\boldsymbol{S}|\boldsymbol{k}^0;\boldsymbol{p}^0;\boldsymbol{q}^0), \tag{A·10}$$

where x stands for a particular point, for example the centre of mass, of the system. At first we confine the domain of the variable x to a four-dimensional box:

$$-\frac{1}{2}L_\mu\leqq x_\mu\leqq\frac{1}{2}L_\mu\qquad(\mu=0,\,1,\,2,\,3).$$

Then we have

$$\begin{aligned} &|u(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})|^2\Pi\,d\boldsymbol{k}\,\Pi\,d\boldsymbol{p}\,\Pi\,d\boldsymbol{q}=\\ &2^8\prod_\mu\frac{\sin^2(P_\mu-P_\mu^0)(L_\mu/2)}{P_\mu-P_\mu^0}|(\boldsymbol{k};\boldsymbol{p};\boldsymbol{q}|\boldsymbol{S}|\boldsymbol{k}^0;\boldsymbol{p}^0;\boldsymbol{q}^0)|^2\Pi\,d\boldsymbol{k}\,\Pi\,d\boldsymbol{p}\,\Pi\,d\boldsymbol{q}, \end{aligned} \tag{A·11}$$

where the repeated indices must not be summed.

Next we change the variables $(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ into $(\boldsymbol{P},P_0,\lambda)$ and define $\rho(\lambda)$ as,

$$\Pi\,d\boldsymbol{k}\,\Pi\,d\boldsymbol{p}\,\Pi\,d\boldsymbol{q}=d\boldsymbol{P}\,dP_0\,\rho(\lambda)\,\Pi\,d\lambda.$$

If we choose λ, for instance, as,

$$(\lambda_i)=(k_{2y},\,k_{2z},\,\boldsymbol{k}_3,\,\ldots\,;\boldsymbol{p};\boldsymbol{q}),$$

then

$$\rho(\lambda)=\partial(\boldsymbol{k}_1,\,k_{2x})/\partial(\boldsymbol{P},P_0).$$

The scattering probability per unit volume and per unit time is

$$w=\lim_{L\to\infty}\int|u|^2d\boldsymbol{P}\,dP_0/VT,$$

where $V=L_1L_2L_3$ and $T=L_0/c$, whence we have

$$w\rho(\lambda)\,\Pi d\lambda=(2\pi)^4 c\,|(\boldsymbol{P}^0, P_0^0, \lambda|\boldsymbol{S}|\boldsymbol{k}^0, \boldsymbol{p}^0, \boldsymbol{q}^0)|^2\rho(\lambda)\,\Pi d\lambda, \qquad (A\cdot 12)$$

which we may write as

$$w\,\Pi\, d\boldsymbol{k}\,\Pi\, d\boldsymbol{p}\,\Pi\, d\boldsymbol{q}=$$

$$c\,|(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}|\boldsymbol{S}|\boldsymbol{k}^0, \boldsymbol{p}^0, \boldsymbol{q}^0)|^2(2\pi)^4\delta(P-P^0)\,\Pi\, d\boldsymbol{k}\,\Pi\, d\boldsymbol{p}\,\Pi\, d\boldsymbol{q}. \qquad (A\cdot 13)$$

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On the Reaction of the Mesonic Proper Field

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The processes of the pion nucleon scattering and the photo production of positive pion are calculated with the pseudovector coupling in the pseudoscalar theory by using the corrected nucleon propagation function obtained from the prescription of the non singular theory proposed by Kamefuchi and Umezawa. It is shown by this calculation that, in this non singular theory, (i) the pole appears in the corrected propagation function and (ii) difficulties with respect to the gauge invariance happen. The circumstances of these problems and the methods to overcome them are discussed.

§ 1. Introduction

When the renormalization procedure, although its brilliant successes in the quantum electrodynamics, is applied to the meson theory, as known to all, whether it can give the close non singular theory or not, depends on the type of the interaction[1]. In the pseudoscalar meson theory, e. g., this procedure is completely successful for the pseudoscalar interaction (the first kind interaction) but it is not the case for the pseudovector interaction (the second kind interaction), which seems to be required from the latest many experimental results. Recently, Kamefuchi and Umezawa[2] have proposed a new prescription obtaining the close non singular theory also for the second kind interaction. The fundamental idea of their method is to take into account some part of the reaction of the proper field from the beginning, while the remaining part is considered later on. That is to say, the suitably corrected propagation functions are used for internal lines. Many possibilities, however, are permitted for interactions introduced in order to correct the propagation function. Hereupon, the purpose of this paper is as follows: in such infinite manifold of many prescriptions for getting the non singular theory, we adopt one that the propagation function S_F of nucleon is corrected by the pseudovector coupling itself in the pseudoscalar meson theory without introducing any other second kind interactions (§ 2). This method belongs to the (A) type in K. U.'s terminology. Using this cor-

rected S_F' function, we calculate then the cross sections for the pion nucleon scattering (§ 3) and for the photo positive pion production (§ 4), and we investigate general characteristic features of our one particular case of K. U. prescription in these concrete examples. The reason why we correct the nucleon propagation function alone but not the meson's, consists firstly in that at the former case, considering the physical systems which consist of merely nucleons and mesons, any divergence does not appear except renormalized terms : (proper nucleon self energy part $\sum$)−(its finite part $\sum_C$); but it is not the case for the latter. Secondly, one may expect that the former procedure would give better tendency for the anomalous magnetic moment of the nucleon (cf. § 5). In this prescription which we used here, there exist however various difficulties with regard to the zero point newly occurring in S'_F function and to the gauge invariance of theory. On details of these situations, general characteristics of our prescription independent of above difficulties and possibilities ((B_1) and (B_2) type in K. U.) of overcoming these obstacles, we shall discuss in the last section (§ 5).

§ 2. Calculations of the corrected propagation function

The corrected propagation function of nucleon, S_F', is obtained from the uncorrected one, S_F, by the following relation[3] :

$$S_F'(p) = S_F(p) + S_F(p)\sum_C^{(2)}(p)\,S_F'(p),$$

or

$$S_F'(p) = \frac{S_F(p)}{1 - S_F(p)\cdot\sum_C^{(2)}(p)}, \tag{1}$$

where $\sum_C^{(2)}(p)$ is the finite nucleon proper self energy part of the second order after performing the renormalization procedure for the divergent parts and p is the energy momentum four vector of the nucleon. Using the above defined S_F', as long as we consider only the interaction between the pseudoscalar mesons and the nucleons, we can obtain the non singular theory as verified in K. U.

Now the nucleon self energy part of the second order, $\sum^{(2)}(p)$, is given by the following expression in the symmetrical pseudoscalar theory with the pseudovector coupling :

$$\sum^{(2)}(p) = G^2\int(dk)\frac{\gamma_5 \boldsymbol{k}\,[i(\boldsymbol{p}-\boldsymbol{k})-\varkappa]\,\gamma_5\boldsymbol{k}}{[(p-k)^2+\varkappa^2]\,[k^2+\mu^2]} = G^2\cdot I, \tag{2}$$

where $G^2 = 3ig^2/(2\pi)^4\mu^2$, $\boldsymbol{k} = k_\lambda\cdot\gamma_\lambda$, g is the coupling constant of the pseudovector interaction, and $\varkappa$ and μ are the masses of the nucleon and the meson respectively. We can separate the convergent parts from this divergent integral I by the conventional method, namely,*)

*) The method of separation of finite parts from the divergent integral is not unique. But, our standpoint in this paper, as described in the introduction, is to refer to the qualitative results rather than the quantitative; hence we have calculated the divergent integral straightforwardly without any cautions to the "ambiguities" as if for the convergent integral.

$$I=C_0+C_1\Lambda+C_2\Lambda^2+C_3\Lambda^3+I_C\,;\quad \Lambda=i\not{p}+\varkappa,$$
$$I_C=-i\pi^2\int_0^1 dx\,[\,a\Lambda^3+b\Lambda^2+c\Lambda+d\,]\log\frac{L_0+x(1-x)N}{L_0}-I_r, \tag{3}$$

where all the coefficients $c_i(i=0,1,2,3)$ are the divergent constants which are renormalized by adding the counter parts into the Lagrangian, and

$$\begin{aligned} &a=-x(1-x)(1+4x), \qquad b=\varkappa x(5+6x-12x^2),\\ &c=\varkappa^2x(9x^2-x-6)+(1+3x)L_0,\\ &d=-\varkappa^3x^3-3(1+x)\varkappa L_0,\\ &L_0=x^2x^2+\mu^2(1-x), \qquad N=p^2+x^2. \end{aligned} \tag{4}$$

I_r in (3) is also the terms to be renormalized, which can be obtained by retaining only the terms of $\Lambda^n(n=1,2,3)$ when we expand the logarithmic term in I_c to the power series with respect to Λ. Thus we have

$$\Sigma_C^{(2)}(p)=G^2\cdot I_C. \tag{5}$$

Since $S_F(p)=-\Lambda^{-1}$, $S_F'(p)$ becomes as follows

$$S_F'(p)=-[\Lambda+G^2\cdot I_C]^{-1}. \tag{6}$$

As shown in the Appendix, when we rationalize the denominator with respect to γ-matrix, we can write also as

$$S_F'(p)=\frac{u(i\not{p})-v}{u^2p^2+v^2}, \tag{7}$$

where u and v are both the functions of p^2, $\varkappa$ and μ.

§ 3. Pion nucleon scattering

The cross section of the pion nucleon scattering can be easily obtained using the expression (7) of S_F' and we have the results:

$$\begin{aligned} d\sigma=&8\pi\frac{(g^2/4\pi\mu^2)^2(d\Omega/4\pi)}{(\epsilon+E)^2\,|\,u^2p^2+v^2\,|^2}\\ &\times\{2\,|\,u(2B^2-\mu^2B-\varkappa^2\mu^2)-\varkappa\mu^2v\,|^2-N^2[(k_1\cdot k_2)+\mu^2][\,|u|^2p^2+|v|^2]\}, \end{aligned} \tag{8}$$

in the center of mass system. But for the exchange scattering the cross section can not be expressed by this formula and the abbreviations in Eq. (8) must be understood as the next tables:

	initial state		final state	
	pion	nucleon	pion	nucleon
energy-momentum	k_1	p_1	k_2	p_2
energy	ϵ	E	ϵ	E

	p	B
$\pi^++P\to\pi^++P$	p_1-k_2	$-(p_1\cdot k_2)$
$\pi^-+P\to\pi^-+P$	p_1+k_1	$(p_1\cdot k_1)$

Since in the case of the scattering of negative pion (π^-) by proton (P) u and v are both angular independent as they are the functions of p^2 and so is of B, it is easy to obtain the total cross section. The angular dependent part is only the following:

$$(k_1 \cdot k_2) + \mu^2 = k^2 \cos\theta - k^2 = -k^2(1-\cos\theta).$$

This fact shows immediately that our results do not agree with the experimental ones and this will be discussed in the section 5.) The total cross section must be calculated by numerical integration in the case of $\pi^+ + P \to \pi^+ + P$. The results are shown in Figs. 1 and 2 by putting the coupling constant $g^2/4\pi = 0.18$ which is adjusted to the total cross section of the meson energy 135 Mev to the experimental results.

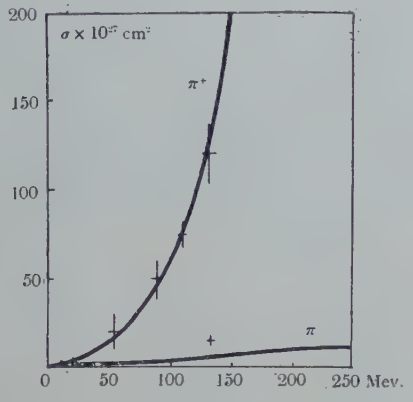

Fig. 1. Excitation functions for pion nucleon scattering. The experimental points were obtained by H. L. Anderson *et al*[4].

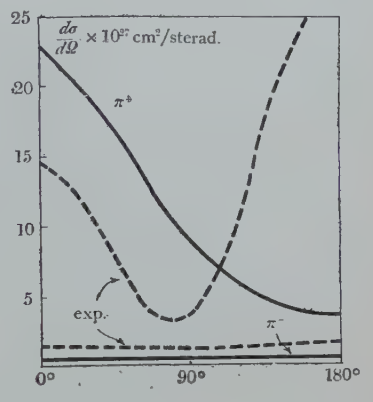

Fig. 2. Angular distributions in C. M. system at the meson energy 135 Mev.

§ 4. Photo positive pion production

When using our corrected S_F' function, we cannot get gauge invariant matrix element for the process in question by summing up Feynman graphs which are permitted to the eg-order. (Since our prescription is equivalent to take into consideration a part of contribution of higher order in lower order, indeed there remains a possibility of making it a gauge invariant form by adding suitable class of higher order graph.) Here, we carried out the calculation by taking only such parts of S_F' function that satisfy the relation for ordinary propagation function S_F[5]; *i.e.*,

$$S_F'(p+q)\gamma_\mu A_\mu(q) S_F'(p) \to S_F'(p+q)\eta_p - \eta_{p+q} S_F'(p), \qquad (9)$$

when $A_\mu \to iq_\mu$. ($\eta_P = 1$ if P is virtual and $\eta_P = 0$ if P is actual.) Then we have,

$$\frac{d\sigma}{d\Omega} = \frac{g^2}{4\pi}\frac{e^2}{4\pi}\frac{1}{\mu^2}\frac{\varkappa^2 k}{E_1 q(q+E_1)}\left[\frac{k^2\sin^2\theta}{q^2(\epsilon - k\cos\theta)^2}\left\{E_1E_2 - \varkappa^2 + \frac{1}{2}q^2(q+E_1)\epsilon U - qk\left(1+\frac{1}{2}q(q+E_1)U\right)\cos\theta\right\} + 2(q/\varkappa^2)(q+E_1)^2 V(E_1E_2 - \varkappa^2 + qk\cos\theta)\right], \qquad (10)$$

where U and V are the real part and the absolute square of $2\pi i u/(u^0 p^0 + v^0)$, respectively. E_1 and E_2 are the initial and final energies of the nucleon; ϵ and k are the energy and the momentum of a meson, severally. q is the photon momentum. These all quantities refer to the center of mass system. The magnitude of the coupling constant $g^2/4\pi = 0.93$ is adjusted to the recent experimental data[6] (at 90°, photon energy 255 Mev.).

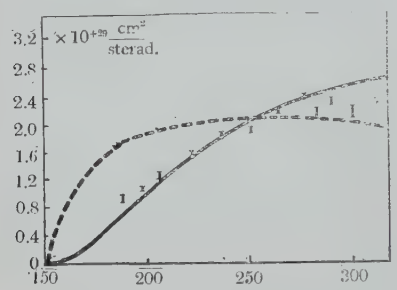

Fig. 3. The excitation function of photo positive pion production. The solid line corresponds to our result and the broken line to the ordinary perturbation result in eg-order. The experimental points were obtained by J. Steinberger and A. S. Bishop[6]. The new curve shows somewhat better agreement with the experiment.

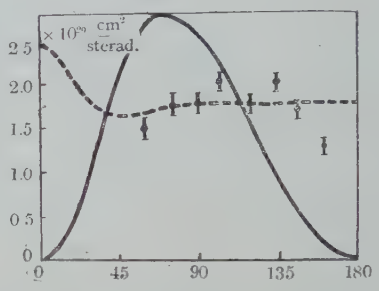

Fig. 4. The angular distribution in the C. M. system. Our result lacks the flatness of curve compared with the ordinary perturbation result.

§ 5. Discussions

(i) *Excitation functions for the pion nucleon scattering and the pole which has appeared in* S_F'

As seen in Fig. 1, relation $\sigma(\pi^+ + P) > \sigma(\pi^- + P)$, exists concerning to the total cross section. This is the relation which was not obtained lowest approximation of the perturbation theory and results from the pole, discussed below, of S_F'. But quantitatively, in the present case,

$$\sigma(\pi^+ + P)/\sigma(\pi^- + P) \simeq 20 \qquad \text{(at 135 Mev)}.$$

which does not agree with experimental result[4].

A very remarkable character is that the zero point appears in the denominator of $\sigma(\pi^+ + P)$. We do not find yet any suitable method to solve this difficulty which was presumed from the beginning by form of Eq. (1)[7]. But this may be considered as the "infrared catastrophe-like"[8] phenomenon since this is a divergence at the intermediate state. In such a case, this difficulty might be solved by considering graphs of higher order, but we did not try this prescription since it was practically impossible for us to do it with S_F' of (7).

This fact might also inherit the incompleteness of method of the higher order correction assumed in this formalism as will be seen in the discussion of the gauge invariance (cf. (iii)). Because the above mentioned appearance of the pole only for the case of

$\sigma(\pi^+ + P)$ and not for $\sigma(\pi^- + P)$ is due to the fact that the energy conservation can be satisfied in intermediate states for the former but not for the latter when the calculation is carried out by using our S_F'. From this circumstance the damping effect exists for $\sigma(\pi^- + P)$, which depresses the resonance to be finite. This corresponds to the fact that integral I_c has the $i\pi\delta_N$-term. (See Appendix). Hence, if we could correct any "vertex part" we might avoid this difficulty by introducing also the damping effect for $\sigma(\pi^+ + P)$. This problem, however, is essentially the "two body problem" which is not solved yet. From this point of view, we did not calculate processes; the charge exchange scattering and photoproduction of negative and neutral pion, in which can not be avoided the effect of the pole. (Moreover, there exist difficult problems with respect to correct definition dealing with the poles of S_F'. This will be discussed in K. U..)

(ii) *Angular distributions for pion nucleon scattering*

The angular distributions are foward-up for the pion nucleon scatterings, as seen in Fig. 2. It can be understood from numerical calculations that this fact is due to pole. There may be possibility of backward scattering by adjusting position of the pole appropriately (of course, if it could be only assumed that the problem of (i) had solved). It must be noticed here, however, that the factor $(1+3\cos^2\theta)$ obtained in the strong coupling theory inherits the spin isobar states, and so, even if we could change suitably the position of the pole, it should be still infered to be unable to obtain the angular distribution of the $\cos^2\theta$-type. This fact does not agree with the experimental results, but this defect may be avoided by adopting a method in which one uses, for instance, a complementary field of spin 3/2 discussed in K. U. (B_2-type method named by them) or corrects the vertex part as described in (i).

(iii) *The gauge invariance of the theory*

The other obstacle for our prescrition concerns to the gauge invariance of the theory. The renormalization procedure for divergencies $\sum - \sum_0$, requires additional counter terms $\sum_{i=0}^{3} \lambda_i \bar{\psi}(\gamma_\mu \partial_\mu + \varkappa)^i \psi$ to the Lagrangian. In the case of taking the interaction with the electromagnetic field into consideration, we must introduce new electromagnetic interactions $\lambda_1(\bar{\psi}\partial_\mu\psi - \partial_\mu\bar{\psi}\psi)A_\mu$, $\lambda_2\partial_\mu\bar{\psi}\partial_\nu\psi A_\mu A_\nu$, $\cdots$ etc. which leads to divergent results again because of their infinite coefficients λ_i. This situation is almost always accompanied by the introduction of the second kind interaction with higher derivatives for the correction of propagation function. In spite of adopting the gauge invariant Lagrangian, moreover, when using the corrected propagation function, we cannot get gauge invariant matrix elements merely by summing up Feynman-Dyson graphs for process in question which are allowed to certain fixed order in the coupling constant. It is the reason for above fact that the relation (9) is not satisfied for our S_F' because of its complicated momentum dependence, with which finite results were warranted and from which the difficulty (i) with respect to the newly appeared zero points of S_F' happened.

(iv) *Diminishing of the contribution from the nucleonic current*

In our prescription we corrected only the propagation function of the nucleon different from Hu's one[9]. Accordingly it may be expected the reduction of contributions from

the nucleonic current and the increase of effects of the mesonic current. This aspect of our method, we can see most clearly in results for the photo production of positive pion. Although the separation of gauge invariant term on the basis of Eq. (9) is, of course, not unique and this convention takes out only one part of gauge invariant terms and does not exhaust the all, this circumstances would not alter essentially above tendency. Indeed, as results of remarkable diminishing of the contribution from the nucleonic current, the angular distribution of Fig. 4 was wanting in the flatness such as ordinary perturbation result and the necessity happened that the coupling constant $g^2/4\pi$ must be set equal to a fairly large value 0.93 compared with that for the case of pion nucleon scattering.

(v) *Another possibilities*

From aforesaid characteristics it might be expected that gauge invariant prescription with similar nature as ours would be promissing for, e. g., the anomalous magnetic moment of the nucleon. Hence the investigation of the (B_1) and (B_2) type of K. U. prescription and the method of correction by using the vertex part are desirable, though for the latter we do not know how to dispose of.

Acknowledgements

We should like to express our cordial thanks to Prof. S. Sakata, Dr. H. Umezawa, Merrs. S. Kamefuchi and J. Fujimura for their kind interests and valuable discussion to this work. Especially, Merrs. H. U. and S. K. showed kindly to us their manuscript before publication and suggested the present problems.

Appendix

Derivation of eq. (7).

Since I_r is the terms of $\Lambda^n (n=0, 1, 2, 3)$ in expanding the logarithmic term in I_c to Taylor series with respect to Λ, using the relation $N=\Lambda(2x-\Lambda)$, we can easily obtain

$$I_r=-i\pi^2\left[c_0'+c_1'\Lambda+c_2'\Lambda^2+c_3'\Lambda^3\right],$$

where

$$c_0'=0, \qquad c_1'=\int_0^1 dx\cdot 2xd\cdot a,$$

$$c_2'=\int_0^1 dx a(2xc-d-2x^2d\cdot a),$$

$$c_3'=\int_0^1 dx a(2xb-c-2x(xc-d)a+8x^2d\cdot a^2/3),$$

$$a=x(1-x)/L_0.$$

These c_i' are all the numerical constants and can be calculated by elementary integration, thus we have

$$I_c=-i\pi^2\left\{\int_0^1 dx\left[a\Lambda^3+b\Lambda^2+c\Lambda+d\right]\log\frac{L_0+x(1-x)N}{L_0}-\left[c_1'\Lambda+c_2'\Lambda^2+c_3'\Lambda^3\right]\right\}.$$

Using the identities,

$$A^2=2x(i\boldsymbol{p})+(x^2-p^2),\quad A^3=(3x^2-p^2)(i\boldsymbol{p})+x(k^2-3p^2),$$

this can be written as

$$I_c=-i\pi^2[Q(i\boldsymbol{p})+R],$$

where

$$Q=Q_l-Q_r,\qquad R=R_l-R_r,$$

$$Q_l=\int_0^1 dx[(3x^2-p^2)a+2xb+c]\log\frac{L_0+x(1-x)N}{L_0},$$

$$R_l=\int_0^1 dx[x(x^2-3p^2)a+(x^2-p^2)b+xc+d]\log\frac{L_0+x(1-x)N}{L_0},$$

$$Q_r=(3x^2-p^2)c_3'+2xc_2'+c_1',$$

$$R_r=x(x^2-3p^2)c_3'+(x^2-p^2)c_2'+xc_1'.$$

It is necessary only to calculate Q_l and R_l, because Q_r and R_r are easily obtained from c_i which are already calculated. If we define integrals;

$$I_n=\int_0^1 dx\cdot x^n\log\frac{L_0+x(1-x)N}{L_0},\qquad (n=0,1,2,3),$$

Q_l and R_l are transformed into as follows:

$$Q_l=\mu^2I_0+(N+2\mu^2)I_1+3(N-\mu^2)I_2-4p^2I_3,$$
$$R_l=-x[2\mu^2I_0+2(N-\mu^2)I_1-3p^2I_2].$$

By integration by parts

$$I_n=[I_n^{(0)}-I_n^{(1)}]/(n+1),$$

where

$$I_n^{(0)}=\int_0^1 dx\cdot x^{n+1}\left(\frac{dL_0}{dx}\right)/L_0$$

$$I_n^{(1)}=\int_0^1 dx\cdot x^{n+1}\frac{2x^2x-\mu^2+(1-2x)N}{L_0+x(1-x)N},$$

in which I_n is also calculated by elementary integration. Thus I_n remains only. Writing the two roots of equation,

$$f(x)=L_0+x(1-x)N=(x^2-N)x^2+(N-\mu^2)x+\mu^2=0,$$

as x_1 and x_2 (say, $x_1>x_2$), this integral becomes as follows:

$$I_n^{(1)}=x_1^{n+1}\left[\log\left|\frac{1-x_1}{x_1}\right|+\sum_{l=1}^{n+1}\frac{{}_{n+1}C_l}{l}\left\{\left(\frac{1}{x_1}-1\right)^l-(-1)^l\right\}\right]$$

$$+x_2^{n+1}\left[\log\left|\frac{1-x_2}{x_2}\right|+\sum_{l=1}^{n+1}{}_{n+1}C_l\left\{\left(\frac{1}{x_2}-1\right)^l-(-1)^l\right\}\right]$$

$$+i\pi\delta_N\left[x_1^{n+1}-x_2^{n+1}\right],$$

where

$$\delta_N=\begin{cases}1 & \text{for } N=0,\\ 0 & N>0.\end{cases}$$

Since we have thus obtained Q_l and R_l, and so I_c, using abbreviation,

$$f^2=-i\pi^2G^2=\frac{1}{4\pi}\left(\frac{3g^2}{4\pi\mu^2}\right),$$

S_F' becomes finally from equation (6) as follows:

$$S_F'=\frac{-1}{i\not{p}+\varkappa+f^2\left[Q(i\not{p})+R\right]}=\frac{-1}{\left[1+f^2Q\right](i\not{p})+\left[\varkappa+f^2R\right]}=\frac{-1}{u(i\not{p})+v},$$

which is the equation (7), and

$$\left.\begin{aligned}u&=1+f^2Q,\\ v&=\varkappa+f^2R.\end{aligned}\right\}$$

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Letters to the Editor

On the "Optical Method" for the Scattering of High Energy Particles by Complex Nuclei

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March 3, 1953

The relation between the optical method[1] and the quantum mechanical many body problem for the scattering of high energy particles by complex nuclei is investigated.

The total Hamiltonian of our problem is

$$H=H_0+V, \tag{1}$$

with

$$H_0=H_0{}^I+H_0{}^{II}, \tag{2}$$

and

$$H_0{}^{II}=K+U. \tag{2}'$$

$H_0{}^I$ is the kinetic energy operator of the incident particle, K the total kinetic energy operator of nuclear constituents, U the nuclear potential energy, and V is the total interaction potential between the incident particle and the nucleons. We shall suppose V to be of the form

$$V=\sum_{k=1}^{A} V_k, \tag{3}$$

where V_k is the interaction potential between the incident particle and the kth nucleon. We start from the exact solution of our problem

$$\Psi(\boldsymbol{r};\boldsymbol{r}_1\cdots\boldsymbol{r}_A)=\varphi(\boldsymbol{r})\,\Phi_i(\boldsymbol{r}_1\cdots\boldsymbol{r}_A)$$
$$+(E+i\eta-H_0)^{-1}T^{(+)}\varphi(\boldsymbol{r})\,\Phi_i(\boldsymbol{r}_1\cdots\boldsymbol{r}_A), \tag{4}$$

where $\varphi(\boldsymbol{r})$ is the eigenfunction of $H_0{}^I$ belonging to E_l, the energy of the incident particle, and $\Phi_i(\boldsymbol{r}_1\cdots\boldsymbol{r}_A)$ is the eigenfunction of $H_0{}^{II}$ belonging to E_i, the initial nuclear energy. There is the relation:

$$E=E_l+E_i.$$

Using the relation

$$(E+i\eta-H_0-V)^{-1}V_k=(\omega_k{}^{(+)}-1)$$
$$+(E+i\eta-H_0-V)^{-1}\{[U,\omega_k{}^{(+)}]$$
$$+(V-V_k)(\omega_k{}^{(+)}-1)\}, \tag{5}$$

the transition matrix $T^{(+)}$ is represented, according to Chew Goldberger[2], as follows:

$$T^{(+)}=\sum_{k=1}^{A}\{t_k{}^{(+)}+V^{\nu}(E+i\eta-H_0-I^{\nu})^{-1}[U,\omega_k{}^{(+)}]$$
$$+[1+I^{\nu}(E+i\eta-H_0-I^{\nu})^{-1}](V-I^{\nu}{}_k)(\omega_k{}^{(+)}-1)\}, \tag{6}$$

where

$$\omega_k{}^{(+)}=1+[E_l'+i\eta-(H_0{}^I+K)-V_k]^{-1}V_k,$$

$$t_k{}^{(+)}=V_k\omega_k{}^{(+)},$$

and E_l' is the eigenvalue of $H_0{}^I+K$.

Assuming the impulse approximation:

$$[U,\omega_k{}^{(+)}]=0^{2)}, \tag{I}$$

we obtain

$$(E+i\eta-H_0-V)^{-1}V_k=(\omega_k{}^{(+)}-1)$$
$$+(E+i\eta-H_0-V)^{-1}(V-V_k)(\omega_k{}^{(+)}-1), \tag{5}'$$

and by applying (5)′ successively (6) becomes as follows:

$$T^{(+)}=\sum_{k=1}^{A}t_k{}^{(+)}+\sum_{k=1}^{A}\sum_{k'\neq k}t_k{}^{(+)}(\omega_{k'}{}^{(+)}-1)$$
$$+\sum_{k=1}^{A}\sum_{k'\neq k}\sum_{k''\neq k'}t_k{}^{(+)}(\omega_{k'}{}^{(+)}-1)(\omega_{k''}{}^{(+)}-1)+\cdots \tag{6}'$$

In order to obtain the correspondence of the above theory to the optical method, it is necessary to introduce the following approximation:

$$[K,V_k]=0. \tag{II}$$

With these approximations, (I) and (II), (4) is expressed by

$$\Psi(\boldsymbol{r};\boldsymbol{r}_1\cdots\boldsymbol{r}_A)=\psi(\boldsymbol{r};\boldsymbol{r}_1\cdots\boldsymbol{r}_A)\,\Phi_i(\boldsymbol{r}_1\cdots\boldsymbol{r}_A)$$
$$=[\varphi(\boldsymbol{r})+\sum_k(E_l+i\eta-H_0{}^I)^{-1}t'_k{}^{(+)}(\boldsymbol{r}_k)\psi^k(\boldsymbol{r})]$$
$$\times\Phi_i(\boldsymbol{r}_1\cdots\boldsymbol{r}_A), \tag{4}'$$

where

$$\psi^k(\boldsymbol{r})=\varphi(\boldsymbol{r})+\sum_{k'\neq k}(E_l+i\eta-H_0{}^I)^{-1}$$
$$\times t'_{k'}{}^{(+)}(\boldsymbol{r}_{k'})\psi^{k'}(\boldsymbol{r}),$$

and

$$t'_k{}^{(+)} = V_k \omega_k{}^{(+)}$$
$$= V_k[1+(E_l+i\eta-H_0{}^I-V_k)^{-1}V_k].$$

Integral eq. (4)′ is the basic equation of the self-consistent procedure given by Lax[3] for the multiple scattering problem, and is the starting point of the optical method. $(E_l+i\eta-H_0{}^I)^{-1}t'_k{}^{(+)}\psi^k(\boldsymbol{r})$ represents the wave scattered by the kth nucleon, and $\psi^k(\boldsymbol{r})$ the wave incident on the scatterer k.

The particle wave which has been used in the optical method is the absolute coherent wave, which can be represented by:

$$\langle\psi(\boldsymbol{r})\rangle \equiv \int \Phi_i{}^*(\boldsymbol{r}_1 \ldots \boldsymbol{r}_A)\psi(\boldsymbol{r};\boldsymbol{r}_1 \ldots \boldsymbol{r}_A) \times \Phi_i(\boldsymbol{r}_1 \ldots \boldsymbol{r}_A)d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_A. \qquad (7)$$

It can be shown[3] by solving the eq. (4′) with some approximations that the coherent wave obeys the wave equation

$$[\nabla^2+k^2+4\pi ncf(0)]\langle\psi(\boldsymbol{r})\rangle = 0,$$

$$f(0) \equiv \frac{Z}{A}f_p(0)+\frac{(A-Z)}{A}f_n(0),$$

where n represents the average density of nucleons, $f_p(0)$ and $f_n(0)$ are the forward scattering amplitudes by the single proton and neutron, and c is a constant depending on the correlation between pairs of nucleons, and further that the effect of the incoherent scattering, the wave density of which is represented by $\langle|\psi|^2\rangle-|\langle\psi\rangle|^2$, gives the nuclear medium an turbidity, i.e., the absorption coefficient $\boldsymbol{K}$ of the coherent wave. Thus the absolute coherent wave travels in the nuclear medium with the index of refraction $(Re\,k')/k$, where $(k')^2=k^2+4\pi ncf(0)$, and the absorption coefficient $\boldsymbol{K}$, which agrees with that given by Fernbach, Serber and Taylor[1] provided that the nuclear system can be described in Hartree approximation.

We have seen that the many body problem for the scattering of the high energy particles has been reduced to the optical method by using the two main approximations; (I) the impulse approximation, and (II) which may mean that the change of the nuclear state is neglected.

The details and the discussion of the validity of these approximations, i.e., the applicability of the optical method will be given in the later issue of this journal. The writer wishes to express his thanks to Mr. E. Yamada and Dr. O. Hara for their valuable discussions.

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Electromagnetic Shifts of Energy Levels and C-meson Hypothesis

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March 10, 1953

Since the first discovery of $2S_{\frac{1}{2}}-2P_{\frac{1}{2}}$ level shift by Lamb and Retherford[1], and its theoretical explanation by Bethe[2], both the experimental and theoretical determinations have been improved considerably. The most recent values are[3),4)]

Experimental;

$$S_H = (1057.77\pm0.16)\ \text{Mc/sec.}$$
$$S_D = (1059.00\pm0.10)\ \text{Mc/sec.} \qquad (1)$$

Theoretical;

$$S_H = (1057.19\pm0.16)\ \text{Mc/sec.}$$
$$S_D = (1058.49\pm0.16)\ \text{Mc/sec.} \qquad (2)$$

Although the agreements seem rather striking, the remaining discrepancies are well outside the estimated error (the experimental values are about 0.5 Mc/sec. larger than the theoretical ones in the case of hydrogen as well as in the case of deuterium), and the explanation has not been given within the framework of the current quantum electrodynamics.

In this connection it seems of some interest to examine the effect produced by the C-meson field, which was proposed by Pais[5], Sakata, and the present author[6] to solve the divergence difficulties of the self-energy problem of charged particles.

The C-meson field produces a short range repulsive potential

$$2e^2\frac{e^{-\mu r}}{r} \qquad (3)$$

between the proton and the electron[7], where μ is the reciprocal of the Compton wave length of the C-meson. The level-shift due to (3) can be calculated easily by taking (3) as a small perturbation. Extra contributions to (2) is thus obtained as

$$\Delta S_H = \Delta S_D = 2\alpha^2(\lambda/\mu)^2 Ry$$

$$\simeq 3.2\times 10^5(\lambda/\mu)^2 \text{ Mc/sec.}$$

where α, λ, and Ry denotes the fine-structure constant, reciprocal of the Compton wave length of the electron, and the Rydberg frequency, respectively. Therefore, by taking

$$(\lambda/\mu) \simeq 10^{-3},$$

we can fill the gap between (1) and (2). This value of (λ/μ) corresponds to the C-meson mass of about $1000\times$ (electron mass). It is not in contradiction to previous result of Ogawa[8] who, from the analysis of cosmic ray data concluded that the mass of the C-meson must be larger than about $500\times$ (electron mass).

Although the above estimation is very rough, we may expect that further experiments would confirm the existence of the C-meson.

The author would like to express his sincere gratitude to Messrs. S. Ogawa, S. Kamefuchi and H. Okonogi for their helpful discussions.

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On the χ-meson

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March 25, 1953

Recently O'Ceallaigh[1] has found in the cosmic ray a new heavy meson (χ-meson), which seems to disintegrate into one charged μ-meson and one or more neutral particles. He also has found the lifetime of a new particle is about 10^{-9} sec. As the experimental facts seem to indicate us that χ-meson disintegrates at least into three secondary particles, we assume tentatively a following decay mode for the χ-meson.

$$\chi^{\pm}\to\mu^{\pm}+N+\bar{N}\to\mu^{\pm}+\pi^0+N'+\bar{N}'\to\mu^{\pm}+\pi^0+\pi^0,$$

where N, N', and $\bar{N}$ mean the nucleons and anti-nucleon respectively. Though we shall calculate the lifetime of χ-meson according to the above assumption by the covariant formalism, here we report the results of the preliminary work, obtained by the more simple method[2], in order to estimate the order of magnitude of the disintegration rate.

The momentum integral of nucleons in the intermediate states was cut off at (Mc). After the usual computation we has obtained the results as follows;

$$\sigma\sim G^2(e_2^2/\hbar c)^2(mc/\hbar)^4 12^{-2}2^{-1}\pi^{-7}m^{-4}\hbar^{-3}M^6(\mu_1/\mu)^{3/2}$$
$$\times(\mu_\chi-\mu_1-2\mu)^2/\{(\mu_\chi-2M-\mu_1)(\mu_\chi-2M-\mu_1-\mu)\}^2$$
$$\times[(2\mu\mu_1)^{1/2}/(2\mu+\mu_1)\{2^{-1}-\mu_1(2\mu)^{1/2}/(2\mu+\mu_1)^{3/2}\}$$
$$+2^{-1}\sin^{-1}\{(2\mu)/(2\mu+\mu_1)\}^{1/2}]$$

where, G is the coupling constant of χ-N-μ, e_2 the coupling constant of N-π; and m, M, μ, μ_1, and μ_χ mean the mass of electron, nucleon, π-meson, μ-meson, and χ-meson respectively.

Using the above equation, experimental value of the life of natural decay of χ-meson, $\tau(=1/\sigma)\sim 10^{-9}$ sec., and the following values, $\mu_\chi\sim 1300$m, $e_2^2/\hbar c\sim 3$, $\mu\simeq 280$m, and $\mu_1\simeq 210$m, we may estimate the value of the coupling constant G as follows;

$$G\sim 15\times 10^{-49}\text{ erg cm}^3.$$

It is interesting fact that the value of G has the same order of magnitude as the coupling constants

for the β-, μ-, and forced μ-decay, which belong to the same types of decay mode as the present one for χ-meson.

The writer wishes to express his gratitude to Mr. T. Inoue for his stimulating discussions.

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Symmetric Fission

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March 1, 1953

This is a preliminary report on the theory and calculation of fission. Many theories,[1)-15)] being based upon the conception of the uniformly distributed charge of droplet, has been proposed since the discovery of fission. Especially, the celebrated paper of Bohr and Wheeler[1)]* has been regarded as a complete one on the subject, and quoted and referred on many books[16)-18)] and papers without any remark. But as it seems that their paper made an error in calculation, the writers will criticize it in the following, shutting their eyes for the moment to the farther problems, for example, of the relation between incident energy and mode of fission, incident particle and threshold, or and cross section etc.

According to B—W, for small deformation the surface energy plus the electro-static energy of the drop has increased to a value:

$$\begin{aligned}\Delta E_{S+E}=4\pi r_0^2 OA^{2/3}&[1+2a_2^2/5+5a_3^2/7+\cdots\cdots\\&+(n-1)(n+2)a_n^2/2(2n+1)+\cdots\cdots]\\&+3(Ze)^2/5r_0A^{1/3}[1-a_2^2/5-10a_3^2/49-\cdots\cdots\\&-5(n-1)a_n^2/(2n+1)^2-\cdots\cdots].\end{aligned}$$

(B—W, 9)

This is well. "A more accurate calculation to the fourth order in a_2" obtained by B—W, is

$$\begin{aligned}\Delta E_{S+E}=4\pi r_0^2 OA^{2/3}&[2a_2^2/5+116a_2^3/105\\&+101a_2^4/35+2a_2^2a_4/35+a_4^2]\\&-3(Ze)^2/5r_0A^{1/3}[a_2^2/5+64a_2^3/105\\&+58a_2^4/35+8a_2^2a_4/35+5a_4^2/27].\end{aligned}$$

(B—W, 22)

Against (B—W, 22) the writers[19)] obtained a expression

$$\begin{aligned}\Delta E_{S+E}=4\pi r_0^2 OA^{2/3}&[\sum_{n=2}(n-1)(n+2)a_n^2/2(2n+1)\\&-4a_2^3/105-8a_2a_3^2/105-4a_2^2a_4/35-4a_3^2a_4/77\\&-40a_2a_4^2/693+(\sum_{n=2}a_n^2/(2n+1))^2\\&-(1/16)\int_{-1}^{+1}\sum_{n=2}a_n^4(dP_n/d\mu)^4(1-\mu^2)^2d\mu]\\&-3(Ze)^2/5r_0A^{1/3}[\sum_{n=2}5(n-1)a_n^2(2n+1)^2\\&+4a_2^3/105+6a_2^2a_4/35+92a_3^2a_2/735\\&+60a_3^2a_4/539+80a_4^2a_2/693\\&-(5/2)(\sum_{n=2}a_n^2/(2n+1))^2\\&+(45/2)\sum_n\sum_m a_n^2a_m^2/(2n+1)^2(2m+1)\\&-(5/4)\sum_{n=2}(n^2+n+1)/(2n+1)\\&\times\int_{-1}^{+1}a_nP_n(\sum_{l=2}a_lP_l)^3d\mu+(5/8)\\&\times\int_{-1}^{+1}(\sum_{l=2}a_lP_l)^4d\mu+(15/32)[\sum_{n=0}(n+2)(n-1)\\&\times\{\int_{-1}^{+1}P_n(\sum_{l=2}a_lP_l)^2d\mu\}^2]],\end{aligned}\qquad(1)$$

or neglecting dissimilar terms from B—W,

$$\begin{aligned}\Delta E_{S+E}=4\pi r_0^2 OA^{2/3}&[2a_2^2/5-4a_2^3/105\\&-38a_2^4/175+a_3^2]-3(Ze)^2/5r_0A^{1/3}[a_2^2/5\\&+4a_2^3/105-157a_2^4/1225+6a_2^2a_4/35\\&+5a_4^2/27].\end{aligned}\qquad(2)$$

Clearly the coefficients of terms higher than third power in a of (1) or (2) different from those of B—W, but coincides with the result of Present and Knipp,[11)] though the latter gives the coefficients of terms with decimals and lacks some terms.

Another defect of B—W, it seems, is the result of (B—W, 23). (B—W, 22) leads to (B—W, 23), only if the coefficient of $a_2^2a_4$ in the bracket of the Coulomb terms is $-8/35$, instead of $8/35$. Present

* This paper of Bohr-Wheeler will be quoted as B-W in the following.

and Knipp[11] has pointed out on this point, though it seems that they did not found the origin of the fault.

Whether " (B—W, 22) is correct or not, the writers feel some incompleteness in the manner of derivation of (B—W, 23), which is derived for $x=1$, where x stands for $(Z^2/A)/(Z^2/A)_{\lim}$. Because of the fission threshold E_f calculated from such expression as (B—W, 22) or (2) is only of accuracy to the number of two or three significant figures, the writers propose the relation

$$a_4=\frac{1+3x}{27-10x}\,\frac{54}{35}\,a_2^2, \qquad (3)$$

in place of (B—W, 23).

For such heavy nuclei as uranium, the sign of a_2 in the nieder saddle point is positive and that of a_4 is also positive as predicted from (3). Thus a tradition that " a_4 must also develop a concavity about its equatorial belt such as to lead continuously with variation of the nuclear charge to the dumb-bell shaped figure " is no more correct.

If one takes 5.8 Mev for the value of E_f of U^{239}, which is given by the experiment of Shoupp and Hill[21] plus the theory of Seaborg,[20] 1.5 $A^{1/3}$ 10^{-13} cm for the nuclear radius, surface tension coefficient becomes $4\pi r_0^2 O=14$ Mev. Using these constants, the expression (2) leads to the values of Table I, which show not so different values from those of B—W.[2]

Table I

Compound Nucleus	x	a_2	a_4	our E_f (Mev)	B—W[2] E_f (Mev)	B. E[20]	Exp.
Th^{233}	0.717	0.516	0.0653	6.30	6.9	4.8	5.9[21,23]
Pa^{233}	0.733	0.495	0.0616	5.41		6.7	
U^{235}	0.740	0.485	0.0596	5.10	5.0	5.4	
U^{236}	0.737	0.491	0.0608	5.28	5.4	6.5	
U^{239}	0.727	0.503	0.0630	5.82	5.9	4.8	5.8[21] 5.1[22]

The last column : Experimental value+B.E.[20]

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2) N. Bohr and J. A. Wheeler, Phys. Rev. **56** (1939), 1065.
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Errata

Principle of Detailed Balance in Quantum Field Theory, K. Goto, Prog. Theor. Phys. **8** (1952), 565. The adjoint wave functions of spinor fields are misprinted in some places. They must be written as $\tilde{\psi}$ always. Moreover we may be allowed to add, on this occasion, the explicit definition of them: $\tilde{\psi}$ should be defined as a contravariant spinor (of the same kind(*) as ψ) which is equal to $\bar{\psi}^*\gamma_4$ ($\bar{\psi}^*$: Hermit conjugate of ψ) in our coordinates and transforms as

$$\tilde{\psi}' = \varepsilon\bar{\psi}'^*\gamma_4 = \psi S^{-1} \qquad (\psi' = S\psi)$$

where ε is $+1$ or -1 for orthochronous or antichronous Lorentz transformations respectively. (*) $\bar{\psi}^*\gamma_4$ is a spinor of the third kind if ψ is a spinor of the first kind.)

On the Meson Wave Equation in de Sitter Space, K. Goto, Prog. Theor. Phys. **8** (1952), 672. There are misprints in the sentence lying in the lines from 24th to 27th of the right half of p. 674. They should be rectified as follows "Thus $S^{\lambda\mu}$ is a special representation of $I^{\lambda\mu}$, and we can say that if irreducible representations of β^λ were seeked for such ones will be obtained, in the same way as in the investigation of Kemmer[6], that give the matrices of (12, a) or (12, b) as the form of $\beta^\lambda\beta^\mu - \beta^\mu\beta^\lambda$. This note gives a concrete form to....."

On the Group of Transformations in Six-dimensional Space, Y. Murai, Prog. Theor. Phys. **9** (1953), 147. Figures 1, 2 and 3 (p. 163) should be read as follows:

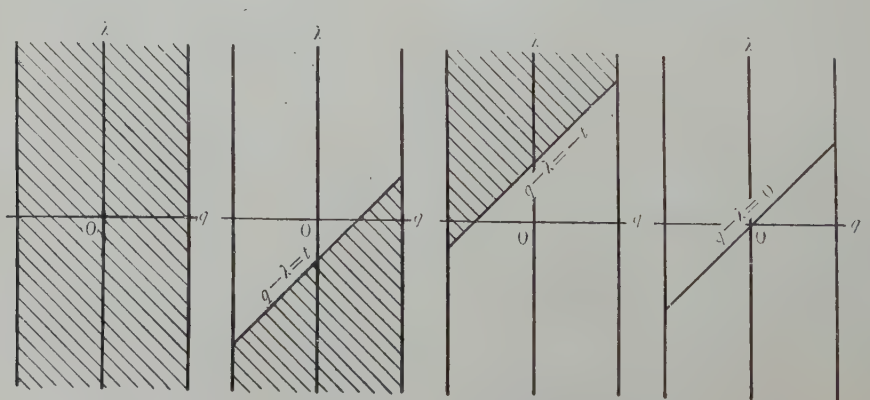

Fig. 1 Fig. 2a Fig. 2b Fig. 3

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On Some Properties of the Interactions between Elementary Particles*

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To secure the sufficient stability of matter, the *Law of Conservation of Heavy Particles* is postulated. From this postulate, a sort of intrinsic charge called nucleonic charge is assigned to every elementary particle uniquely (§ 2). In analogy to electric charge, our law can be stated as *the conservation law of nucleonic charge* and the properties of this charge are investigated (§ 3). Availing of this charge and recently obtained knowledge about heavy mesons, the possible existence of families of elementary particles is presented. Discussions about the interactions between families which seem to have primary importance are made in § 4. Bearing the *conservation law of nucleonic charge* in mind, the production mechanism of heavy mesons is analysed, and the necessity of the ordering of the interactions between families is stressed (§ 5). In appendix I, some speculations about heavy mesons and in appendix II, the threshold energies for productions are discussed.

§ 1. Introduction and summary

From the discoveries by Rochester and Butler[1], a sizable number of heavy mesons[2] have been reported. While definite statements do require more experimental elucidation, the following established facts seem to be full of suggestion from the theoretical point of view.

(a) While their productions seem to be copious, they have rather long lifetimes $\lesssim 10^{-10}$ sec.

(b) They may be classified as follows by their decay schemes,

(1) V^0 which transforms into proton.

(2) Heavy π-mesons which seem to decay ultimately into π-mesons or posssibly γ-rays without passing through μ-mesons. We may have $\zeta^{\pm}$, υ^0, $\tau^{\pm}$ and $\chi^{\pm}$ in this group at present.

(3) Heavy mesons which decay into μ-mesons directly. The decay modes of $\varkappa$-mesons are reported to be $\varkappa \to \mu + 2$ neutral particles.

Confronting with such a many megalomorphs, the first problem will be whether they are composite particles or elementary particles. As discussed by many authors, the point (a) seems to make it impossible to treat V^0 as strong coupling isobar in the usual nucleon-π-meson interaction, because it will be difficult to stabilize V^0 against π-decay or γ-decay into lower states. The attempts to treat heavy mesons as composite particles[3]

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* Read at the Kyoto Meeting of the Physical Society of Japan, October 31, 1952. A preliminary account of the present paper was given in Prog. Theor. Phys. 8 (1952), 255, 568.

will be very interesting from the standpoint of uniformity* and simplicity, though our poor knowledge about bound states does prevent clear conclusions. Here, however, as was proposed recently by Pais**[4], we may take an another viewpoint that the point (b) will be interpreted to indicate the possible existence of *families* of elementary particles. Indeed the appearance of such unstable particles makes us imagine there are some sorts of regularities among the mass spectra and the constitutions of the elementary particles.

If we prefer to such standpoint, we must look for the substance which discriminates these families and for the structures of the interactions between them. In setting up elementary interactions, we should, first of all, guarantee the stability of matter. So the *law of conservation of heavy particles* is postulated. From this law every elementary particle has a sort of unique intrinsic property called "*nucleonic charge*" (§ 2) and our law becomes very analogous to the conservation law of electric charge in any reaction. The nature of nucleonic charge is discussed in § 3. Every elementary particle can be characterized by its nucleonic charge. We call family with unit nucleonic charge as nucleon family and the members of it are heavier than proton and must always have complex amplitude even when they have no electric charge. The family with no nucleonic charge is divided into two families, that is, π-meson family and electron family phenomenologically. These families may be described by real fields when they have no electromagnetic properties. Our present knowledge about elementary particles seems to suggest unique statistics to each family. In assigning elementary particles to each family, photon seems to play a particular behaviour. Namely, charged particles and neutral particles in each family are distinguished by electromagnetic interactions. Recently Nambu[5] presented a proposition that the mass spectra of elementary particles are realized in unit of 137 $m(e^2/\hbar c = 1/137)$, where m is the electron mass. This proposition seems interesting, because all our means to specify elementary particles are electromagnetic and this proposition implies close connections between mass spectra and charge which allows its specification. In § 4, some remarks are also made about the simple interactions which seem to exist between families of elementary particles and to play a role of primary importance. Generalized Furry's theorem is also discussed from the different standpoint. In § 5, keeping the conservation law of heavy particles in mind the possible production schemes of heavy mesons are analysed. The strong interactions for copious productions are liable to cause unwanted processes and we may expect to learn the structures of the possible couplings between nucleon and π-meson families. It is pointed out that promising production schemes are apt to be *pair-wise* producton of newly found heavy mesons. So if these *pair-wise* productions are not checked, recourse to the models of unobservable parents of observed heavy mesons will be needed. In these points, the possible existence of more members of nucleon family is suggested. In appendices the possible assignments of spin, parity etc. to the π-meson groups are speculated, and the threshold energies for their productions are discussed briefly.

* Heisenberg proposed a model of unified field theory. W. Heisenberg, ZS, f. Naturf. **5a** (1950), 251, 367.

** The author wishes to express his thanks to Dr. A. Pais for sending his manuscript before publication.

§ 2. The law of conservation of heavy particles*[6]

Nambu et al. and the present author[7] proposed a theory which treats the V^0 as a heavy nucleon and extensively discussed the problem of newly found heavy mesons. Adopting the same standpoints as these authors, Pais[4] attempted the ordering of the various sorts of coupling schemes and the magnitudes of the coupling constants introduced there. In these theories, attention was given to *the conservation of heavy particles***. Here this problem will be discussed from more general point of view. In constructing the couplings between elementary particles we should, first of all, notice the stability of matter. So far as we know, our universe seems to have been stable and no phenomena has yet been observed in which a heavy particle has disappeared without the formation of another heavy particle and, in particular, the proton seems to be entirely stable. So actually, we may not be far from truth, even if we assume hereafter *the absolute stability of matter*. That is, no reaction should occur which causes the decay of proton or diminishes the number of nucleons in matter ultimately.

As regards the stable particles, which never undergo natural decays, only two sorts of them are now believed to be present. One is proton and the others are leptons such as electron, neutrino, photon and possibly μ^0-meson. This fact may be suggestive and full of interest from the viewpoint of families of elementary particles in § 3, but the possible existence of other sorts of stable particles cannot be denied completely in the present state of our knowledge. The observation of some elementary processes such as collisions or natural decays of unstable particles will establish the empirical interactions between particles participating in these reactions and we have known the existence of the various sorts of close connections between all known elementary particles. In our present formalism of the elementary interactions, it is possible that two or more interactions may couple so as to cause new interaction between some particles and sometimes this interaction may be observed experimentally. For instance, the introduction of π-meson-nucleon interaction will cause the $\gamma-\pi$ production. Therefore, every interaction which we want to introduce should be consistent with each other so as not to cause the unwanted processes. That is, they should not destroy some conservation law ultimately and in the problem of heavy mesons we must decouple the strong interactions for their copious production from the weak interactions for their longevity. As unstable particle will, in the end, disintegrate into lighter stable particles, there exist the ultimate interactions between this unstable particle and its stable decay products. So if there are no stable particles heavier than proton, every unstable particle will disintegrate into proton or stable particles lighter than proton. Taking these circumstances into account, we are tempted to postulate the following *ad hoc* but acceptable conservation law for guranteeing sufficient*** stability of matter.

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* The analogus results were obtained independently by E. P. Wigner Proc. N. A. S. 38 (1952), 449. The author wishes to express his thanks to Prof. S. Tomonaga for noticing him Wigner's work.

** J. R. Oppenheiner, discussion remark at the Rochester Conference (1951).

*** It may not be a necessary condition.

"*Postulate*"

"In every possible interaction scheme between elementary particles* which are not heavier than proton, the difference of the number of protons and anti-protons should be conserved"

Consider a certain elementary particle A, and suppose the existence of the following type of interaction is established.

$$A \rightleftarrows m \text{ protons} + n \text{ anti-protons} + \text{other particles* lighter than proton} \quad (1)$$

Assign a characteristic number $\lambda = m - n$ to A as $A\ (\lambda)$.

Then the following consequences will be obtained from our postulate.

Consequence (α)

The assignment of λ-value to A is unique.

This is because otherwise by the intermediary of A, reactions will occur which are incompatible with the above *postulate*. By the coupling scheme (1).

Consequence (β)

If the field A is described by complex amplitude, the hermitian conjugate field A^* of $A(\lambda)$ has $-\lambda$.

On the contrary, if A is described by real field it will have both signs of λ and cannot have unique λ-values except when $\lambda = 0$. So we have the following remarkable consequences for electrically neutral fields.

Consequence (γ)

Neutral particle which has non-zero λ-value could not be described by real amplitude.

For instance, from the decay schemes $V^0 \to p + \pi^-$ and β-decay $n \to p + e^- + \nu$, V^0 and neutron should have $\lambda = 1$, because π-meson, electron and neutrino are lighter than proton. Then $l^{\prime 0}$ and neutron should be described by Dirac fields. That is, if $l^{\prime 0}$ is Majorana particle, the decay $l^{\prime 0} \to p + \pi^-$ causes also the unwanted reaction $p + p \to \pi^+ + \pi^+$ through $l^{\prime 0}$. However, our postulate will not decide whether neutral particle with $\lambda = 0$ should be described by real field or complex field. Usually, γ-ray and π^0-meson which are shown to have $\lambda = 0$ (see (δ)) are treated as real fields. As regards the neutrino which has also $\lambda = 0$, the existence of its anti-particle will be decided by investigating, for instance, its *electromagnetic property* or the phenomena of *double β-decay* which will favour the *distinguishable* or *indistinguishable* theory of neutrino respectively.

Consequence (δ)

Those particles which are lighter than proton should have $\lambda = 0$.

This is evident because if A which is lighter than proton has non-zero λ in scheme (1), this reaction is incomptible with our postulate. So we can state,

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* In reference (5), we stated "stable particles which are not heavier than proton" instead of "elementary particles which are not heavier than proton." Also in scheme (1) we write "other stable particles lighter than proton" in place of "other particles lighter than proton". This is equivalent in its contents as is easily shown by *consequence* (δ).

(δ 1) *Unstable particles which will ultimately decay into elementary particles lighter than proton will have* $\lambda=0$.

Thus, it may be noted that our consequence (δ) does not mean that all particles heavier than proton should have $\lambda \neq 0$. It means only that

(δ 2) *among particles with* $\lambda \neq 0$, *proton is the lightest one.*

Note also that the assignment of λ does not relate *a piiori* to the spin of that particle. For instance, both decay schemes $V^0 \rightarrow p+\pi^-$ and $B^+ \rightarrow n+\mu^+$ (of course, this decay scheme is speculative) will lead to $\lambda=1$ for V^0 and B^+ but they have different statistics. Next consider the general interaction $A(\lambda_a)*B(\lambda_b)*C(\lambda_c)*\cdots\cdots L(\lambda_l)M(\lambda_m)N(\lambda_n)\cdots\cdots+h.c.$ which is assumed to satisfy the conditions for elementary interaction such as Lorentz invariancy, gauge invariancy, etc.

Consequence (ε)

This interaction is allowed in only the cases when

$$-(\lambda_a+\lambda_b+\lambda_c+\cdots\cdots)+(\lambda_l+\lambda_m+\lambda_n+\cdots\cdots)=0. \tag{2}$$

It is clear from the *consequence* (α) that only under this condition our postulate is satisfied. That is, this condition is the concrete realization of our postulate and is analogous in its structure to the conservation of charge in any reaction. Note that this condition (ε) is also of use to decide λ of the elementary particle from the known λ-values of other particles which participate in the experimentally established or assumed interaction containing that particle, even if there exist stable particles heavier than proton. The electromagnetic interaction gives $\lambda=1$ for proton. From the above considerations we can make explicit decision of λ of every elementary particle and λ may be regarded as the *intrinsic* property of elementary particle. So we will call it as "*nucleonic charge*" in the following discussions, and *consequence* (ε) is nothing but the conservation of it in any reaction. It must also be noted that electrons are also the stable constituents of matter. But, at present, besides the neutral stable particles such as γ-ray and neutrino no existence of other charged particles lighter than electron is known. So conservation of electric charge and nucleonic charge seem to be enough to prevent the reactions which destroy the absolute stability of matter. Of course, another stable particle which has the same stable properties as proton and electron may exist. In that case another λ' and its conservation law will be required and this stable particle will define a sequence of elementary particles which contain this particle as its ultimate stable decay product. Our present knowledge, however, seems not to assert the necessity of the existence of such sorts of stable particles.

§ 3. Electromagnetic properties and nucleonic charge of elementary particles

(a) *Conservation of electric charge*[8]

Usually, we postulate that Lagrange function L of the complex field φ under the presence of electromagnetic field A_μ is invariant with respect to gauge transformation

$$\varphi(x)\to\varphi(x)e^{iQ\alpha(x)},\ \varphi^*(x)\to\varphi^*(x)e^{-iQ\alpha(x)},\ A_\mu(x)\to A_\mu(x)-\frac{i}{Q}\frac{\partial\alpha(x)}{\partial x_\mu},$$

where $\alpha(x)$ is any real space-time function and Q is the charge carried by this field. The electromagnetic current is defined by

$$j_\mu=iQ\left\{\partial L/\partial\left(\frac{\partial\varphi}{\partial x_\mu}\right)\cdot\varphi-\varphi^*\partial L/\partial\left(\frac{\partial\varphi^*}{\partial x_\mu}\right)\right\}.$$

There remains the possibility of introducing additional terms in L which depend explicitly on the field strength $F_{\mu\nu}=\partial A_\nu/\partial x_\mu-\partial A_\mu/\partial x_\nu$ and which are consistent with the postulate of gauge invariance. The new current vector is $j'_\mu=j_\mu-\partial/\partial x_\nu(\partial L/\partial F_{\mu\nu})$ which also satisfies $\partial j'_\mu/\partial x_\mu=0$.

For spin one meson φ_μ

$$j'_\mu=j_\mu-iK\frac{\partial}{\partial x_\mu}(\varphi_\nu^*\varphi_\mu-\varphi_\mu^*\varphi_\nu) \tag{3}$$

and for spinor field ψ $(\bar\psi=\psi^*\gamma_4)$

$$j'_\mu=j_\mu+\frac{1}{2}l\frac{\partial}{\partial x_\nu}[\bar\psi\gamma_\mu\gamma_\nu,\psi]. \tag{4}$$

Note that these expressions cannot be constructed from real fields. Consider the i-th particle with chage Q_i in terms of e as a unit. Introduce dichotomic variable M for each particle; $M=-1$ when the particle is absorbed and $M=1$ if the particle is emitted. Define also the dichotomic variable $K=MQ$. Thus if φ_i has K_i, φ_i^* will have $-K_i$. We use the notation φ_i^{Ki} for φ_i with K_i. For securing the conservation of charge in a simple way we may postulate the gauge invariance of the following restricted type.

"The Lagrangian of the system containing several interacting fields should at least de invariant under the following gauge transformation U;

$$U\varphi_i^{Ki}U^{-1}\to\varphi_i^{Ki}e^{iKi\alpha},$$

where α is a real constant."

The Hamilton function and other measurable quantities constructed from Lagrangian will then be gauge invariant in the above sense. Especially, transition matrix T which rules the reaction will become gauge invariant $UTU^{-1}=T$. This means $\sum_i K_i=\sum_i M_iQ_i=0$ where summation is extended over all particles associated with this reaction. Thus conservation of charge is secured in any reaction.

(*b*) *Conservation of nucleonic charge*

In analogy to the conservation of electric charge, our law of conservation of heavy particles in § 2 can be stated in another way.

"*Postulate*"

(α) Every elementary particle has definite value λ of the nucleonic charge where proton has unit positive nucleonic charge $(\lambda=1)$ and other elementary particles* lighter than proton have zero nucleonic charge $(\lambda=0)$.

* It is equivalent if we replace "elementary particles lighter than proton" by "stable elementary particles lighter than proton".

(β) The Lagrangian of the system containing interacting fields should be invariant under a sort of gauge transformation U'

$$U'\varphi_i^I U'^{-1}=\varphi_i^I e^{\beta L_i^3},$$

where β is an arbitrary real constant and $L_i=M_i\lambda_i$.

From these *postulates* (α) and (β) our consequences (α),(β),(γ), (δ 1) and (δ 2) in § 2 follow at once. Our postulate of gauge invariance requires

$$\sum_i L_i=\sum_i M_i\lambda_i=0$$

in any reaction where summation i is over the fields participating in this reaction. This is nothing but *the conservation of heavy particles.*

(ε) *Remarks on neutral fields*

As remarked in § 2, neutral field with *non-zero* nucleonic charge cannot be described by real field. In such case we are forced to use complex field and put electric charge Q equal to zero. If the two body decay $V^0\to p+\pi^-$ is excluded and $V^0\to p+\pi^-+\nu$ (ν... neutrino with *Fermi* statistics) is true, this l'^0 with integer spin should be described by such field*. As is evident from (1), the charge conjugation also means the *nucleonic charge conjugation* for field with *non-zero nucleonic charge*. From (3) and (4), charge conjugate field (also nucleonic charge conjugate field when this field has nucleonic charge) is also *conjugate* with respect to the *magnetic moment* of the particle. (K and l change their sign. See also § 4, (IV)). As regards nucleonic charge it is not yet known which field plays the role of electromagnetic field in the case of electric charge. Neutral field can still be discriminated by the *nucleonic charge* unless it is zero. Thus V^0 and *anti-*V^0 can be distinguished by their decay modes $V^0\to p+\pi^-$ and *anti-*$V^0\to$ *anti-proton*$+\pi^+$. Moreover, if $V^-\to n+\pi^-$ occurs, V^- has unit *positive* nucleonic charge in spite of its *negative* electric charge, and it is remarkable that from our *consequence* (ε), l'^+ will hardly be produced compared with V^- by nucleon-nucleon collisions[9], as is the case of the production of anti-proton. The same situation occurs in the *anti-*V^0 production[9] and we may attribute to this the cause of the fact that no definite examples of the decay of l'^0 into anti-proton are reported. It is quite interesting whether nature permits the existence of *neutral Boson with non-zero nucleonic charge* (such as $l'^0\to p+\pi^-+\nu$) or not. In the case of spinor fields, consistent with our *consequence* (γ), electrically neutral neutron has anomalous magnetic moment and should not be described by Majorana field.

§ 4. Families of elementary particles and interactions between them

So far elementary particles have been characterized by their charge, mass and statistics.

* In the attempts to intermediate β-decay couplings which are the interactions of the second kind by Boson field from the standpoint of *renormalizable* theory, same situations are encountered. S. Tanaka, unpublished.

Here we refer to the possibility of another *phenomenological* classification suggested by the appearance of the socalled megalomorphs.

(I) *Family of elementary particles*

The law of conservation of heavy particles decides the characteristic nucleonic charge of every elementary particle. Then it may be natural to attempt classification with this charge. We have not known elementary particles with $|\lambda| > 1$ and nuclei which have $\lambda > 1$ are regarded as the bound states of nucleons.

(A) *Nucleon family* (we call it as *N. F.*)

This family is characterized by its unit nucleonic charge. The lightest stable member is *proton* and other members are neutron and V^0 at present. As discussed in § 2, our assignment of nucleonic charge does not a priori predict unique statistics. But if $V^0 \to p + \pi^-$ is true, all at present known members of *N. F.* have Fermi statistics. So it may be very likely that *N. F.* has Fermi statistics as its characteristic. Then it cannot contain Majorana particles. Though only small members of this family are known, it will be suggested in § 5 that the existence of more members of this family is likely. We arrange them as N_0 (nucleon), $N_1, N_2, \ldots$ in the increasing orders of their masses. Note also that as discussed in the last section one can only get anti-particles of the existing members of *N. F.*, if enough energy is available to create a pair of heavy Fermions.

(B) *Families with zero nucleonic charge* ($\lambda = 0$)

We have many particles lighter than proton which have zero nucleonic charge. But the possible existence of particles heavier than proton which still have zero nucleonic charge will not be excluded. From the viewpoint of simplicity, the existence of π-meson and τ-meson with Bose-statistics and electron, neutrino and μ-meson with Fermi-statistics seems to make it very desirable that the division of these elementary particles into two groups one of which has *Bose*-and the other has *Fermi-statistics*, may lead to reasonable classification. For the present, we shall follow these lines.

(B. 1) π *-meson family* (*II. F.*)

π_0(π-meson), $\pi_1, \pi_2 \ldots$ etc. with Bose statistics.

We imagine that from the believed decay modes, $\zeta^{\pm}$, v^0, $\tau^{\pm}$ and $\chi^{\pm}$ will belong to this groups. Transitions are likely to take place between these members (such as $\tau \to 3\pi$, $\chi^{\pm} \to \pi^{\pm} + \tau^0$(or v^0) etc.). It may be of interest to inquire into the possibility whether this family has unique spin value and parity (see appendix I)

(B. 2) *Electron family* (*E. F.*)

l_0(neutrino, or electron), $l_1, l_2, \ldots$ etc. with Fermi statistics.

Also μ-meson and possibly μ^0-meson will be the members of *E. F.*. If our assignment of statistics is true, *N. F.* and *E. F.* have the same statistics, but there is remarkable difference that *E. F.* may have *Majorana* particle contrary to *N. F.*. Next we briefly summarize the known interactions between these families.

(a) Interactions between *N. F.* and *II. F.*

Strong interaction (nucleon-π-meson interaction) and week interaction ($V^0 \to p + \pi^-$).

(b) Interactions between *N. F.* and *E. F.*

Only weak interactions such as β-decay coupling and μ^--capture by nuclei ($\mu^- + p \to$

$n+\nu$).

(c) Interaction between *H. F.* and *E. F.*

Only weak interaction $(\pi \to \mu + \nu(\mu^0))$.

The points (a) may suggest that some members of *H. F.* have strong interactions (responsible for their productions in high energy nuclear events) and weak interactions with *N. F.*. As regards the point (c), it may be possible that in analogy to $\pi \to \mu + \nu(\mu^0)$ one may expect $\pi_i \to l_j + l_k$ or $\pi_i \to \pi_j + l_k + l_m$ which will compete with possible transitions $\pi_i \to \pi_j + \pi_k$ etc. between π-family*. The points (b) and (c) may suggest that *E. F.* will interact weakly with both *N. F.* and *E. F.* and will be mere their decay products. But in these points situation is not so clear. If $\varkappa$-mesons which are produced rather copiously in nuclear collisions have Fermi statistics (their decay, for instance, may be $\varkappa \to \mu + \nu + \nu$) there exist, at least, members of *E. F.* which have, whether directly or indirectly, *strong couplings* with *N. F.*[10] contrary to (b). On the other hand, if the decay schemes such as $\varkappa^{\pm} \to \mu^{\pm} + \pi^0 + \nu$ are true, $\varkappa$ becomes Boson (*H. F.*) and $\varkappa$-production is nothing but the strong interaction (a) between *N. F.* and *H. F.* and the above decay which has the form $\pi_i \to l_k + \pi_j + l_m$ will be deduced from the interactions consistent with (a), (b) and (c). Finally it must be remarked that photon plays a particular role. That is, it decides the charge of each family as a fine structure. It seems to be the lightest member of *H. F.* $(\pi^0 \to 2\gamma)$ but has fairly large interaction $(e^2/\hbar c)$ with all charged members of each family and has spin 1.

(II) *Interactions between families*

In discussing the primary interactions between families, we may first of all, give our attention to those of the *first* kind from the standpoint of *renormalizable* theory[11] or to the "$\bar{\psi}\psi\varphi$" type interactions from the viewpoint of simplicity**. Here, however. we take. for the time being, from the familiarity and phenomenological reasons the following interactions as the fundamental interactions. We write Ψ_i, ψ_i and φ_i for the members of *N. F.*, *E. F.* and *H. F.* respectively.

(A) *Universal Fermi interaction*[12] *between N. F. and E. F.*

It is quite remarkable that the following various sorts of phenomena can be explained approximately by Universal Fermi coupling constant $g_F \sim 10^{-49}$ *erg cm*^{3}. β-decay, μ-deeay, μ^--capture by nuclei $(\mu^- + p \to n + \nu)$ and possibly $\varkappa$-decay[6] $(\varkappa \to \mu(e) + \nu + \nu)$***. Note also that the same g_F will lead to $V^0 \to p + \mu^- + \nu$ with life-time comparable with $V^0 \to p + \pi^-$[4]. It is hoped that these arguments will be extended to other members of these families with some selection principles.

(B) "$\Psi\Psi\varphi$" *type interactions between N. F. and H. F.*

* Our selection used in § 1 (b) may not be adequate if members of electron family other than μ exist such as possibly $\varkappa$-meson which might decay as $\varkappa \to \mu(e) + \nu(\mu^0) + \nu(\mu^0)$. Then some π_i may disintegrate into $\varkappa$ instead of μ. Thus, as a possible first step, our procedure to inquire whether the division by *statistics* leads to families with their own features or not will be justified.

** By this types of interactions every possible interaction may be constructed by virtual processes.

***R. E. Marshak and L. Michel, discussion remarks at the Rochester and Copenhagen (1952) Conferences respectively.

Known interactions are the famous nucleon-π-meson interactions $G^2/\hbar c \sim 1$ and the weak interactions responsible for $V^0 \to p + \pi^-$ ($g^2/\hbar c \sim 10^{-11}$). There may be other orders of magnitude of the coupling constants. Of course, other sorts of primary interactions, such as *Boson-Boson interactions* "$\varphi_i \varphi_j \varphi_k$" or "$\varphi_i \varphi_j \varphi_k \varphi_l$" (which are renormalizable) as indicated by the decay modes between the members of π-family, may be considered. But it is very interesting to note the combinations of "$\bar{\Psi}\Psi\varphi$" and Universal Fermi interactions might be enough to cause the decays of *H. F.*. It is remarkable that we are succeeded in $\pi^0 \to 2\gamma$[13] and $\tau \to 3\pi$[14] decays by the intermediary of "$\bar{\Psi}\Psi\varphi$" type interactions. Moreover, $\pi \to \mu + \nu$ might occur through nucleon-π-meson interactions and nucleon-μ-meson-neutrino interactions with Universal Fermi couplings[15]. In the same way, it should be noted strong "$\Psi_i \bar{\Psi}_j \varphi_k$" interactions $G^2/\hbar c \sim 1$ couple with Universal Fermi interactions $(\bar{\Psi}_i \Psi_j)(\bar{\psi}_l \psi_m)$ and lead to the decay $\pi_i \to l_m + l_n$ etc.. Thus primary "$\bar{\psi}\psi\varphi$" interactions (such as $\pi \to \mu + \nu$) between *H. F.* and *E. F.* might not be so necessary*. Of course, some *selection principles* of the couplings between families are needed to forbid unwanted processes (see reference[15]).

(III) *Remarks on Pais' even-odd rule*[4]

As stated in (II), typical orders of magnitude of the coupling costant of fundamental interactions are known. However, Pais's even-odd rule, as he stressed, must be regarded as provisional and be replaced by more complicated ones, because strong even interactions $\bar{N}_0 N_0 \pi_0$, $\bar{N}_0 N_0 \pi_2$, $\bar{N}_0 N_0 \pi_4$..... and $\bar{N}_0 N_1 \pi_1$, $\bar{N}_0 N_1 \pi_3$ ... will lead fast decays $\pi_2 \to \pi_0 + \pi_0$, $\pi_4 \to \pi_0 + \pi_0$, $\pi_3 \to \pi_1 + \pi_0$ etc. and these rules will only stabilize π_1 and N_1 against π_0 or *γ-instability*. Nevertheless, we seem to have more π-meson groups which are fairly stable.

(IV) *Note on nucleonic charge conjugation*

In setting up primary interactions discussed in (II) of this section *the conservation of nucleonic charge* excludes the possibities such as $\bar{\Psi}\psi\varphi$ and $(\Psi^{I_i}_i \Psi^{I_j}_j)(\bar{\psi}\psi')$ with $L_i + L_j \neq 0$. Moreover, the notion of families of elementary particles seems to prefer $(\Psi^{I_i}_i \Psi^{I_j}_j)(\bar{\psi}\psi')$ to $(\Psi^{I_i}_i \psi)(\Psi^{I_j}_j \psi')$ where $L_i + L_j = 0$. Thus the interactions we are fond of are $\Psi^{I_i}_i \Psi^{I_j}_j \varphi$ and $(\Psi^{I_i}_i \Psi^{I_j}_j)(\bar{\psi}\psi')$ type with $L_i + L_j = 0$ and we investigate some properties of them. Describing the i-th member and j-th member of nucleon family, $\Psi^{I_i}_i$ and $\Psi^{I_j}_j$ by 8-component spinor $\Psi = \begin{pmatrix} \Psi^{I_i}_i \\ \Psi^{I_j}_j \end{pmatrix}$ we define

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to *nucleonic charge conservation*, our interactions must be of the form

$$G(\Psi^{-I} \Omega \tau \Psi^I)\varphi + h.c. \text{ and } g(\Psi^{-I} \Omega \tau \Psi^I)(\bar{\psi} \Omega' \tau' \psi') + h.c.,$$

where Ω and Ω' are appropriate functions of γ_μ and τ' is the isotopic variable between

* Of course, it may also be possible that "$\bar{\Psi}\Psi\varphi$" and "$\bar{\psi}\psi\varphi$" interactions are primary and other interactions will be deduced from them.

E.F. G and g are interaction constants.

Consider a *nucleonic charge conjugation* (this is equivalent to charge conjugation when field under consideration carries electric charge.),

$$V\Psi^L V^{-1}=C\Psi^{-L} \text{ and } V\Psi^{-L}V^{-1}=C^{-1}\Psi^L(\Psi^{-L}=(\Psi^L)^*\gamma_4)$$

where $C^{-1}\gamma_\mu C=-\gamma_\mu{}^T$, $C^*C=1$ and $C^TC^{-1}=-1$.

Then we get dropping an infinite C-number by usual procedure,

$$V\Psi^{-L}\Omega\tau\Psi^L V^{-1}=\Psi^{-L}\Omega\tau^T\Psi^L$$

in the cases; *scalar*, *pseudovector* and *pseudoscalar* couplings, and

$$V\Psi^{-L}\Omega\tau\Psi^L V^{-1}=-\Psi^{-L}\Omega\tau^T\Psi^L$$

in the cases; *vector* and *tensor* couplings.

Next, consider the interchange of the particle i and j. This is realized by

$$W\Psi^L W^{-1}=\tau_1\Psi^L.$$

Using the properties of $\tau_i(i=0,1,2,3)$, we get

$$W\Psi^{-L}\Omega\frac{\tau_1\pm i\tau_2}{2}\Psi^L W^{-1}=\Psi^{-L}\Omega\frac{\tau_1\mp i\tau_2}{2}\Psi^L,$$

$$W\Psi^{-L}\Omega\tau_3\Psi^L W^{-1}=-\Psi^{-L}\Omega\tau_3\Psi^L,\quad W\Psi^{-L}\Omega\tau_0\Psi^L W^{-1}=\Psi^{-L}\Omega\tau_0\Psi^L.$$

Thus our interactions can be written in the form

$$G(\Psi^{-L}\Omega\tau\Psi^L)\varphi+h.c.$$

$$=\frac{1}{2}\{G\Psi^{-L}\Omega\tau\Psi^L+\theta GWV\Psi^{-L}\Omega\tau\Psi^L V^{-1}W^{-1}\}\varphi+h.c.,$$

$$g(\Psi^{-L}\Omega\tau\Psi^L)(\bar\psi\Omega'\tau'\psi)+h.c.$$

$$=\frac{1}{2}\{g(\Psi^{-L}\Omega\tau\Psi^L)+\theta gWV(\Psi^{-L}\Omega\tau\Psi^L)V^{-1}W^{-1}\}(\bar\psi\Omega'\tau'\psi)+h.c.,$$

where

$$\theta=1 \quad \text{when } \Omega=s,\ pv,\ ps \text{ and } \tau=\frac{\tau_1\pm i\tau_2}{2},\ \tau_0,$$

$$\text{or when } \Omega=v,\ t \quad \text{and } \tau=\tau_3,$$

$$\theta=-1 \quad \textit{other cases.}$$

Fig. I shows these procedures schematically and arrow means the *conservation of nucleonic charge*. These properties of sign changes of the coupling constants under both the *nucleonic charge conjugation* and *interchange* of the associated members of *nucleon family* will lead to Generalized Furry's theorem[16] and were extensively used in the discussions of $\pi^0\to 2\gamma$[13] and $\tau\to 3\pi$[14]. If we take the viewpoint that transitions of π-meson

Fig. 1.

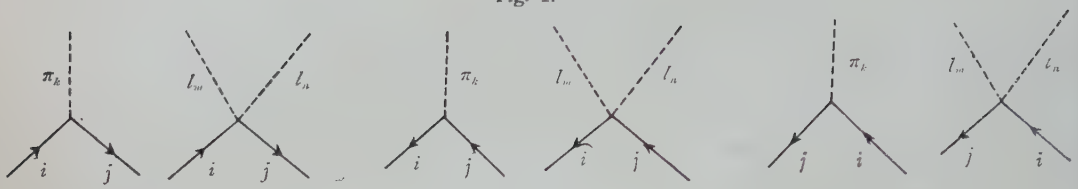
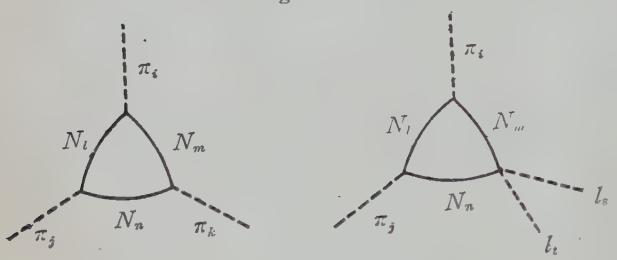


Fig. 2.



groups are caused by the intermediary of these interactions as Fig. 2, these properties will give various absolute and relative selection rules. If Majorana particles may exist in *E. F.* these properties will not appear in such cases.

§ 5. Productions of *N. F.* and *H. F.*

In our current meson theory, we have no good tools for strong or intermediate couplings and, moreover, there appear many divergences which cannot be renormalizable by simple procedure. Our arguments, therefore, are forced to be qualitative or order estimations by simple but reasonable procedure. In the following analysis, however, we can use confidently the fact that *pion* is *pseudoscalar* and pion-nucleon interaction is very likely *charge independent* and the magnitude of the coupling constant $G^2/\hbar c \sim 1$. Heavy mesons seem to be produced rather copiously in high energy nuclear collisions. So the empirical coupling constants of their production schemes will be fairly large. Combining with known strong interactions such as nucleon-π-meson and electromagnetic interaction, these interactions for production are apt to cause the π_0 or γ-instability of heavy meson and we may expect to find a clue to the concrete structure of the interactions between *N. F.* and *H. F.* Our arguments assume that *N. F.* and *H. F.* have Fermi statistics and Bose statistics respectively, but the cases when each family has no definite statistics can be discussed similarly with few modifications[10]. *Conservation law of heavy particles* restricts the possible production schemes and we will analyze each case briefly.

(I) $$N_0 + N_0 \rightarrow N_i + N_j,$$

(Ia) $$N_0 + N_0 \rightarrow N_0 + N_i + (\pi_0) \quad (i \neq 0),$$

(Ia′) $$\pi_0 + N_0 \rightarrow \pi_0 + N_i + (\pi_0) \quad (i \neq 0).$$

These fairly strong interactions will couple with large interactions $\bar{N}_0 N_0 \pi_0$ or $\bar{N}_0 N_0 \gamma$ and lead to $N_i \rightarrow N_0 + \pi_0(\gamma)$ at once, since the existence of Fermions will not permit

selection rules. The order of these decay lifetimes is in general* much shorter than 10^{-10} sec. which is the order of decay life-time of $V^0 \to p + \pi^-$. It should be stressed that these single production models of V^0 are not successful.

(Ib) $$N_0 + N_0 \to N_i + N_j \quad (i \neq 0,\ j \neq 0),$$

(Ib′) $$\pi_0 + N_0 \to N_i + \pi_j \quad (i \neq 0, j \neq 0).$$

Unless we do introduce strong interactions such as $\bar{N}_0 N_i \pi_0$ or $\bar{N}_0 N_j \pi_0$ etc., we need not trouble with the γ or π_0 *instability* of N_i or N_j for (Ib). Pais[4] solved this problem by his *even-odd rule*. But of course, pair-wise prodution such as (N_i, N_i) or (N_i, N_j) should be checked by experiments. The case (Ib′) will be discussed in (IIb′).

(II) $$N_0 + N_0 \to N_i + N_j + \pi_k,$$

(IIa) $$N_0 + N_0 \to N_0 + N_0 + \pi_i \quad (i \neq 0),$$

(IIa′) $$\pi_0 + N_0 \to N_0 + \pi_i + (\pi_0) \quad (i \neq 0).$$

In these cases, at least one of the following fast decays is inevitable;

$$\pi_i \to \pi_0 + \pi_0 + (\pi_0),\ \pi_i \to \pi_0 + \gamma + (\gamma) \text{ and } \pi_i \to \pi_0 + \pi_0 + \gamma,$$

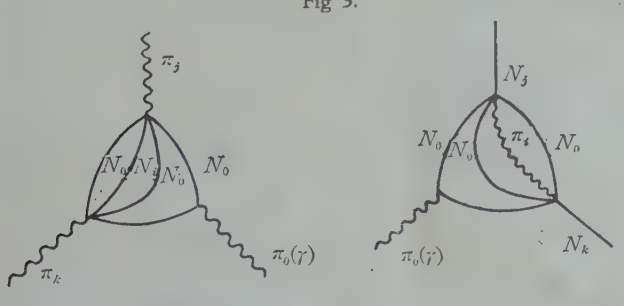
because as a result of *charge independence* and γ-decay, conservation of charge is not useful and selection rules based on Lorentz invariance and gauge invariance and charge symmetry etc.[16] cannot be enough to forbid all the above decays. So this scheme cannot be used for π_i's which are fairly stable (such as, $\zeta^\pm$, v^0, $\tau^\pm$, $\chi^\pm$ etc.).

(IIb) $$N_0 + N_0 \to N_0 + N_i + \pi_j \quad (i \neq, j \neq 0),$$

(IIb′) $$\pi_0 + N_0 \to (\pi_0) + N_i + \pi_j. \quad (i \neq 0, j \neq 0).$$

If we make $\bar{N}_0 N_i \pi_0$, $\bar{N}_0 N_0 \pi_j$ etc. small by some ordering principles, we need not trouble with the π_0 or γ instability of π_j. But these cases are also the pair-wise production of N_i and π_j. Further it must be noted that if the same N_i or π_i participate in two reactions such as

Fig 3.



$$N_0 + N_0 \to N_0 + N_i + \pi_j,$$
$$N_0 + N_0 \to N_0 + N_i + \pi_k$$
or $$N_0 + N_0 \to N_0 + N_j + \pi_i,$$
$$N_0 + N_0 \to N_0 + N_k + \pi_i,$$

these reactions will also lead to fast decay $\pi_j \to \pi_k + \pi_0(\gamma)$ or $N_j \to N_k + \pi_0(\gamma)$ etc. as is shown by Fig. 3.

* In a special case the use of regularization method for the treatment of the ambiguous integrals from the standpoint of gauge invariance and equivalence theorem leads to surprisingly long life-times of the order 10^{-10} sec. K. Aizu and T. Kinoshita, Prog. Theor. Phys. **6** (1951), 63.

(IIc) $N_0+N_0\rightarrow N_i+N_j+\pi_k \quad (i\neq 0,\ j\neq 0,\ k\neq 0),$

(IIc′) $\pi_0+N_0\rightarrow N_i+\pi_j+\pi_k \quad (i\neq 0,\ j\neq 0,\ k\neq 0).$

These do not cause troubles unless we introduce strong $\bar{N}_0N_i\pi_0$, $\bar{N}_jN_0\pi_0$ etc.. But the theshold energies will be larger than previous cases. Next we must proceed to

(III) $N_0+N_0\rightarrow N_i+N_j+\pi_k+\pi_l,$

but we are interested in the schemes which do without large threshold energies. Indeed, if $N_i(i\neq 0)$ does not play a role in the production schemes, as the schemes (Ia), (Ia′), (IIa), (IIa′) are wrong, we must depend on

(IIIa) $N_0+N_0\rightarrow N_0+N_0+\pi_i+\pi_j,$

(IIIb) $\pi_0+N_0\rightarrow N_0+\pi_i+\pi_j.$

If $i=0$ and $j\neq 0$, $\pi_j\rightarrow\pi_0+\pi_0(\gamma)$ or $\pi_j\rightarrow\pi_0+\pi_0(\gamma)+\pi_0(\gamma)$ etc. and if $i\neq 0$, $j\neq 0$ and $i\neq j$, $\pi_j\rightarrow\pi_i+\pi_0(\gamma)$ or $\pi_j\rightarrow\pi_i+\pi_0(\gamma)+\pi_0(\gamma)$ etc. are liable to occur. The harmless case is $i=j\neq 0$, but we must face also with pair-wise production of (π_i,π_i).

In summarizing these results emphasis should be made on the following points.

(1) Heavy particle conservation law and the γ or π_0-*instability* of N_i and π_i impose fairly severe restrictions on the production modes. *Single production* models such as (Ia), (Ia′), and (IIa), (II′a) with the smallest threshold energies must be abandoned by these reasons.

(2) Even in the schemes which do not lead π_0 or γ-decay of N_i or π_j directly, some *ordering principles* of the coupling schemes and their coupling constants are inevitable in order to *stabilize* N_i and π_j against unwanted processes.

(3) If we reject the possible existences of unobservable parents, we are forced to face with the situation that the promising schemes require the *pair-wise* productions such as (N_i,N_j), (N_i,π_j) and (π_i,π_j) with $i\neq 0$ and $j\neq 0$. These should be checked by experiments.

(4) Comparing the schemes (Ib), (Ib′), (IIb), (IIb′) with (IIIa), the existence of other members of nucleon family seems to be desirable to lessen the threshold energies. (see appendix II.)

So far, we have no recourse to the unobservable parents of observable N_i and π_j. But as discussed above, if *pair-wise* productions such as (N_i,N_j), (N_i,π_j) or (π_i,π_j) are not true we are forced to remove to these models. Even in such cases some remarks are also needed.

(a) The strong interactions responsible for the production of parents are also liable to cause π_0 or γ-*instability* of parents and these compete with the decays of parents into daughters we want. To *stabilize* parents against unwanted decays still some *ordering principle* will be needed, and the simplest and safest way will be the *pair-wise* production of parents. For instance, in schemes such as (Ib), (Ib′), (IIb), (IIb′) and (IIIa), we can assume that the N_i and π_i are unobservable parents.

(b) Next, we set up *strong* interactions responsible for the unobservable *fast* decays of parents into daughters N_i or π_j which we want. But some selection principles are needed to *stabilize* observable N_i or π_j against the decays into unwanted daughters through *thus constructed interactions.*

(c) The models of unobervable parents have the advantages that by availing of the freeedom of choosing the decay modes and the masses of the *intermediate* parents, we can make the production of N_i or π_j take the apparent form of *single production* and can *adjust* the threshold energies.

(d) If the unobservable parents of $l^{\prime 0}$ or π_i exist, this parents must belong to *N. F.* or *H. F.*. Thus the models of unobservable parents will inevitably predict the existence of more members of *N. F.* and *H. F.*.

§ 6. Concluding remarks

In view of our poor knowledge about elementary particles the *preliminary* nature of our discussions about families of elementary particles in § 4 must be stressed. There, our arguments prefer to *unique statistics* of each family. Present experimental facts seems to support this viewpoint. This assumption also makes it possible to expect that each member of given family will be distinguished from each other through some *quantization process* of new kind. However, as discussed in the attempts to intermediate β-decay coupling through *renormalizable interactions*[11], *Bosons* with *nucleonic charge* might exist (if $l^{\prime 0} \to p + \pi^- + \nu$ is true $l^{\prime 0}$ may be such field). Also the properties of $\varkappa$-meson seem to be enlightening in our standpoint. Moreover, we must order the various interactions between families so that they may be consistent with each other. So the attempts to get *phenomenological ordering principles* should be pursued*. The increasing number of elementary particles and their complicated interactions may rather give some insight into the hidden structures concerning with the constitution and interactions of elementary particles. In these points, we may say " the more, the better " as regards the occurence of unstable particles.

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Appendix I. Possible properties of *H. F.*

The decay modes of charged τ-mesons seem to be settled; $\tau^{\pm} \to \pi^{\pm} + \pi^{\pm} + \pi^{\mp}$ (or $\tau^{\pm} \to \pi^0 + \pi^0 + \pi^{\pm}$) This implies that τ-meson is very likely *pseudoscalar*[14]. Then we

* Recently Nishijima presented some ordering attempts. K. Nishijima, *Soryushiron Kenkyu* (*mimeographical circular in Japanese*) **5** (1953), 89.

may be impressed that two members (π and τ) of *II. F.* are already *pseudoscalar*. The decay modes of other π_i are suggested to be,

(a) $\zeta^{\pm} \to \pi^{\pm} + \pi^0$ ($\tau \sim 10^{-10}$ sec.), $\zeta^0 \to \pi^+ + \pi^-$ ($\tau < 10^{-14}$ sec.),

(b) $v^0 \to \pi^{\pm} + \pi^{\mp}$ ($\tau \sim 10^{-10}$ sec.), (or may be $v^0 \to \pi^{\pm} + \zeta^{\mp}$),

(c) $\chi^{\pm} \to \pi^{\pm} + \tau^0$ (or v^0) ($\tau \sim 10^{-10}$ sec.).

These modes are not so well established and it may be possible that two or more decay modes appear with comparable frequency or that there may be neutral counterpart among their daughters. As discussed in § 4, if their transitions will occur via "$\bar{\Psi}\Psi\varphi$" interactions we may have many competing processes and also expect various sorts of selection rules to obtain adequate models. But it may be premature to discuss them in detail[17].

Among the competing decay modes, the cases which are most dangerous and easy to occur will be the γ-decays of the type $\pi_i \to \pi_j + \gamma$. To fordid these processes completely we may be tempted to assume,

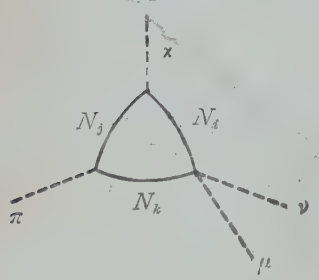
(A) *All members of II. F. have spin* 0

Then γ-decays will be of the form $\pi_i \to \pi_j + \gamma + \gamma$ or $\pi_i \to \pi_j + \pi_k + \gamma$ which will be rare compared with $\pi_i \to \pi_j + \gamma$ and we may not be so worried about γ-decays. Conversely, if γ-decay of the type $\pi_i \to \pi_j + \gamma$ is established, this means that *II. F.* contains the particles which do not have *spin* 0.

(B) If the decays $\pi_i \to \pi_0 + \pi_0$ occur such as (a) and (b), π_i must be *scalar* or *vector*. To make all the members of *II. F. pseudoscalar*, the decay modes must be, at least, *three body decays* such as (a′) $\zeta^{\pm} \to \pi^{\pm} + \pi^0 + \pi^0(\gamma)$ (b′) $v^0 \to \pi^{\pm} + \pi^{\mp} + \pi^0(\gamma)$ (c′) $\chi^{\pm} \to \pi^{\pm} + \tau^0(v^0) + \pi^0(\gamma)$ etc.

Thus definite experimental decisions of decay schemes are hoped. If ζ is *scalar* meson, we may explain the remarkable difference of the life-time of $\zeta^{\pm}$ and ζ^0 by assigning decay schemes; $\zeta^0 \to \pi^+ + \pi^-$ and $\zeta^{\pm} \to \pi^{\pm} + \pi^0 + \gamma$[17]. For this purpose we take ζ meson as *scalar* meson with *scalar* coupling with certain member of *N. F.* (such as nucleon). Moreover we make ζ^0 have τ_0 and π^0 have τ_π. Then Generalized Furry's theorem will forbid $\zeta^{\pm} \to \pi^{\pm} + \pi^0$ as a lowest order process which may otherwise occur similarly to $\zeta^0 \to \pi^{\pm} + \pi^{\mp}$. As Brueckner and Watson[18] suggested, we may attribute $\zeta^0 \to \pi^{\pm} + \pi^{\mp}$ to strong meson-meson interaction and deny ζ^0. Finally, in § 4 we discussed the possible decay schemes $\varkappa^{\pm} \to \mu^{\pm} + \pi^0 + \nu$ which might be desirable to make $\varkappa$ belong to *II. F.* Such decay modes will be deduced by the schemes shown in Fig. 4. But we must order $\bar{\Psi}_i \Psi_j \varphi_k$ and $\bar{\Psi}_i \Psi_j \psi \psi'$ suitably in order not to cause the faster processes than $\varkappa^{\pm} \to \mu^{\pm} + \pi^0 + \nu$ or the other unwanted processes.

Fig. 4.



Appendix II. Threshold energies for the production of *N. F.* and *H. F.*

We write the production reactions in § 5 as $M_0+M\rightarrow M_1+M_2+\dots$. Denote the masses of M_0 (incident nucleon or π_0-meson), M (nucleon in target nuclei) and M_i (member of *N. F.* or *H. F.* produced) as m_0, m and m_i respectively. The threshold energies required for production reactions are

$$E/c_2=1/2m\cdot[(\sum_i m_i)^2-(m_0+m)^2]+Q/2m^2\cdot[(\sum m_i)^2-(m_0^2+m^2)]$$

$$-1/2m^2\cdot\sqrt{Q(2m+Q)}\{(\sum_i m_i)^2-(m_0+m)^2\}\{(\sum_i m_i)^2-(m_0-m)^2\},$$

where Q is the kinetic energy of the internal motion of nucleon in the target nuclei. First of all, we shall show the orders of the magnitude of E in the promising but *pair-wise* production schemes. For brevity we put $Q=0$ (nucleon in target nuclei is rest) and take approximately $m_{N0}=1800m$, $m_{V0}=2200m$, $m_{\tau 0}=800m$, $m_{\chi\pm}=1500m$ and $m_{\pi 0}=300m$.

$N_0+N_0\rightarrow V^0+V^0$,	$E=0.9$ Bev.,	(Ib)
$\pi_0+N_0\rightarrow V^0+\pi_i$,	$E=0.64$ Bev. $(\pi_i=\tau^0)$, $E=1.3$ Bev. $(\pi_i=\chi^\pm)$,	(Ib′)
$N_0+N_0\rightarrow N_0+V^0+\pi_i$,	$E=1.4$ Bev. $(\pi_i=\tau^0)$, $E=2.4$ Bev. $(\pi_i=\chi^\pm)$,	(IIb)
$\pi_0+N_0\rightarrow V^0+\pi_0+\pi_i$,	$E=0.9$ Bev. $(\pi_i=\tau^0)$, $E=1.6$ Bev. $(\pi_i=\chi^\pm)$,	(IIb′)
$N_0+N_0\rightarrow N_0+N_0+\pi_i+\pi_i$,	$E=2.0$ Bev. $(\pi_i=\tau^0)$, $E=4.3$ Bev. $(\pi_i=\chi^\pm)$,	(IIIb)
$\pi_0+N^0\rightarrow N_0+\pi_i+\pi_i$,	$E=1.0$ Bev. $(\pi_i=\tau^0)$, $E=2.6$ Bev. $(\pi_i=\chi^\pm)$.	(IIIb)

It may be noted the schemes (Ib′) and (II) require smaller threshold energies than schemes (III). Thus the appearance of $N_i(i\neq 0)$ in the production reactions seems to be convenient to lessen the threshold energies. If these *pair-wise* productions are not favoured by experiments, recourse to the *unobservable parents* must be made. For instance, we may regard the above reactions as the productions of parents and replace V^0 and π_i by the unobservable parents N_i' and π_i'. One of the members of the pair must decay into N_0 or π_0 and the other into *observable* N_i (such as V^0) or π_i quickly in order to pretend the form of *single production*. Thus by choosing suitable modes for these decays we can lessen the threshold energies. For example, take $N_0(\pi_0)+N_0\rightarrow N_j'(\pi_j')+N_i'$. To lessen the threshold energies as far as possible, we may take the unobservable fast decays as $N_j'(\pi_j')\rightarrow N_0(\pi_0)+$*leptons* and $N_i'\rightarrow N_i$ (such as V^0)$+$*leptons*. Then the threshold energies become comparable with those of the reactions

$N_0(\pi_0)+N_0 \rightarrow N_0(\pi_0)+N_i$ (such as V^0). (When $N_i = V^0$, $E_{N_0} \sim 0.42$ Bev. or $E_{\pi_0} \sim 0.26$ Bev). If we take into account the kinetic energy Q of the internal motion of nucleon in target nuclei we can further decrease the threshold energy. Of course, as noted in § 5, some ordering principles are needed to make these procedures without contradictions. It must be stressed that in these cases also N_i' and N_j' belong to $N.F.$

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Note added in proof

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On the Electron-Neutrino Angular Correlation

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The electron-neutrino angular correlation functions are calculated for the allowed and first forbidden transitions in the Fermi theory of beta decay taking into account the effect of the nuclear charge. The interferences which arise due to squaring of terms in the interaction are worked out perfectly. The Majorana theory is also handled for the allowed transition. The differences observed between both theories are not so serious. The experimental data of P^{32} are consistent with the tensor interaction.

§ 1. Introduction

There are five relativistic invariants of the interaction Hamiltonian in the theory of beta-decay. Recently several efforts[1] are concentrated to diminuate this arbitrariness of the types of the interaction. The main tool of these arguments is to find the certain nuclear matrix elements which make the Kurie plots straight. But the straightness of the Kurie plot is not so sensitive[2] to the change of the correction factors that we can diminuate the arbitrariness of types of interaction not completely. The other one is the β-γ angular correlation which is more effective than the Kurie plot but not sufficient to our purpose.

A latest proceeding of the experimental technique makes the measurements of the electron-neutrino angular correlation possible[3] which is most effective to decide the types of the interaction. For example, in the allowed transition it is definite whether 0—0 transition is S or V and 1—0 transition is T or A when the spin changes of the nucleus are well known by any other method. In spite of the extreme difficulties of the technique the experiment of e-ν angular correlation will be widely done in near future. This is the principal motive by which we calculate the e-ν angular correlation functions $\mathfrak{W}(\theta, W, K)$.

In the approximation $Z=0$, Hamilton[4] calculated $\mathfrak{W}(\theta, W, K)$ up to the first forbidden except the interferences between two types. In the approximation $\alpha Z \sim 1$, Greuling and Meeks[5] calculated $\mathfrak{W}(\theta, W, K)$ up to the first forbidden except all the interferences in the same types or between other types.

Therefore we calculate $\mathfrak{W}(\theta, W, K)$ by its present formulation for the allowed and first forbidden transitions including all the interferences in the same types and between other types, and taking into account the effect of the nuclear charge perfectly. For the second forbidden transition the calculations will be easily performed when they are necessary. It does not happen the interference between two nuclear matrix elements which

belong to the different ranks. The Majorana theory* is also handled in the case of the allowed transition. The differeces observed between both theories are not so serious on the limits of nowaday experimental technique.

The data of P^{32} that Scherwin[3] explained as the first forbidden are consistent with the allowed and tensor interaction.

§ 2. Theory

In this section we introduce the forms of the e-ν angular correlation functions $\mathfrak{W}(\theta, W, K)$ which are convenient in the cases of the mixed nuclear matrix elements. Mainly we follow the peocedures of Yamada and Morita.[7]

In the β-decay of the Fermi theory an antineutrino disappears, (this is equivalent to creation of an neutrino), and an electron appears the same instant, then the nucleus changes $J, m, \alpha \rightarrow J', m', \alpha'$. According to the usual quantum mechanical considerations the transition probability $p_{mm'}$ is

$$p_{mm'} = \sum_{\nu} (\alpha', J', m' \mid H \mid \alpha, J, m), \qquad (1)^{**}$$

where J, m and α are the total angular momentum, its z-component and other quantum numbers of the initial nucleus respectively, and J', m' and α' are those of the final nucleus, H is the interaction Hamiltonian. $\sum_{\nu}$ indicates the sum over the spin of the neutrino, (in the case of $Z \neq 0$, where the spin of the electron can not be a good quantum number), or the sums over the spins of the neutrino and electron, (in the case of $Z=0$), but without the integrations over each of their directions because the energies of the electron and the neutrino, W and K, and the angle θ between their directions of emission are known by the experiment.

From eqs. (5), (6), (7) and (8) of YM the interaction Hamiltonian is

$$H = \sum_{L} \sum_{T_L(\boldsymbol{X}_i)} a\,T(\boldsymbol{X}_i)\Big] \sum_{M=-L}^{L} \mathfrak{Y}_{LM}(\boldsymbol{X}_i)\mathfrak{Y}_{LM}^{*}(\boldsymbol{A}_i). \qquad (2)$$

The notation is as follows: $\boldsymbol{X}_i$ are vector operators for the nucleus, $\boldsymbol{A}_i$ for the electron, $\sum_{T_L(\boldsymbol{X}_i)}$ indicates the sum over various irreducible tensors retaining the L constant, $\sum_{L}$ indicates the sum over various ranks L of the irreducible tensors, $\mathfrak{Y}_{LM}$ are the polarized solid harmonics, $a[T(\boldsymbol{X}_i)]$ are numerical coefficients.

$$p_{mm'}(\theta, W, K) = \sum_{\nu} \Big| \sum_{L} \sum_{T_L(\boldsymbol{X}_i)} a[T(\boldsymbol{X}_i)]\, \mathfrak{M}(\boldsymbol{X}_i)\,(JLmM \mid JLJ'm')\,\mathfrak{Y}^{*}_{LM}(\boldsymbol{A}_i)\Big|^2$$

$$= \sum (JLmM \mid JLJ'm')^2 \sum_{T_L(\boldsymbol{X}_i)} \sum_{\nu} \Big| a[T(\boldsymbol{X}_i)]\, \mathfrak{M}(\boldsymbol{X}_i)\mathfrak{Y}^{*}_{LM}(\boldsymbol{A})\Big|^2$$

$$+ \sum_{L_1 L_2} \sum (JL_1mM \mid JL_1J'm')\,(JL_2nM \mid JL_2J'm') \sum_{T_{L1}(\boldsymbol{X}_i)}{}' \sum_{T_{L2}(\boldsymbol{X}_j)}{}' \sum_{\nu}$$

* The existence of the Majorana particle is somewhat doubtful[9], but we do not enter into this discussion.

** The constant factors are dropped, and the natural units are taken hereafter.

$$\times \{a^*[T(\boldsymbol{X}_i)]a[T(\boldsymbol{X}_j)]\mathfrak{M}^*(\boldsymbol{X}_i)\mathfrak{M}(\boldsymbol{X}_j)\mathcal{Y}_{L_1M}(\boldsymbol{A}_i)\mathcal{Y}_{L_2M}{}^*(\boldsymbol{A}_j)+\text{c.c.}\}, \tag{3}$$

with $m+M=m'$.

The notation is as follows: $(JLmM|JL'J'm')$ are real numbers and called the transformation coefficients for the vector addition which vanish unless $m+M=m'$, and this shows the selection rule for the magnetic quantum number. $\mathfrak{M}(\boldsymbol{X}_i)$ are complex numbers and called the reduced nuclear matrix elements. $\sum_{T_{L_1}(\boldsymbol{X}_i)}{}' \sum_{T_{L_2}(\boldsymbol{X}_j)}{}'$ excludes the square terms. Eq. (3) is somewhat different to eq. (10) of YM where the first term of the last equation includes the interferences in the same rank. To the contrary all the interferences are included in the second term of the last equation of eq. (3).

We do not distinguish the magnetic quantum numbers of the initial and final nucleus so that $p_{mm'}$ is summed up for the final magnetic quantum numbers m' and averaged for the initial ones m. The total transition probability $\mathfrak{W}_{J\to J'}$ is

$$\mathfrak{W}(\theta, W, K)=\frac{1}{2J+1}\sum_{m,m'} p_{mm'}. \tag{4}$$

Using the properties of the transformation coefficient and V function[8] the eq. (4) is simplified.

$$\sum_{\alpha,\beta} V(abc\,;\alpha\beta\gamma)\,V(abc'\,;\alpha\beta\gamma')=\delta_{cc'}\delta_{\gamma\gamma'}/(2c+1),\ (a+b\geqq c\geqq|a-b|,\ c\geqq|\gamma|)$$

$$V(acb\,;\alpha\gamma\beta)=(-1)^{a+b+c}V(abc\,;\alpha\beta\gamma),$$

$$(JLmM|JL'J'm')=(-1)^{J'+M'}(2J'+1)^{\frac{1}{2}}V(JLJ'\,;mM-m').$$

Hence

$$\begin{aligned}
&\sum_m (JL_1mM|JL_1J'm')\,(JL_2mM|JL_2J'm')\\
&=\sum(-1)^{J'+m'}(2J'+1)^{\frac{1}{2}}V(JL_1J';\,mM-m')\,(-1)^{J'+m'}(2J'+1)^{\frac{1}{2}}V(JL_2J';\,mM-m')\\
&=(2J'+1)\,(-1)^{L_1+L_2}\sum V(JJ'L_1\,;\,m-m'M)\,V(JJ'L_2\,;\,m-m'M)\\
&=\delta_{L_1L_2}(2J'+1)/(2L_1+1).
\end{aligned} \tag{5}$$

Therefore the interferences between different ranks all vanish as same as the case of the correction factors. From eqs. (4) and (5) $\mathfrak{W}$ is as follows:

$$\begin{aligned}
\mathfrak{W}(\theta, W, K)=&\frac{2J'+1}{2J+1}\sum_L \sum_{T_L(\boldsymbol{X}_i)}{}' \left\{\frac{1}{2L+1}\,|\,a[T(\boldsymbol{X}_i)]\,\mathfrak{M}(\boldsymbol{X}_i)|^2 \sum_M \mathfrak{F}^M_{T_L(\boldsymbol{X}_i)\,T_L(\boldsymbol{X}_i)}(\theta, W, K)\right\}\\
&+\frac{2J'+1}{2J+1}\sum_L \sum_{T_L(\boldsymbol{X}_i)}{}' \sum_{T_L(\boldsymbol{X}_j)}{}' \frac{1}{2L+1}\\
&\times\{a^*[T(\boldsymbol{X}_i)]\,a[T(\boldsymbol{X}_j)]\,\mathfrak{M}^*(\boldsymbol{X}_i)\mathfrak{M}(\boldsymbol{X}_j)\sum_M \mathfrak{F}^M_{T_L(\boldsymbol{X}_i)\,T_L(\boldsymbol{X}_j)}(\theta, W, K)+\text{c. c.}\},
\end{aligned} \tag{6}$$

where

$$\mathfrak{F}^M_{T_L(\boldsymbol{X}_i)\,T_L(\boldsymbol{X}_j)}(\theta, W, K)=\mathfrak{S}\,\mathcal{Y}_{LM}(\boldsymbol{A}_i)\mathcal{Y}^*_{LM}(\boldsymbol{A}_j)\quad \textit{including}\ i=j. \tag{7}$$

$\mathfrak{F}_L^M$ and $\mathfrak{W}$ are called e-ν angular distribution functions and e-ν angular correlation functions respectively.

Next we calculate the explicit forms of the e-ν angular distribution functions defined by eq. (7) and we do not formulate the relation between $\mathfrak{F}_L^M$, so that each $\mathfrak{F}_L^M$ is calculated and summed up, then $\mathfrak{W}$ is found.

$\mathfrak{F}_L^M$ are obtaind by an analogous method for the correction factors of ${}_i\beta$-decay or ${}_i\beta$-ray angular distribution functions F_L^M. At first we calculate the matrix elements of the interaction Hamiltonian corresponding to the definite change of the magnetic quantum number of nucleus. We denote them with $H_{\varkappa m}$, where $\varkappa$ and m are the quantum numbers of the emitted electron. $H_{\varkappa m}$ is obtained by extracting the corresponding parts of the concerned nuclear matrix element from the entire matrix element of the interaction Hamiltonian.

In order to do this procedure the following simple relations of the solid harmonics are available for the first forbidden transition.

$$
\begin{aligned}
&\mathfrak{Y}_{10}(\boldsymbol{A})\mathfrak{Y}_{10}(\boldsymbol{B}) = (1/\sqrt{6})\mathfrak{Y}_{20}(\boldsymbol{A},\boldsymbol{B}) + (1/3)(\boldsymbol{A}\cdot\boldsymbol{B}),\\
&\mathfrak{Y}_{1\pm1}(\boldsymbol{A})\mathfrak{Y}_{1\mp1}(\boldsymbol{B}) = (1/2)\{(1/\sqrt{6})\mathfrak{Y}_{20}(\boldsymbol{A},\boldsymbol{B}) \pm i\mathfrak{Y}_{10}(\boldsymbol{A}\times\boldsymbol{B}) - (2/3)(\boldsymbol{A}\cdot\boldsymbol{B})\},\\
&\mathfrak{Y}_{1\pm1}(\boldsymbol{A})\mathfrak{Y}_{10}(\boldsymbol{B}) = (1/2)\{(1/\sqrt{2})\mathfrak{Y}_{2\pm1}(\boldsymbol{A},\boldsymbol{B}) \pm i\mathfrak{Y}_{1\pm1}(\boldsymbol{A}\times\boldsymbol{B})\},\\
&\mathfrak{Y}_{10}(\boldsymbol{A})\mathfrak{Y}_{\pm1}(\boldsymbol{B}) = (1/2)\{(1/\sqrt{2})\mathfrak{Y}_{2\pm1}(\boldsymbol{A},\boldsymbol{B}) \mp i\mathfrak{Y}_{1\pm1}(\boldsymbol{A}\times\boldsymbol{B})\},\\
&\mathfrak{Y}_{1\pm1}(\boldsymbol{A})\mathfrak{Y}_{1\pm1}(\boldsymbol{B}) = (1/2)\mathfrak{Y}_{2\pm2}(\boldsymbol{A},\boldsymbol{B}).
\end{aligned}
\tag{8}
$$

The first two equations are already used in the calculation of ${}_i\beta$-ray angular distribution functions $F^0{}_L$ of YM.

Neglecting the common factors the asymptotic wave function of the emitted electron[9] is

$$
\psi = \sum_{\varkappa,m} H_{\varkappa m}\,\psi_{\varkappa m} \tag{9}
$$

$\psi_{\varkappa m}$ are the parts containing $\exp(ikr)$ of the asymptotic forms of the eigenfunctions. We construct $\psi^*\psi$ from eq. (9), sum over all the neutrino states and compare with eqs. (7) and (6), then the $\mathfrak{F}_L^M$ and $\mathfrak{W}$ are obtained.

If we replace $\mathfrak{S}$ with $\mathfrak{S}\int d\Omega_e \int d\Omega_\nu$, $\mathfrak{W}$ become the usual correction factors where the parts with $\varkappa' = \varkappa$ of $\psi^*\psi$ only remain as the results of the orthogonality of $\psi_{\varkappa m}$. In $\mathfrak{F}_L^M$ the cross parts with $\varkappa' \neq \varkappa$ all remain so that the calculation of $\mathfrak{F}_L^M$ and $\mathfrak{W}$ are tedious. To avoid a cumbersome calculation we assume that the neutrino mass is zero, and take that the direction of the emitted neutrino is z-axis, then average on the surface of the nucleus.

The β-ray angular distribution function F_L^M for any nuclear matrix element containing ${}_i\beta$ (Dirac matrix) is obtained from the one for that without ${}_i\beta$ by replacing K (neutrino energy) by $-K$. But in the case of the e-ν angular correlation functions $\mathfrak{F}_L^M$ for that with and without ${}_i\beta$ we replace K by $-K$ if K are in the even power term of $\cos\theta$ and

replace K's sign if K are fulfilled by both conditions that the power of $\cos\theta$ is odd and the power of K is even. In the case of $\mathfrak{F}_K^M$ for the interference between two nuclear matrix elements with and without β, these considerations are not available.

The phases of the nuclear matrix elements are described in YM.

§ 3. Results

Using the method of § 2 and the β-ray interaction Hamiltonian in the Fermi theory

$$H_\beta=\lambda_S S+\lambda_V V+\lambda_T T+\lambda_A A+\lambda_P P,\ \lambda_i \text{ are real}, \tag{10}$$

we obtain the e-ν angular correlation functions.

1) Allowed

a) scalar

$$\mathfrak{W}_{0S}=|\mathfrak{M}(\beta)|^2\{L_0-2\Lambda_1\cos\theta\}. \tag{0S}$$

b) vector

$$\mathfrak{W}_{0V}=|\mathfrak{M}(1)|^2\{L_0+2\Lambda_1\cos\theta\}. \tag{0V}$$

c) tensor

$$\mathfrak{W}_{0T}=|\mathfrak{M}(\beta\sigma)|^2\{L_0+(2/3)\Lambda_1\cos\theta\}. \tag{0T}$$

d) pseudovector

$$\mathfrak{W}_{0A}=|\mathfrak{M}(\sigma)|^2\{L_0-(2/3)\Lambda_1\cos\theta\}. \tag{0A}$$

Interferences do not exist.

2) First forbidden

square terms

a) scalar

$$\begin{aligned}\mathfrak{W}_{1S}=|\mathfrak{M}(\beta\boldsymbol{r})|^2[&\{(1/3)K^2L_0+(2/3)KN_0-(2/3)KL_{12}+2L_1+M_0\}\\&+\{-(2/3)K^2\Lambda_1-(2/3)K\boldsymbol{N}_{11}+(4/3)K\boldsymbol{L}_{12}+(8/3)\boldsymbol{N}_{12}-(4/3)\Lambda_2\\&-(2/3)\mathfrak{M}_1\}\cos\theta+2KL_{12}\cos^2\theta].\end{aligned} \tag{1S}$$

b) vector

$$\begin{aligned}\mathfrak{W}_{1V}=|\mathfrak{M}(\boldsymbol{r})|^2[&\{(1/3)K^2L_0-(2/3)KN_0+(2/3)KL_{12}+2L_1+M_0\}\\&+\{(2/3)K^2\Lambda_1-(2/3)K\boldsymbol{N}_{11}+(4/3)K\boldsymbol{L}_{12}-(8/3)\boldsymbol{N}_{12}+(4/3)\Lambda_2\\&+(2/3)\mathfrak{M}_1\}\cos\theta-2KL_{12}\cos^2\theta]+|\mathfrak{M}(\alpha)|^2\{L_0-(2/3)\Lambda_1\cos\theta\}.\end{aligned} \tag{1V}$$

c) tensor

$$\begin{aligned}\mathfrak{W}_{1T}=|\mathfrak{M}(\beta\sigma\cdot\boldsymbol{r})|^2[&\{(1/9)K^2L_0+(2/3)KN_0+M_0\}\\&+\{-(2/9)K^2\Lambda_1-(2/3)K\boldsymbol{N}_{11}+2\mathfrak{M}_1\}\cos\theta]\\&+|\mathfrak{M}(\beta\alpha)|^2\{L_0+(2/3)\Lambda_1\cos\theta\}\\&+|\mathfrak{M}(\beta\sigma\times\boldsymbol{r})|^2[\{(1/6)K^2L_0-(2/3)KN_0+(1/6)KL_{12}+(1/2)L_1+M_0\}\end{aligned}$$

$$+\{(1/3)K^2A_1+(1/3)KL_{12}-(2/3)KN_{11}-(4/3)N_{12}-(1/3)A_2$$
$$-(2/3)\mathbb{A}_1\}\cos\theta-(1/2)KL_{12}\cos^2\theta]$$
$$+|\mathfrak{M}(B^3_{ij})|^2\{(1/12)K^2L_0+(1/20)KL_{12}+(3/4)L_1\}$$
$$+\{+(1/30)K^2A_1+(1/2)KL_{12}+(3/10)A_2\}\cos\theta-(3/20)KL_{12}\cos^2\theta].$$
(1T)

d) pseudovector

$$\mathfrak{W}_{1A}=|\mathfrak{M}(\boldsymbol{\sigma}\cdot\boldsymbol{r})|^2[\{(1/9)K^2L_0-(2/3)KN_0+M_0\}$$
$$+\{(2/9)K^2A_1-(2/3)KN_{11}-2\mathbb{A}_1\}\cos\theta]$$
$$+|\mathfrak{M}(\gamma_5)|^2\{L_0+2A_1\cos\theta\}$$
$$+|\mathfrak{W}(\boldsymbol{\sigma}\times\boldsymbol{r})|^2[\{(1/6)K^2L_0+(2/3)KN_0-(1/6)KL_{12}+(1/2)L_1+M_0\}$$
$$+\{-(1/3)K^2A_1+(1/3)KL_{12}-(2/3)KN_{11}+(4/3)N_{12}+(1/3)A_2$$
$$+(2/3)\mathbb{A}_1\}\cos\theta+(1/2)KL_{12}\cos^2\theta]$$
$$+|\mathfrak{M}(B_{ij})|^2[\{(1/12)K^2L_0-(1/20)KL_{12}+(3/4)L_1\}$$
$$+\{-(1/30)K^2A_1+(1/2)KL_{12}-(3/10)A_2\}\cos\theta+(3/20)KL_{12}\cos^2\theta].$$ (1A)

e) pseudoscalar

$$\mathfrak{W}_{1P}=|\mathfrak{M}(\beta\gamma_5)|^2\{L_0-2A_1\cos\theta\}.$$ (1P)

interference terms

f) scalar and tensor

$$\mathfrak{W}_{1ST}=\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c.c}\}[\{L_1-M_0\}$$
$$+(2/3)\{KL_{12}+KN_{11}-N_{12}+\mathbb{A}_1-A_2\}\cos\theta]$$
$$-\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\alpha})+\text{c.c.}\}[\{(1/3)KL_0+N_0\}$$
$$+\{-(2/3)KA_1+(4/3)L_{12}+(1/3)N_{11}\}\cos\theta].$$ (1ST)

g) scalar and pseudovector

$$\mathfrak{W}_{1SA}=-\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}[\{(2/3)KN_0^-+(1/3)KL_{12}^-+L_1^-+M_0^-\}$$
$$+\{-(2/3)KL_{12}^--(2/3)KN_{11}^-+2N_{12}^-\}\cos\theta-KL_{12}^-\cos^2\theta].$$ (1SA)

h) vector

$$\mathfrak{W}_{1VV}=\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\alpha)+\text{c. c.}\}[\{(1/3)KL_0-N_0\}$$
$$+\{(2/3)KA_1+(4/3)L_{12}+(1/3)N_{11}\}\cos\theta].$$ (1VV)

i) vector and tensor

$$\mathfrak{W}_{1VT}=\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\beta\alpha)+\text{c.c.}\}[\{(1/3)KL_0^--N_0^-\}$$

$$+\{-(4/3)\boldsymbol{L}_{12}^{-}-(1/3)\boldsymbol{N}_{11}^{-}\}\cos\theta]$$
$$+\{i\mathfrak{M}^{*}(\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}[\{(2/3)KN_{0}^{-}+(1/3)KL_{12}^{-}-L_{1}^{-}-M_{0}^{-}\}$$
$$+\{(2/3)K\boldsymbol{L}_{12}^{-}+(2/3)K\boldsymbol{N}_{11}^{-}+2\boldsymbol{N}_{12}^{-}\}\cos\theta-KL_{12}^{-}\cos^{2}\theta]$$
$$-\{\mathfrak{M}^{*}(\alpha)\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}[\{(1/3)KL_{0}^{-}-N_{0}^{-}\}$$
$$+\{-(2/3)\boldsymbol{L}_{12}^{-}+(1/3)\boldsymbol{N}_{11}^{-}\}\cos\theta]$$
$$+\{\mathfrak{M}^{*}(\alpha)\mathfrak{M}(\beta\alpha)+\text{c. c.}\}L_{0}^{-}. \qquad (1VT)$$

j) vector and pseudovector

$$\mathfrak{W}_{1VA}=\{i\mathfrak{M}^{*}(\boldsymbol{r})\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c.c.}\}[\{L_{1}-M_{0}\}$$
$$+(2/3)\{K\boldsymbol{L}_{12}+K\boldsymbol{N}_{11}+\boldsymbol{N}_{12}-\mathbb{M}_{1}+\Lambda_{2}\}\cos\theta]$$
$$+\{\mathfrak{M}^{*}(\alpha)\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}[\{(1/3)KL_{0}+N_{0}\}$$
$$+\{-(2/3)K\Lambda_{1}+(2/3)\boldsymbol{L}_{12}-(1/3)\boldsymbol{N}_{11}\}\cos\theta]. \qquad (1VA)$$

k) tensor

$$\mathfrak{W}_{1TT}=-\{\mathfrak{M}^{*}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})\mathfrak{M}(\beta\alpha)+\text{c. c.}\}[\{(1/3)KL_{0}-N_{0}\}$$
$$+\{(2/3)K\Lambda_{1}+(2/3)\boldsymbol{L}_{12}-(1/3)\boldsymbol{N}_{11}\}\cos\theta]. \qquad (1TT)$$

l) tensor and pseudoscalar

$$\mathfrak{W}_{1TP}=-\{i\mathfrak{M}^{*}(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r})\mathfrak{M}(\beta\gamma_{5})+\text{c.c.}\}[\{(1/3)KL_{0}+N_{0}\}$$
$$-\{(2/3)K\Lambda_{1}+\boldsymbol{N}_{11}\}\cos\theta]. \qquad (1TP)$$

m) pseudovector

$$\mathfrak{W}_{1AA}=-\{i\mathfrak{M}^{*}(\gamma_{5})\mathfrak{M}(\boldsymbol{\sigma}\cdot\boldsymbol{r})+\text{c. c.}\}[\{(1/3)KL_{0}-N_{0}\}$$
$$+\{(2/3)K\Lambda_{1}-\boldsymbol{N}_{11}\}\cos\theta]. \qquad (1AA)$$

n) pseudovector and pseudoscalar

$$\mathfrak{W}_{1AP}=\{i\mathfrak{M}^{*}(\boldsymbol{\sigma}\cdot\boldsymbol{r})\mathfrak{M}(\beta\gamma_{5})+\text{c. c.}\}[\{(1/3)KL_{0}^{-}-N_{0}^{-}\}$$
$$+\boldsymbol{N}_{11}^{-}\cos\theta]+\{\mathfrak{M}^{*}(\gamma_{5})\mathfrak{M}(\beta\gamma_{5})+\text{c.c.}\}L_{0}^{-}. \qquad (1AP)$$

The L_i, M_i and N_i functions are given by Konopinski and Uhlenbeck[10]. The L_i^{-}, M_i^{-} and N_i^{-} functions are given by Smith[11]. The other functions have to be introduced as follows. The arrow in each case is an approximation good for $\alpha Z \ll 1$. It should be noted that the suffixes of f and g are quantum numbers $\varkappa$.

$$\boldsymbol{L}_{12}=\left(\frac{p^{2}}{2\pi}F\right)^{-1}\frac{f_{1}f_{2}\sin(\delta_{1}-\delta_{2})+g_{-1}g_{-2}\sin(\delta_{-1}-\delta_{-2})}{4\pi\rho}\rightarrow\frac{p}{3}$$

$$\boldsymbol{N}_{11}=\left(\frac{p^{2}}{2\pi}F\right)^{-1}\frac{(f_{-1}f_{1}-g_{-1}g_{1})\sin(\delta_{-1}-\delta_{1})}{4\pi\rho}\rightarrow-\frac{p}{lV}\cdot\frac{\alpha Z}{2\rho}-\frac{p}{3},$$

$$N_{12}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_{-1}g_{-2}\sin(\delta_{-1}-\delta_{-2})-g_1f_2\sin(\delta_1-\delta_2)}{4\pi\rho^2}\to-\frac{p}{3}\frac{\alpha Z}{2\rho}-\frac{p^3}{9W}$$

$$L_{12}^{-}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_1f_2\sin(\delta_1-\delta_2)-g_{-1}g_{-2}\sin(\delta_{-1}-\delta_{-2})}{4\pi\rho}\to-\frac{p}{3W},$$

$$N_{11}^{-}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{(f_{-1}f_1+g_{-1}g_1)\sin(\delta_{-1}-\delta_1)}{4\pi\rho}\to\frac{p}{3W},$$

$$N_{12}^{-}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_{-1}g_{-2}\sin(\delta_{-1}-\delta_{-2})+g_1f_2\sin(\delta_1-\delta_2)}{4\pi\rho^2}\to-\frac{p}{3W}\frac{\alpha Z}{2\rho},$$

$$L_{12}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_{-1}f_2\cos(\delta_{-1}-\delta_2)-f_1g_{-2}\cos(\delta_1-\delta_{-2})}{4\pi\rho}\to-\frac{p^2}{3W},$$

$$L_{12}^{-}=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_{-1}f_2\cos(\delta_{-1}-\delta_2)+f_1g_{-2}\cos(\delta_1-\delta_{-2})}{4\pi\rho}\to 0,$$

$$A_1=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_{-1}f_1\sin(\delta_{-1}-\delta_1)}{4\pi}\to\frac{p}{2W},$$

$$A_2=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{g_{-2}f_2\sin(\delta_{-2}-\delta_2)}{4\pi\rho^2}\to\frac{p^3}{18W},$$

$$M_1=\left(\frac{p^2}{2\pi}F\right)^{-1}\frac{f_{-1}g_1\sin(\delta_{-1}-\delta_1)}{4\pi\rho^2}\to-\frac{p}{2W}\frac{\alpha^2Z^2}{4\rho^2}-\frac{p}{3}\frac{\alpha Z}{2\rho}-\frac{p^3}{18W}.$$

Therefore in the approximation $\alpha Z\ll 1$ f), g), ...n) become :

f′) scalar and tensor

$$\mathfrak{W}_{1ST}=\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}\left\{-\frac{\alpha^2Z^2}{4\rho^2}\left(1+\frac{p}{3W}\cos\theta\right)-\frac{\alpha Z}{3\rho}\frac{p}{W}(p+K\cos\theta)\right\}$$

$$-\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\alpha})+\text{c. c.}\}$$

$$\times\left\{-\frac{\alpha Z}{2\rho}\left(1+\frac{p}{3W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)-\frac{p}{3W}(p+K\cos\theta)\right\},\quad(1\text{ST})'$$

g′) scalar and pseudovector

$$\mathfrak{W}_{1SA}=-\{i\mathfrak{M}^*(\beta\boldsymbol{r})\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}\left\{\frac{\alpha^2Z^2}{4\rho^2}\frac{1}{W}-\frac{\alpha Z}{3\rho}\frac{1}{W}(K+p\cos\theta)\right\},\quad(1\text{SA})'$$

h′) vector

$$\mathfrak{W}_{1VV}=\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\boldsymbol{\alpha})+\text{c. c.}\}$$

$$\times\left\{\frac{\alpha Z}{2\rho}\left(1-\frac{p}{3W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)+\frac{p}{3W}(p+K\cos\theta)\right\},\quad(1\text{VV})'$$

i′) vector and tensor

$$\mathfrak{W}_{1VT}=\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\alpha})+\text{c. c.}\}\left\{\frac{aZ}{2\rho}\frac{1}{W}+\frac{1}{3W}(K+p\cos\theta)\right\}$$

$$+\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c.c.}\}\left\{-\frac{a^2Z^2}{4\rho^2}\frac{1}{W}-\frac{aZ}{3\rho}\frac{1}{W}(K+p\cos\theta)\right\}$$

$$-\{\mathfrak{M}^*(\boldsymbol{\alpha})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c.c.}\}\left\{\frac{aZ}{2\rho}\frac{1}{W}+\frac{1}{3W}(K+p\cos\theta)\right\}$$

$$+\{\mathfrak{M}^*(\boldsymbol{\alpha})\mathfrak{M}(\beta\boldsymbol{\alpha})+\text{c.c.}\}(1/W), \qquad \text{(1VT)}'$$

j′) vector and pseudovector

$$\mathfrak{W}_{1VA}=\{i\mathfrak{M}^*(\boldsymbol{r})\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}\left\{-\frac{a^2Z^2}{4\rho^2}\left(1-\frac{p}{3W}\cos\theta\right)-\frac{aZ}{3\rho}\frac{p}{W}(p+K\cos\theta)\right\}$$

$$+\{\mathfrak{M}^*(\boldsymbol{\alpha})\mathfrak{M}(\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c.c.}\}$$

$$\times\left\{-\frac{aZ}{2\rho}\left(1-\frac{p}{3W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)-\frac{p}{3W}(p+K\cos\theta)\right\}, \qquad \text{(1VA)}'$$

k′) tensor

$$\mathfrak{W}_{1TT}=-\{\mathfrak{M}^*(\beta\boldsymbol{\alpha})\mathfrak{M}(\beta\boldsymbol{\sigma}\times\boldsymbol{r})+\text{c. c.}\}$$

$$\times\left\{\frac{aZ}{2\rho}\left(1+\frac{p}{3W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)+\frac{p}{3W}(p+K\cos\theta)\right\}, \qquad \text{(1TT)}'$$

l′) tensor and pseudoscalar

$$\mathfrak{W}_{1TP}=-\{i\mathfrak{M}^*(\beta\boldsymbol{\sigma}\cdot\boldsymbol{r})\mathfrak{M}(\beta\gamma_5)+\text{c.c.}\}$$

$$\times\left\{-\frac{aZ}{2\rho}\left(1-\frac{p}{W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)-\frac{p}{3W}(p+K\cos\theta)\right\}, \qquad \text{(1TP)}'$$

m′) pseudovector

$$\mathfrak{W}_{1AA}=-\{i\mathfrak{M}^*(\gamma_5)\mathfrak{M}(\boldsymbol{\sigma}\cdot\boldsymbol{r})+\text{c. c.}\}$$

$$\times\left\{\frac{aZ}{2\rho}\left(1+\frac{p}{W}\cos\theta\right)+\frac{1}{3}(K+p\cos\theta)+\frac{p}{3W}(p+K\cos\theta)\right\}, \qquad \text{(1AA)}'$$

n′) pseudovector and pseudoscalar

$$\mathfrak{W}_{1AP}=\{i\mathfrak{M}^*(\boldsymbol{\sigma}\cdot\boldsymbol{r})\mathfrak{M}(\beta\gamma_5)+\text{c.c.}\}\left\{\frac{aZ}{2\rho}\frac{1}{W}+\frac{1}{3W}(K+p\cos\theta)\right\}$$

$$+\{\mathfrak{M}^*(\gamma_5)\mathfrak{M}(\beta\gamma_5)+\text{c.c.}\}(1/W). \qquad \text{(1AP)}'$$

The square terms are as same as given by Greuling and Meeks.[5]

The case where neutrino mass is not zero is handled by the approximation $Z=0$ only for the allowed transition in the Fermi and Majorana theories.

$$\mathfrak{W}_{0S}=|\mathfrak{M}(1)|^2\left\{1-\frac{pq\cos\theta}{WK-\varepsilon m_e m_\nu}\right\},$$

$$\mathfrak{W}_{0V}=|\mathfrak{M}(1)|^2\left\{1+\frac{pq\cos\theta}{WK-\varepsilon m_e m_\nu}\right\},$$

$$\mathfrak{W}_{0T}=|\mathfrak{M}(\beta\sigma)|^2\left\{1+\frac{1}{3}\frac{pq\cos\theta}{WK-\varepsilon m_e m_\nu}\right\},$$

$$\mathfrak{W}_{0A}=|\mathfrak{M}(\sigma)|^2\left\{1-\frac{1}{3}\frac{pq\cos\theta}{WK-\varepsilon m_e m_\nu}\right\},$$

where

$\varepsilon=1$ for Fermi theory,

$\varepsilon=0$ for Majorana theory.

$m_e\equiv 1$, $p=\sqrt{W^2-1}$, $q=\sqrt{K^2-m_\nu^2}$.

The differences observed between both cases is not so significant because of $m_\nu<0.05m_e$.[12]

In spite of the several studies[1] on the determination of the interaction types we take only the well known Fierz condition which excludes the mixtures of S and V or T and A.

§ 4. Comparison with experiment

Scherwin[3] measured the angles between the emitted directions of an electron and a neutrino by the coincidence experiments of the β-ray and the recoil of S^{32} nucleus. The results are in Figs. 1, 2, 3 and 4, where

$H\rho=3150$ is equivalent $W=2.2$,

$H\rho=2150$ is equivalent $W=1.6$.

He assumed that the transition is first forbidden and parity change yes, and calculated $\mathfrak{W}$ with the results of Hamilton. His conclusions are:
if P^{32} has spin 1,

$\int\beta\sigma\times r$; $\mathfrak{W}\approx 1+(4/3)\cos\theta+(1/3)\cos^2\theta$ is preferable,

if P^{32} has spin 0,

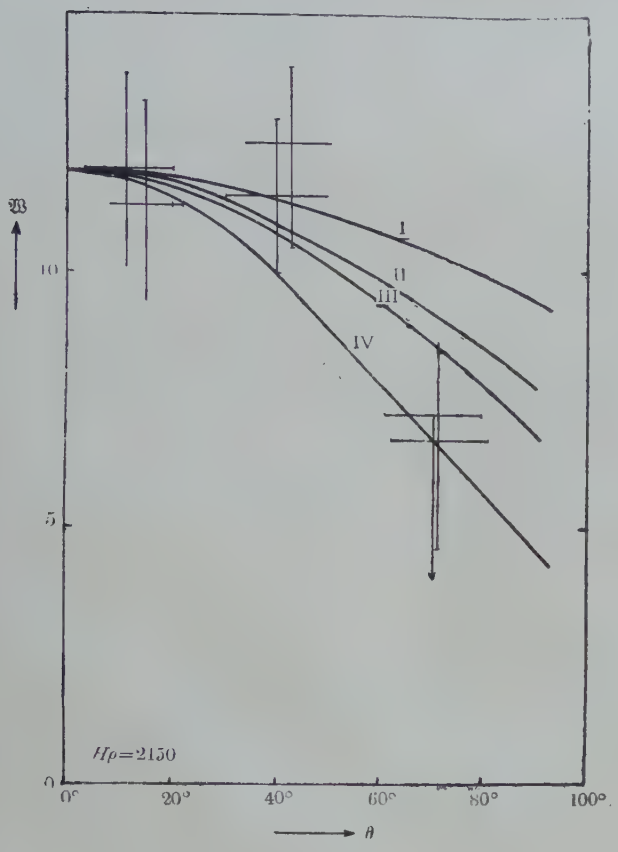
$\int\gamma_5$; $\mathfrak{W}=1+\frac{v}{c}\cos\theta$

and

$\int\sigma\cdot r$; $\mathfrak{W}\approx 1+\cos\theta$ are preferable,

and in these cases β-spectrum shapes are shifted within 4% from the allowed shape.

But it is seemed that we should make the Kurie plot more straight. Also the $Z=0$



Angle between electron and neutrino θ

Fig. 1 Curves of $\mathfrak{W}$ of P^{32} for the first forbidden transition.
$H\rho=2150$.
I: $\int\beta\alpha$,
II: $\int\beta\sigma\times\boldsymbol{r}(Z\neq0)$,
III: $\int\gamma_5$, $\int\sigma\cdot\boldsymbol{r}(Z=0$ and $Z\neq0)$
three curves are not distinguished,
IV: $\int\beta\sigma\times\boldsymbol{r}(Z=0)$
The experimental values are given by Scherwin.[3]

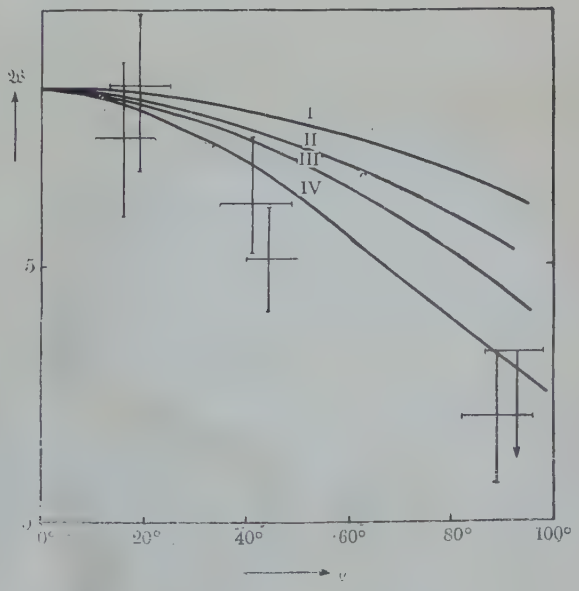


Fig. 2 Curves of $\mathfrak{W}$ of P^{32} for the first forbidden transition.
$H\rho=3150$.
I: $\int\beta\alpha$,
II: $\int\beta\sigma\times\boldsymbol{r}(Z\neq0)$,
III: $\int\gamma_5$, $\int\sigma\cdot\boldsymbol{r}(Z=0$ and $Z\neq0)$,
three curves are not distinguished,
IV: $\int\beta\sigma\times\boldsymbol{r}(Z=0)$.

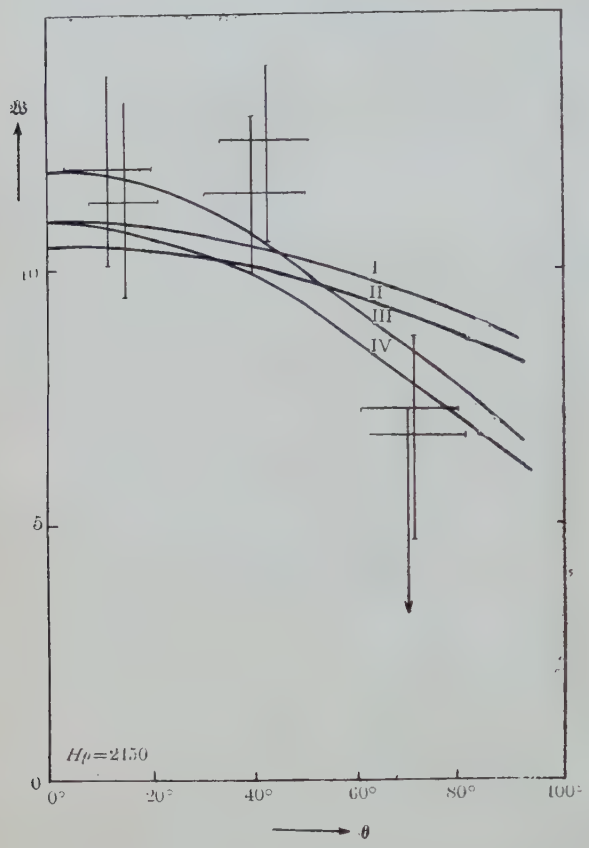


Fig. 3 Curves of $\mathfrak{W}$ of P^{32} for the allowed transition. $H\rho=2150$.
I, II: $\S\beta\tau$,
III, IV: $\S 1$.
An arbitrary normalization is taken in each curve.

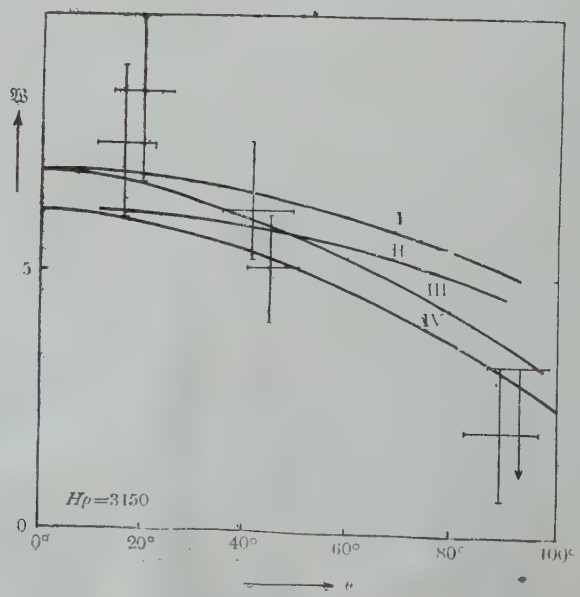


Fig. 4 Curves of $\mathfrak{W}$ of P^{32} for the allowed transition. $H\rho=3150$.
I, II: $\S\beta\sigma$,
III, IV: $\S 1$.
An arbitrary normalization is taken in each curve.

approximation is crude one*. If $\alpha Z \ll 1$ approximation is taken, Scherwin's results change as follows (See Figs. 1 and 2):

$$\int \beta\boldsymbol{\sigma}\times\boldsymbol{r}\,; \quad \mathfrak{W}\approx 1+0.57\cos\theta \quad (W=2.2),$$

$$\mathfrak{W}\approx 1+0.51\cos\theta \quad (W=1.6),$$

$\int \boldsymbol{\sigma}\cdot\boldsymbol{r}$; $\mathfrak{W}$ is almost as same as the case of $Z=0$,

$\int \gamma_5$; $\mathfrak{W}$ does not depend on Z.

We assume that the transition is allowed because of the experimental spectrum. And as in the present situation the experiment is very difficult, it is seemed that the sign of $\cos\theta$ is only probable. In allowed transition T and V which have the positive sign of $\cos\theta$ are drawn with arbitrary normalization (two curves per each type and each energy) in Figs. 3 and 4. According above considerations the curves are well fit to the experimetal data. It is correct that S^{32} has spin 0, and P^{32} has spin 1 probably, so that the interaction of β-decay is tensor. But if P^{32} has spin 0 it is vector. Recently it is indicated that tensor is preferable in the case of He^6.[3]

If the measurement of the e-ν angular correlation is performed in 0–0 transition,[13] we will decide S-T-P or V-T-P as the interaction of β-decay perfectly. We hope a more developement of experimental technique in these region. The analysis of these combinations is proceeding by us.

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* Although $Z\neq 0$ calculation was not published at the time.

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Note added in proof :

We take that the λ_i are real, for example see L. C. Biedenharn and M. E. Rose, Phps. Rev. **83** (1951), 456, where the Wigner type time inversion is used. If the time reversal operator has not a complex conjugate operator as the Pauli type[6], the λ_i could be complex numbers. For the last case we must eliminate $1, H^{-}$ terms of the allowed spectrum by the adjustment of the phase factors of λ_i. But all five invariants can exist simultaneously, so that in the case of P^{2} T and A are mixed, then the tensor interaction is not definite but predominant.

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A Theory of Antiferromagnetism

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A theory of antiferromagnet based on Heisenberg model is developed by a method which is much more directly the analogue of the original Bethe method than Weiss-Li's. By the present method, the calculations can be carried out over all temperatures. Especially, the situation at lower temperatures which is obscure in Weiss-Li's papers is made clear. Thus we can see that an anti-Curie point which Anderson has proposed, does not occur, if we calculate the density matrix of a cluster exactly without the perturbation method. By considering only a small cluster, being characteristic of the Bethe approximation, the long wavelength spin waves, which are important at lower temperatures, are excluded, so that the magnetization of the sublattice is completely saturated at 0°K in just the same way as in the Ising model. It is shown that, in the Bethe approximation, Curie point of antiferromagnet is higher than that of ferromagnet, for the same magnitude of $|J|$.

§ 1. Introduction

The results of neutron beam analysis of various antiferromagnetic substances[1] seem to indicate that we can rely upon the Heisenberg model of superexchange interaction with sufficient accuracy in the theoretical treatment of antiferromagnets. Theories of antiferromagnets using Heisenberg model have been developed by Kubo *et al.*[2] and Li[3] respectively. The former is an expansion method of the density matrix of the entire crystal using the general expansion theorem introduced by Kubo, while the latter is the application of Weiss method[4], in which the internal field parameter is determined self-consistently. In contrast to the Bethe approximation in classical systems, in the Heisenberg model, owing to the non-commutability of the operators, there are several possibilities in introducing the internal field, besides Weiss-Li's one. Here we shall develope the theory based on another alternative of introducing the internal field. Which is the best will be made clear in future, when the rigorous solution is obtained. The present method, however, is much more directly the analogue of the original Bethe method than Weiss-Li's, since the molecular field introduced in the present method acts only on quantum mechanical averages instead of changing the eigenstates of the cluster such as in Weiss-Li's method. At present we can only remark that in every method we must make some assumption when we represent the effect of surroundings by the internal field. The accuracy of our assumption may be of the same order as that of Weiss-Li's. In practical evaluation, however, our method is far feasible to mathematical manipuration than the rests. The

calculation at lower temperatures which is unsurmountable owing to the mathematical complexity in Weiss-Li's, can be carried out in our treatment. Anderson's remarks[7] on anti-Curie point, using the perturbation treatment of the Weiss model, are not appropriate for our case. Indeed, in our treatment there can be found no anti-Curie point, in fair agreement with experiments.

§ 2. The outline of the method

According to the Heisenberg model, the partition function of the antiferromagnetic crystal, with no external magnetic field, can be written;

$$Z=tr.\, e^{-2|J|\beta\sum_{(k,l)}(\boldsymbol{S}_k,\boldsymbol{S}_l)}, \tag{1}$$

where $\boldsymbol{S}_k$ represents the spin operator of the k-th atom, $|J|$ is the superexchange integral, $\beta=1/kT$, $\sum_{(k,l)}$ shall mean that the sum is carried out over all pairs (k,l) which are direct neighbors. The lattices to be considered in this paper are confined to those which can be divided into two sublattices a and b, in which all the nearest neighbors of any site (a- or b-site) are on the other sublattice (b- or a-site). The crystals which break these limitations cannot be ordered antiferromagnetically, and therefore will be excluded in our consideration.

If we take traces of the density matrix (1) about all the spin variables except those of the central atom on the a-site ($\boldsymbol{S}_0$) and of $2n$ neighboring atoms ($\boldsymbol{S}_i$, $i=1, 2.....2n$), the reduced density matrix can be written in the form;

$$e^{-2|J|\beta\sum_{i=1}^{2n}(\boldsymbol{S}_0,\boldsymbol{S}_i)}f(\boldsymbol{S}_0,\boldsymbol{S}_1,\ldots,\boldsymbol{S}_{2n}), \tag{2}$$

where $f(\boldsymbol{S}_0,\boldsymbol{S}_1,\ldots,\boldsymbol{S}_2)$ is a complicated function of the spin variables $\boldsymbol{S}_0$ and $\boldsymbol{S}_i$, expressing the effect of the surrounding, and we can assume that the function can be expanded in power series in $\boldsymbol{S}_0$ and $\boldsymbol{S}_i$. This function is, of course, symmetrical about the $2n$ spin operators of the b-site atoms, but the expansion coefficients of them are, in general, different from that of the central spin. Further reducing (2) about the spin operators except those of the central atom ($\boldsymbol{S}_0$) and of one b-site atom ($\boldsymbol{S}_j$), we get the distribution function of one pair of spins in the following form:

$$\begin{aligned}e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S}_j)}&[1+a(\boldsymbol{S}_0^x+\boldsymbol{S}_0^y+\boldsymbol{S}_0^z)+a'(\boldsymbol{S}_j^x+\boldsymbol{S}_j^y+\boldsymbol{S}_j^z)\\&+b(\boldsymbol{S}_0^x\boldsymbol{S}_j^x+\boldsymbol{S}_0^y\boldsymbol{S}_j^y+\boldsymbol{S}_0^z\boldsymbol{S}_j^z)+c(\boldsymbol{S}_0^x\boldsymbol{S}_j^y+\boldsymbol{S}_0^y\boldsymbol{S}_j^z+\ldots)\\&+c'(\boldsymbol{S}_0^y\boldsymbol{S}_j^x+\boldsymbol{S}_0^z\boldsymbol{S}_j^y+\ldots)+\ldots].\end{aligned}$$

From the symmetrical considerations we can conclude that $a=a'$, $c=c'$, ..., but these relations are insufficient to determine all the expansion coefficients. Therefore, giving up treating this expansion faithfully, we assume that it can be written in the form $e^{\beta\boldsymbol{S}^zH^b}$. Here $\boldsymbol{S}^z$ is the z-component of the resultant b-site spins, and H^b is the internal field acting on the b-site in the z-direction, which is to be determined self-

consistently. Here and henceforth, the symbol H stands for a magnetic field in gauss multiplied by μ_0 where μ_0 is the product of the Bohr magneton and the Lande g-factor. Thus the partition function of the system consisting of the central (a-site) and its nearest-neighbor spins (we shall call the system the a-cluster) is given by

$$Z_a = tr.\, e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S})}\, e^{\beta \boldsymbol{S}^z H^b}. \tag{3}$$

Thus from our standpoint, the density matrix of a-cluster in Bethe approximation consists of two factors. One is $e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S})}$, i. e. the Hamiltonian of the cluster and the other is $e^{\beta \boldsymbol{S}^z H^b}$, the effect of the surroundings is taken as a statistical factor.

When the uniform external field H_0 is acting in the z-direction, we must add $-(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H$ to the Hamiltonian of the cluster, and the partition function of a-cluster is

$$Z_a = tr.\, e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S}) + \beta(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H_0}\, e^{\beta \boldsymbol{S}^z H^b}. \tag{4}$$

Similarly, for the b-cluster

$$Z_b = tr.\, e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S}) + \beta(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H_0}\, e^{\beta \boldsymbol{S}^z H^a}. \tag{5}$$

These partition functions are different from Li's in that $e^{\beta \boldsymbol{S}^z H^b}$ or $e^{\beta \boldsymbol{S}^z H^a}$ is separated from the first exponential factor. Our density matrices are not hermitian, but the trace of them are equal to the hermitized operator

$$\frac{1}{2}\left(e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S}) + \beta(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H_0}\, e^{\beta \boldsymbol{S}^z H^b} + e^{\beta \boldsymbol{S}^z H^b}\, e^{-2|J|(\boldsymbol{S}_0, \boldsymbol{S}) + \beta(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H_0}\right).$$

Following Weiss and Li, we use the representation system in which the superexchange interaction term of the cluster is diagonalized. The basic functions of this representation are (we assume each atom has only one magnetic electron)

$$= \pm \sqrt{\frac{S \pm m + \frac{1}{2}}{2S+1}}\, \theta\, \alpha_0 + \sqrt{\frac{S \mp m + \frac{1}{2}}{2S+1}}\, \theta \beta_0, \tag{6}$$

where α_0 and β_0 are the spin functions of the central atom, $\theta_S^{m \pm \frac{1}{2}}$ are the spin functions of the $2n$ resultant spins, and (6) is the recurrence formula for increasing spin one by one. S is the total spin quantum number of the outer atoms and m is the magnetic quantum number of the cluster. Using (6) as the basis of representation, we get

$$e^{-2|J|\beta(\boldsymbol{S}_0, \boldsymbol{S}) + \beta(\boldsymbol{S}_0^z + \boldsymbol{S}^z)H_0} = \begin{pmatrix} e^{-\beta(|J|S - mH_0)} & 0 \\ 0 & e^{-\beta(-|J|(S+1) - mH_0)} \end{pmatrix}. \tag{7}$$

Since $e^{\beta \boldsymbol{S}^z H^b}$ does not commute with the expression (7), the representation of which is not diagonal, that is,

$$\boldsymbol{S}^z=\begin{pmatrix} m-\dfrac{m}{2S+1}, & -\dfrac{1}{2}\sqrt{1-\dfrac{4m^2}{(2S+1)^2}} \\ -\dfrac{1}{2}\sqrt{1-\dfrac{4m^2}{(2S+1)^2}}, & m+\dfrac{m}{2S+1} \end{pmatrix}. \tag{8}$$

The eigenvalues of the operator (8) are, of course, $m\pm\frac{1}{2}$.

Now, let $\boldsymbol{T}$ be any operator having n eigenvalues T_i, that is,

$$\prod_i(\boldsymbol{T}-T_i)=0$$

and $f(\boldsymbol{T})$ be any function of $\boldsymbol{T}$, then $f(\boldsymbol{T})$ can be reduced to a polynomial of $(n-1)$th degree as follows:

$$f(\boldsymbol{T})=\sum_{i=1}^{n}f(T_i)A_i\prod_{j\neq i}(\boldsymbol{T}-T_j), \tag{9}$$

where A_i is defined by

$$\frac{1}{\prod_{i=1}^{n}(\boldsymbol{T}-T_i)}=\sum_{i=1}^{n}\frac{A_i}{\boldsymbol{T}-T_i}. \tag{10}$$

Making use of the relation (9), we get

$$\begin{aligned} e^{\beta\boldsymbol{S}^zH^b}&=\{e^{\beta H^b\left(m-\frac{1}{2}\right)}-e^{\beta H^b\left(m+\frac{1}{2}\right)}\}\boldsymbol{S}^z+\left\{e^{\beta H^b\left(m-\frac{1}{2}\right)}\left(m+\frac{1}{2}\right)\right. \\ &\qquad\left.-e^{\beta H^b\left(m+\frac{1}{2}\right)}\left(m-\frac{1}{2}\right)\right\}\mathbf{1} \\ &=e^{m\beta H^b}\begin{pmatrix} -\dfrac{2m}{2S+1}\operatorname{sh}\dfrac{\beta H^b}{2}+\operatorname{ch}\dfrac{\beta H^b}{2}, & \cdots\cdots \\ \cdots\cdots, & \dfrac{2m}{2S+1}\operatorname{sh}\dfrac{\beta H^b}{2}+\operatorname{ch}\dfrac{\beta H^b}{2} \end{pmatrix}. \end{aligned} \tag{11}$$

After all,

$$\begin{aligned} &e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})+\beta(\boldsymbol{S}_0{}^z+\boldsymbol{S}^z)H_0}e^{\beta\boldsymbol{S}^zH^b} \\ &=e^{m\beta H^b}\begin{pmatrix} e^{-\beta(|J|S-mH_0)}\left\{-\dfrac{2m}{2S+1}\operatorname{sh}\dfrac{\beta H^b}{2}+\operatorname{ch}\dfrac{\beta H^b}{2}\right\}, & \cdots\cdots \\ \cdots\cdots, & e^{-\beta(-|J|(S+1)-mH_0)}\left\{\dfrac{2m}{2S+1}\operatorname{sh}\dfrac{\beta H^b}{2}+\operatorname{ch}\dfrac{\beta H^b}{2}\right\} \end{pmatrix}. \end{aligned} \tag{12}$$

The non-diagonal elements are not explicitly written down, because they are unnecessary

for our calculations. The non-diagonal element $\sqrt{1-\frac{4m^2}{(2S+1)^2}}$ in (8), has been quite eliminated in the expression (12), which is the reason why our calculation is far simpler than Weiss-Li's.

The density matrix of the cluster is the direct product of the expression (12), and, taking the trace of it, we get the partition function

$$Z_a=\sum_{S=0}^{n}\omega(S)\left\{\sum_{m=-\left(S+\frac{1}{2}\right)}^{S+\frac{1}{2}} e^{m\beta H^b-\beta|J|S+\beta m H_0}\left(-\frac{2m}{2S+1}\mathrm{sh}\frac{\beta H^b}{2}+\mathrm{ch}\frac{\beta H^b}{2}\right)\right.$$

$$\left.+\sum_{m=-\left(S-\frac{1}{2}\right)}^{S-\frac{1}{2}} e^{m\beta H^b+\beta|J|(S+1)+\beta m H_0}\left(\frac{2m}{2S+1}\mathrm{sh}\frac{\beta H^b}{2}+\mathrm{ch}\frac{\beta H^b}{2}\right)\right\}, \quad (13)$$

$$\omega(S)=\frac{(2n)!}{(n+S)!\,(n-S)!}-\frac{(2n)!}{(n+S+1)!\,(n-S-1)!}. \quad (14)$$

(13), of course, contains a characteristic of the lattice type, but it is obtained in the closed form for any crystal.

§ 3. The transition temperature, the specific heat and the susceptibility

From now we shall confine the calculation to the case of a simple cubic lattice. The partition function of the cluster is, from (13) and (14)

$$Z_a=10\,\mathrm{ch}\frac{\beta H_0}{2}+18\,e^{-3|J|}\mathrm{ch}\,\beta\left(H_f^b+\frac{H_0}{2}\right)+6\left(-e^{-3|J|}+e^{2\beta|J|}\right)\times$$

$$\times\,\mathrm{sh}\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{sh}\frac{\beta H_f^b}{2}$$

$$+18\left(e^{-3|J|}+e^{2\beta|J|}\right)\mathrm{ch}\,\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{ch}\frac{\beta H_f^b}{2}+10\,e^{-2\beta|J|}\mathrm{ch}\left(2H_f^b+\frac{H_0}{2}\right)$$

$$+4\left(-e^{-2\beta|J|}+e^{3\beta|J|}\right)\mathrm{sh}\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{sh}\frac{\beta H_f^b}{2}(2+3\mathrm{ch}\,\beta H_f^b)$$

$$+20\left(e^{-2\beta|J|}+e^{3\beta|J|}\right)\mathrm{ch}\,\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{ch}\frac{\beta H_f^b}{2}\,\mathrm{ch}\,\beta H_f^b+2\,e^{-3\beta|J|}\,\mathrm{ch}\,\beta\left(3H_f^b+\frac{H_0}{2}\right)$$

$$+\frac{2}{7}\left(-e^{-3\beta|J|}+e^{4\beta|J|}\right)\mathrm{sh}\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{sh}\frac{\beta H_f^b}{2}\left(-1+16\,\mathrm{ch}\,\beta H_f^b+20\,\mathrm{ch}^2\beta H_f^b\right)$$

$$+2\left(e^{-3\beta|J|}+e^{4\beta|J|}\right)\mathrm{ch}\frac{\beta}{2}(H_f^b-H_0)\,\mathrm{ch}\frac{\beta H_f^b}{2}\left(-1+4\mathrm{ch}^2\beta H_f^b\right), \quad (15)$$

where

$$H_f^b=H^b+H_0, \quad (16)$$

and quite similar for Z_b. Though H_f^b and H_f^a have the physical meaning as the sum of the external and internal fields, here they play a role as a selector of the spin of the outer shell. H_0, also can be regarded as a selector of the central spin. Thus the magnetizations of the central and outer shell atoms are given respectively

$$m_0^a = \frac{\mu_0}{\beta}\left(\frac{\partial}{\partial H_0}\log Z_a\right)_{H_0=0}, \tag{17a}$$

$$m^b = \frac{\mu_0}{\beta}\left(\frac{\partial}{\partial H_f^b}\log Z_a\right)_{H_0=0}. \tag{17b}$$

For the b-cluster

$$m_0^b = \frac{\mu_0}{\beta}\left(\frac{\partial}{\partial H_0}\log Z_b\right)_{H_0=0}, \tag{18a}$$

$$m^a = \frac{\mu_0}{\beta}\left(\frac{\partial}{\partial H_f^a}\log Z_b\right)_{H_0=0}. \tag{18b}$$

Or, one can get the magnetization directly, without evaluating the partition function

$$m_0^a = \mu_0 \bar{\boldsymbol{S}}_0^z = \mu_0 \frac{tr.\, \boldsymbol{S}_0^z e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})+\beta \mathbf{S}_0^z H_0} \cdot e^{\beta \mathbf{S}^z H^b}}{tr.\, e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})+\beta \mathbf{S}_0^z H_0} \cdot e^{\beta \mathbf{S}^z H^b}}, \tag{19a}$$

$$m^b = \mu_0 \bar{\boldsymbol{S}}^z = \mu_0 \frac{tr.\, \boldsymbol{S}^z e^{-|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})+\beta \mathbf{S}_0^z H_0} \cdot e^{\beta \mathbf{S}^z H^b}}{tr.\, e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})+\beta \mathbf{S}_0^z H_0} \cdot e^{\beta \mathbf{S}^z H^b}}. \tag{19b}$$

If the density matrix commutes with $\boldsymbol{S}_0^z$ and $\boldsymbol{S}^z$, the expressions (17), of course, give the same result as (19). But even if it does not commute with $\boldsymbol{S}_0^z$ and $\boldsymbol{S}^z$, both expressions give the same results. The proof will be given in the Appendix.

The requirements of self-consistency are expressed as,

$$m_0^a = \frac{1}{2n} m^a, \qquad m_0^b = \frac{1}{2n} m^b. \tag{20}$$

These equations have two solutions $H^a = H^b$, and $H^a = -H^b$, and the latter is the stable one. From now we shall write $H^a = -H^b = H$. For the simple cubic lattice, the equation (20) is rewritten explicitly

$$\begin{aligned}\mathrm{sh}\,\beta H\big[&(15\, e^{-\beta|J|} - 6\, e^{2\beta|J|}) + (1+4\mathrm{ch}\,\beta H)(6e^{-2\beta|J|} - e^{3\beta|J|}) \\ &+ (-2+4\,\mathrm{ch}\,\beta H + 12\,\mathrm{ch}^2\beta H) e^{-3\beta|J|}\big] = 0.\end{aligned} \tag{21}$$

At higher temperatures (21) has only one real root $H=0$, but when the temperature is lowered below certain point, there appears non-trivial root $H \neq 0$, which makes the free energy minimum. This temperature is interpreted as the Curie point. Therefore the Curie point is determined by

$$15\, e^{-\beta_c|J|} + 30\, e^{-2\beta_c|J|} + 14\, e^{-3\beta_c|J|} - 6\, e^{2\beta_c|J|} - 5\, e^{3\beta_c|J|} = 0. \tag{22}$$

In Table I, the Curie tempratures of various lattices are listed. For comparison the results of Li and Kubo *et al.* and Opechowski are shown. According to our results the two and three-dimensional lattices can become antiferromagnetic.

Table I. The Curie point of different lattices. The numerical values are those of $kT_c/|J|$.

	present authors	Li	Kubo (The fourth approximation)	Opechowski
Linear chain	—	—	1.387	0.313
Quadratic layer	1.582	—	2.073	0.439
Simple cubic	2.618	2.004	2.710	0.654
Body centered cubic	3.636	3.18	3.278	0.613

If we treat the ferromagnet by our model, every lattice has no Curie point. It is concluded quite generally that in treating both ferromagnet and antiferromagnet by the models of Weiss-Li's and ours, the antiferromagnet always has a higher Curie point than the ferromagnet, for the same magnitude of $|J|$. For, in ferromagnet the lowest energy of the cluster is $-JS$, while in antiferromagnet it is $-|J|(S+1)$. For the first, second, ..., excited states antiferromagnet has lower energies than ferromagnet. This is the reason why antiferromagnet has a higher Curie point. In this point, the quantum-mechanical Bethe's method gives quite different results from those of the Ising model. As will be discussed later, at extremely lower temperatures antiferromagnet has larger zero-point energy, and the treatment of only one cluster becomes very poor approximation. For the discussion of the Curie temperatures, however this zero-point energy matters little, i. e. treating one cluster becomes a good approximation, the above conclusion that antiferromagnet has a higher Curie point seems to be correct. In most real substances, the magnitude of $|J|$ is much smaller for antiferromagnet, so the Curie temperatures of which lie lower than that of ferromagnet.

The internal energy E is given by

$$\begin{aligned} E=&\frac{N|J|}{Z_a}\Big[6(1+2\,\mathrm{ch}\,\beta H)(e^{-\beta|J|}-e^{2\beta|J|}) \\ &+6(-1+2\,\mathrm{ch}\,\beta H+4\,\mathrm{ch}^2\beta H)(e^{-2\beta|J|}-e^{3\beta|J|}) \\ &+\frac{12}{7}(-1-4\,\mathrm{ch}\,\beta H+4\,\mathrm{ch}^2\beta H+8\,\mathrm{ch}^3\beta H)(e^{-3\beta|J|}-e^{4\beta|J|})\Big] \\ \equiv&\frac{N|J|}{Z_a}f(H,\beta), \end{aligned} \tag{23}$$

and the specific heat is

$$C_v=-k\beta^2N|J|\frac{1}{Z_a^2}\left\{\left(\frac{\partial f}{\partial\beta}+\frac{\partial f}{\partial H}\frac{\partial H}{\partial\beta}\right)Z_a-f\left(\frac{\partial Z_a}{\partial\beta}+\frac{\partial Z_a}{\partial H}\frac{\partial H}{\partial\beta}\right)\right\}. \tag{24}$$

The Curie point is the branching point of the partition function so that the phase change is of second order. The discontinuity of the specific heat is

$$\Delta C_v = \left| \frac{k\beta^2 N|J|}{Z_a^2} f \frac{\partial Z_a}{\partial H} \frac{\partial H}{\partial \beta} \right|_{\substack{H=0 \\ \beta=\beta_c}} = 3.56\, kN. \tag{25}$$

This is larger than Li's value 1.0 kN.

If we expand the susceptibility above the transition point in terms of β, it coincides with the result of Weiss-Li's. This fact is derived from the general theorem of the density matrix[6]. Consequently the paramagnetic Curie point θ, which is defined as the crossing point of the asymptote of inverse susceptibility at $T \to \infty$ and of the abscissa in the $1/\chi - T$ diagram, coincides in both methods. On the other hand, as can be seen from the Table I, the Curie point is higher in our model, so θ/T_c is 1.15 which is less than 1.50 of Li's. The experimental values are often larger than these. But taking into account of the fact that the experimental values are derived from the inclination of the susceptibility little higher than T_c and that almost all the antiferromagnetic atoms have larger spin values than $S=1/2$, this disagreement is not unsatisfactory.

§ 4. Considerations at lower temperatures

Using our model, the calculations can be carried out exactly to the lower temperatures. The internal field is given by

$$\begin{aligned} \operatorname{ch} \beta H = \frac{1}{6} e^{3\beta|J|} \big[& -(6e^{-2\beta|J|} + e^{-3\beta|J|} - e^{3\beta|J|}) \\ & + (7e^{-6\beta|J|} - 6e^{-5\beta|J|} - 9e^{-4\beta|J|} + 18e^{-3\beta|J|} \\ & + 1 - 12e^{\beta|J|} + 6e^{6\beta|J|})^{\frac{1}{2}} \big]. \end{aligned} \tag{26}$$

In Fig. 1 the temperature variation of the internal field is shown. The field increases monotonically as the temperature is lowered. This implies that the magnetization of the sublattice increases monotonically, and there are no anti-Curie points.

At very lower temperatures the susceptibility in the z-direction varies as

$$\chi_{\parallel} \propto e^{-\text{const}/kT}. \tag{27}$$

This temprature dependence is different from the results of spin wave theory, $\chi_{\parallel} \propto T^2$ and of the classical theory, $\chi_{\parallel} \propto T$.

Near the 0°K, the magnetization of each sublattice is given by

$$m_0 \sim \frac{\mu_0}{2}(1 - \text{const}\cdot e^{-\beta|J|}). \tag{28}$$

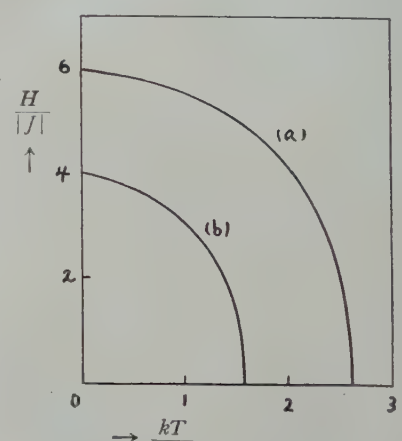

Fig. 1. $H/|J|$ *vs.* $kT/|J|$ diagram. Curve (a) simple cubic lattice, (b) body centered cubic lattice.

This shows that the magnetization of each sublattices saturates completely. The situation is similar to the case of the Ising model. But in the quantum-mechanical antiferromagnetic system, the state in which the each sublattice is completely saturated is not the eigenstate of the Hamiltonian and the above result is incorrect. As Anderson pointed out, the reason for this is that in considering one cluster only, we omit the effect of the long wavelength spin waves, which plays a dominant role at lower temperatures. The action of this long wave is to decrease the magnetization of the sublattice. As we exclude this long waves, we get the saturated magnetization at 0°K. This is the weak point of our model.

Anderson's remarks on the Weiss' method, saying that the exclusion of the long wavelength spinwaves is the cause of the appearance of the anti-Curie point, are difficult to understand. His reasoning is the following[7]: It is only through the medium of thermal excitation overcoming the tendency towards preferential alignment by H^b of the outer atoms of the cluster that the Curie point occurs. As the cluster size increase the states become more closely spaced and thermal effects become more important relative to the perturbation by H^b. Thus the anti-Curie point tends to disappear. On the other hand, our opinion is that the long waves oppose to the tendencies towards preferential alignment of outer atoms and also center atom of the cluster, so the solution of $m_0^a = \frac{1}{2n} m^a$ always exists below the Curie point, that is, an anti-Curie point does not occur, and the exclusion of the long waves works so as to make the sublattice magnetization greater than the real case as shown above.

To the authors the derivation of the anti-Curie point by Anderson seems to be questionable. As shown in Fig. 1, the energy due to the internal field becomes comparable with the exchange interaction energy at the immediately lower temperature than the Curie point. Probably this is also true for Weiss-Li's method. So Anderson's method treating the internal field as the second order perturbation will not give a correct result. In our case, the off-diagonal elements of S^z do not cause the effect, so we cannot discuss the perturbation treatment in the same way as Anderson does. His perturbation treatment, however, utilizes the Weiss' fomulae which are the power series of the molecular field divided by the exchange integral. In our case, if we do not calculate density matrix of the cluster exactly, but expand it in the power series of the molecular field, we can follow the same procedure as Anderson's, and obtain an anti-Curie point. This shows that the perturbation treatment is not allowed for.

In conclusion the authors wish to express their gratitude to Dr. R. Kubo who has kindly criticized the work in letters, and also to Dr. P. W. Anderson who communicated his opinion in detail and commented on our work.

Appendix

Since $\beta(S_0^z + S'^z)H_0$ commutes with $2|J|\beta(S_{0'}, S)$ and $\beta S^z H^b$, we can write the density matrix (4) as

$$e^{-2|J|\beta(S_0,\,S)}\cdot e^{\beta(S_0{}^z,\,S^z)H_0+\beta S^zH^h}$$

$$=e^{-2|J|\beta(S_0,\,S)}\cdot e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}.$$

Further we hermitize it as following :

$$\frac{1}{2}\left\{e^{-2|J|\beta\,(S_0,\,S)}\cdot e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}+e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}\cdot e^{-2|J|\beta(S_0,\,S)}\right\}. \quad \text{(A.1)}$$

If we regard (A.1) as the density matrix, no changes occur in the results of this paper, because the trace of (A.1) is the rame as that of (4).

Let the eigenvalue and the normalized eigenfunction of the density matrix (A.1) be denoted by $e^{-\beta E_n}$ and ψ_n respectively. Then (17a) is given by

$$m_0{}^a=\frac{\mu_0}{\beta}\cdot\frac{\partial}{\partial H_0}\log\sum_n e^{-\beta E_n}=\mu_0\frac{\sum_n\left(-\dfrac{\partial E_n}{\partial H_0}\right)e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}. \quad \text{(A.2)}$$

While, (19a) can be written as

$$m_0{}^a=\mu_0\frac{\sum_n{}'(\psi_n,\,S_0{}^z\psi_n)e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}, \quad \text{(A.2)}$$

where

$$\sum_n e^{-\beta E_n}=tr.\frac{1}{2}\{e^{-2|J|\beta(S_0,\,S)}\cdot e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}+e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}\cdot e^{-2|J|\beta(S_0,\,S)}\}$$

$$=\sum_n\left(\psi_n,\ \frac{1}{2}\left\{\quad\right\}\psi_n\right), \quad \text{(A.4)}$$

and we have uued an abbreviation $\frac{1}{2}\left\{\quad\right\}$. If we differenciate the both sides of (A.4) with H_0,

$$\sum_n\left(-\beta\frac{\partial E_n}{\partial H_0}\right)e^{-\beta E_n}=\sum_n\left(\frac{\partial\psi_n}{\partial H_0},\ \frac{1}{2}\left\{\quad\right\}\psi_n\right)+\sum_n\left(\psi_n,\ \frac{1}{2}\left\{\quad\right\}\frac{\partial\psi_n}{\partial H_0}\right)$$

$$+\sum_n\Big(\psi_n,\ \frac{1}{2}\{e^{-2|J|\beta(S_0,\,S)}\beta S_0{}^z\,e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}$$

$$+\beta S_0{}^z\,e^{\beta S_0{}^zH_0+\beta S^zH_f{}^h}\cdot e^{-2|J|\beta(S_0,\,S)}\}\,\psi_n\Big). \quad \text{(A.5)}$$

$\frac{1}{2}\left\{\quad\right\}$ is hermitic so that the second term of (A.5) is equal to $\sum_n\left(\frac{1}{2}\left\{\quad\right\}\psi_n,\ \frac{\partial\psi_n}{\partial H_0}\right)$ and cancels out with the first term as follows ;

$$\sum_n e^{-\beta E_n}\left\{\left(\frac{\partial \psi_n}{\partial H_0},\ \psi_n\right)+\left(\psi_n,\ \frac{\partial \psi_n}{\partial H_0}\right)\right\}$$
$$=\sum_n e^{-\beta E_n}\frac{\partial}{\partial H_0}(\psi_n,\ \psi_n)=0. \tag{A.6}$$

The third term of (A.5) is

$$tr.\frac{1}{2}\{e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\beta\boldsymbol{S}_0^z e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}_z H_f^h}+\beta\boldsymbol{S}_0^z e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}^z H_f^h}\cdot e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\}$$
$$=tr.\Big[\beta\boldsymbol{S}_0^z\frac{1}{2}\{e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\cdot e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}^z H_f^h}+e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}_z H_f^h}\cdot e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\}$$
$$-\beta\boldsymbol{S}_0^z\frac{1}{2}\cdot e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\cdot e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}^z H_f^h}$$
$$+\frac{1}{2}e^{-2|J|\beta(\boldsymbol{S}_0,\boldsymbol{S})}\beta\boldsymbol{S}_0^z e^{\beta\boldsymbol{S}_0^z H_0+\beta\boldsymbol{S}_z H_f^h}\Big]$$
$$=\sum_n\beta(\psi_n,\ \boldsymbol{S}_0^z\psi_n)e^{-\beta E_n}. \tag{A.7}$$

After all, from (A.5), (A.6) and (A.7), we get

$$\sum_n\Big(-\frac{\partial E_n}{\partial H_0}\Big)e^{-\beta E_n}=\sum(\psi_n,\ \boldsymbol{S}_0^z\psi_n)e^{-\beta E_n}. \tag{A.8}$$

This shows the identities of (17) and (19).

Similar proof can be obtained using the perturbation method.

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On a Correlation between the Electron Assembly and the Lattice Waves in the Crystal

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The Hamiltonian of the system, consisting of the electron assembly and of the lattice ion assembly interacting with each other in the crystal, is studied, by making use of a kind of a canonical transformation, called Bohm-Pines' transformation. Under our interpretation deduced from the results of this treatment, Fröhlich's theory of superconductivity is regarded as reasonable despite of the important objection raised by Wentzel and others. Futhermore our approximate treatment is quite accurate in the one-dimensional case, and this fact possibly seems to suggest that our treatment is tolerably well also in the three dimensional case.

§ 1. Introduction

The problem of the interaction between the electron assembly and the lattice waves in the crystal, on the ground of which Fröhlich[1] and Bardeen[1] once tried to discuss the phenomena of superconductivity, has been often argued recently[2]. At the same time the treatment of Fröhlich and Bardeen has been often criticized[2]. Though some objections have been raised on the perturbation calculation in Fröhlich's theory, no inspection has been made on the starting point of his theory, viz. on the starting Hamiltonian of the system in his calculation. Hoping that this kind of inspection may also be instructive, we have studied this problem by basing upon the method of the collective description of the electron assembly Bohm and Pines[3] recently studied in the case of the co-existing electromagnetic field. In our case their electromagnetic field is replaced by the vibration field of the crystalline lattice or in other words, we deal with the lattice phonon assembly instead of that of photon. As a result, we arrive at a clearer interpretation of the calculation by Fröhlich, and in the light of this interpretation the above-mentioned objection proposed by Wentzel and others seems not necessarily to attack the essential weak point of Fröhlich's theory, which therefore is considered to remain to be reasonable.

In § 2 we discuss the Hamiltonian of the system consisting of the electron assembly and of the lattice ion assembly interacting each other and establish a simpler form for it, on which we have exclusively our arguments in the following paragraphs. This Hamiltonian of a simpler form is formally the same as Fröhlich's, but the interpretation seems to be different from his.

In § 3 we apply Bohm-Pines' transformation, a kind of canonical transformations, to this simplified Hamiltonian, where we pass into the scheme of the second quantization of the electron assembly for the purpose of automatically taking into account the Pauli principle.

In § 4 we perform the perturbation calculation to the Hamiltonian resulting from this transformation and on this basis we examine critically Fröhlich's theory.

In § 5 we apply our method to the one-dimensional problem which can be solved strictly by Tomonaga's theory[4] as shown by Wentzel[2] and show that consequences of our theory coincide just with the strict one as regards the one-dimensional problem. Furthermore we make comments on Wentzel's criticism pointing to the unreasonableness of the perturbation.

In § 6 we conclude our theory summarizing the main points of our considerations.

§ 2. Hamiltonian of the system.

Hamiltonian of the system consisting of the electron assembly and of the lattice-ion assembly interacting with each other is in Hartree units,

$$H=-\frac{1}{2}\sum_i \Delta_i+\sum_{i>j} v(\boldsymbol{r}_i-\boldsymbol{r}_j)+\sum_i\sum_\alpha U(\boldsymbol{r}_i-\boldsymbol{R}_\alpha)$$

$$-\frac{1}{2M}\sum_\alpha \Delta_\alpha+\sum_{\alpha>\beta} V(\boldsymbol{R}_\alpha-\boldsymbol{R}_\beta), \qquad (2\cdot1)$$

where M is the mass of ions, the Roman and the Greek indices refer to electrons and ions respectively and where $\boldsymbol{r}_i$ and $\boldsymbol{R}_\alpha$ represent their cartesian coordinates. Introducing a fictitious potential $\bar{U}$, we may assume the following form:

$$H=-\frac{1}{2}\sum_i \Delta_i+\sum_{i>j} v(\boldsymbol{r}_i-\boldsymbol{r}_j)+\sum_i\sum_\alpha \{U(\boldsymbol{r}_i-\boldsymbol{R}_\alpha)-\bar{U}(\boldsymbol{R}_\alpha)\}$$

$$+\sum_\alpha\{-\frac{1}{2M}\Delta_\alpha+N\bar{U}(R_\alpha)\}+\sum_{\alpha>\beta} V(\boldsymbol{R}_\alpha-\boldsymbol{R}_\beta), \qquad (2\cdot1')$$

where
$$\bar{U}(\boldsymbol{R}_\alpha)=\int U(\boldsymbol{r}-\boldsymbol{R}_\alpha)\rho(\boldsymbol{r})\,d\boldsymbol{r}/\int\rho(\boldsymbol{r})\,d\boldsymbol{r},$$
$$\int\rho(\boldsymbol{r})\,d\boldsymbol{r}=N.$$

The quantity $\rho(\boldsymbol{r})$ represents a total distribution of all electrons, N in total, and therefore the last two summands of $(2\cdot1')$ represent the Hamiltonian of the system of ions interacting one another immersed in the sea of this electron distribution. This part of Hamiltonian possibly leads to the state of motions of the ion system where each ion oscillates about its own equilibrium point; the range of oscillation will be very small on account of the large mass M.

If we write $\boldsymbol{R}_\alpha^{(0)}$ for the equilibrium value* of $\boldsymbol{R}_\alpha$ and use the normal coordinates, we arrive at the following expression of Hamiltonian,

---

* The value $\boldsymbol{R}_\alpha^{(0)}$ is determined from the equation, $\left[\frac{\partial}{\partial \boldsymbol{R}_\alpha}\left\{NU(\boldsymbol{R}_\alpha)+\sum_\eta V(\boldsymbol{R}_\alpha-\boldsymbol{R}_\beta^{(0)})\right\}\right]$ $\boldsymbol{R}_\alpha-\boldsymbol{R}_\alpha^{(0)}=0$. This value can be adjusted by means of adopting an appropriate static electron distribution $\rho(\boldsymbol{r})$, and this does not feel any effect coming from the difference between the static and the dynamic distribution due to the interaction between the electron assembly and the vibration field.

$$H=\sum_i\{-\frac{1}{2}\Delta_i+u(\boldsymbol{r}_i)\}+\sum_{i>j}v(\boldsymbol{r}_i-\boldsymbol{r}_j)$$

$$+\sum_i\sum_k \boldsymbol{q}_k\cdot\boldsymbol{V}_k(\boldsymbol{r}_i)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_i)+\sum_k\Big(-\frac{1}{2M}\frac{\partial}{\partial\boldsymbol{q}_k^*}\cdot\frac{\partial}{\partial\boldsymbol{q}_k}+\frac{M}{2}\boldsymbol{q}_k^*\cdot\Omega_k^2\cdot\boldsymbol{p}_k\Big)\ , \qquad (2\cdot2)$$

where

$$\boldsymbol{q}_k=N^{-\frac{1}{2}}\sum_\alpha \exp(-i\boldsymbol{k}\cdot\boldsymbol{R}_\alpha^{(0)})\cdot(\boldsymbol{R}_\alpha-\boldsymbol{R}_\alpha^{(0)}),$$

$$u(\boldsymbol{r})=\sum\{U(\boldsymbol{r}-\boldsymbol{R}_\alpha^{(D)})-U(\boldsymbol{R}_\alpha^{(D)})\},$$

$$\boldsymbol{V}_k(\boldsymbol{r})=N^{-\frac{1}{2}}\sum_\alpha\exp[-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{R}_\alpha^{(0)})]\boldsymbol{V}\{U(\boldsymbol{r}-\boldsymbol{R}_\alpha^{(0)})-\bar{U}(\boldsymbol{R}_\alpha^{(0)})\},$$

$$\Omega_k^2=\frac{1}{MN}\sum[N^\alpha\boldsymbol{V}_\alpha\boldsymbol{V}_\alpha\bar{U}(\boldsymbol{R}_\alpha^{(0)})+\sum_\beta\{1-\cos(\boldsymbol{k}\cdot(\boldsymbol{R}_\alpha^{(0)}-\boldsymbol{R}_\beta^{(0)}))\}$$
$$\times\nabla_\alpha\nabla_\beta V(\boldsymbol{R}_\alpha^{(0)}-\boldsymbol{R}_\beta^{(0)}).$$

We omit unharmonic terms and terms of higher than the first order of q_k which have the meaning of the collisions of the lattice waves by the intermediation of electrons. We shall neglect the transverse waves only small interacting with electrons and also neglect the static lattice field $u(\boldsymbol{r}_i)$ acting on the electrons and the Coulomb interaction between the electrons in order to simplify the problem. Then we have

$$H=-\frac{1}{2}\sum_i\Delta_i+\sum_k\Big(-\frac{1}{2M}\frac{\partial^2}{\partial q_k^*\partial q_k}+\frac{M\Omega_k^2}{2}q_k^*q_k\Big)$$
$$+\sum_i\sum_k q_kV_k(\boldsymbol{r}_i)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_i), \qquad (2\cdot3)$$

where q_k is now only the coordinate of the longitudinal wave and Ω_k^2 is one component of the original dyadics, Ω_k^2. This Hamiltonian coincides formally with that adopted by Fröhlich. We must now call the readers' attention to the following fact: " the frequency of the lattice wave actually observed is that of the vibration under the dynamical influence of the electrons in motion and not the one, Ω_k, of the vibration in the sea of a static electron distribution $\rho(\boldsymbol{r})$". The difference between the dynamical and the static influences of the electron distribution is prominent on account of the fact that the state of the electron assembly is extremely apt to adapt itself to the displacements of the ions. We can not regard that Ω_k, in the equation of the form (2·3) is the frequency actually observed. Bardeen[6] noted this fact, but seemed not to try to clarify Fröhlich's theory to discuss the objection raised by Wentzel.

§ 3. Bohm-Pines' transformation

The presence of the virtual frequency Ω_k does not please us from the physical viewpoint. We can eliminate this by making use of the canonical transformation of the same sort, by means of which Bohm and Pines[4] studied the collective description of the system consisting of the electron assembly and the electromagnetic field interacting with each other.

If we neglect what Peierls called "Umklapprozesse" in the same way as Fröhlich did, we can rewrite Eq. (2·3)

$$H=-\frac{1}{2}\sum_i \Delta_i+\sum_k\left(-\frac{1}{2M}\frac{\partial}{\partial q_k^*\partial q_k}+\frac{M\Omega_k^2}{2}q_k^*\,q_k\right)$$
$$+\sum_i\sum_k C_k q_k \exp\,(i\boldsymbol{k}\cdot\boldsymbol{r}_i), \qquad (2\cdot3')$$

where the constant C_k is the mean value of $V_k(\boldsymbol{r}_i)$ over the entire crystal. We now pass to the formalism of the second quantization, which affords the facility for introducing automatically the Pauli principle of the electron assembly into the theory. Then we obtain

$$H=\sum_k\frac{\boldsymbol{K}^2}{2}a_k^*\,a_k+\sum_k\frac{\omega_k}{2}(b_k^*\,b_k+b_k\,b_k^*)$$
$$+\sum_k\frac{\Omega_k^2-\omega_k^2}{4\omega_k}(b_k^*\,b_k+b_k\,b_k^*+b_k\,b_{-k}+b_k^*b_{-k}^*) \qquad (3\cdot1)$$
$$+\sum_k\sum_k\frac{C_k}{\sqrt{2M\omega_k}}(a_{\boldsymbol{K}+k}^*\,a_{\boldsymbol{K}}\,b_k+a_{\boldsymbol{K}+k}^*\,a_{\boldsymbol{K}}\,b_{-k}^*),$$

where

$$q_k=q_{-k}^*=\frac{1}{\sqrt{2M\omega_k}}(b_k+b_{-k}^*)$$

and $a_{\boldsymbol{K}}^*$ and $a_{\boldsymbol{K}}$ are the creation and the annihilation operator on the electron state of the wave number $\boldsymbol{K}$. Here we have introduced a fictitious parameter ω_k which value is to be determined appropriately afterwards.

If we pass to the new variables according to the canonical transformation,

$$a_{\boldsymbol{K}}=e^{-iS}A_{\boldsymbol{K}}\,e^{iS},$$
$$a_{\boldsymbol{K}}^*=e^{-iS}A_{\boldsymbol{K}}^*\,e^{iS}, \qquad (3\cdot3)$$
$$b_k=e^{-iS}B_k\,e^{iS},$$
$$b_k^*=e^{-iS}\,B_k^*\,e^{iS},$$

the Hamiltonian H of eq. (3·1) also passes to the new $\mathcal{H}$, as follows.

$$H=e^{-iS}\,\mathcal{H}\,e^{iS}$$
$$=\mathcal{H}_0$$
$$-i\,[S,\ \mathcal{H}_0]+\mathcal{H}_1$$
$$-\frac{1}{2}S,\,[S,\ \mathcal{H}_0]]-i[S,\ \mathcal{H}_1]+\mathcal{H}_2 \qquad (3\cdot4)$$
$$+\cdots\cdots\quad\cdots\cdots\quad\cdots\cdots,$$

where

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2,$$

$$\mathcal{H}_0=\sum_{\boldsymbol{K}}\frac{K^2}{2}A^*_{\boldsymbol{K}}A_{\boldsymbol{K}}+\sum_{\boldsymbol{k}}\frac{\omega_{\boldsymbol{k}}}{2}(B^*_{\boldsymbol{k}}B_{\boldsymbol{k}}+B_{\boldsymbol{k}}B^*_{\boldsymbol{k}}),$$

$$\mathcal{H}_1=\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}}{\sqrt{2M\omega_{\boldsymbol{k}}}}A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}}(B_{\boldsymbol{k}}+B^*_{-\boldsymbol{k}}),$$

$$\mathcal{H}_2=\sum_{\boldsymbol{k}}\frac{\Omega_{\boldsymbol{k}}^2-\omega_{\boldsymbol{k}}^2}{4\omega_{\boldsymbol{k}}}(B^*_{\boldsymbol{k}}B_{\boldsymbol{k}}+B_{\boldsymbol{k}}B^*_{-\boldsymbol{k}}+B^*_{\boldsymbol{k}}B^*_{\boldsymbol{k}}+B_{\boldsymbol{k}}B^*_{-\boldsymbol{k}}).$$

The generator S of this transformation will be so chosen that

$$-i\,[S,\ \mathcal{H}_0]+\mathcal{H}_1=0,$$

so that we get

$$S=i\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}}{\sqrt{2M\omega_{\boldsymbol{k}}}}\left(\frac{A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}}B_{\boldsymbol{k}}}{\boldsymbol{K}\cdot\boldsymbol{k}-\frac{k^2}{2}-\omega_{\boldsymbol{k}}}+\frac{A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}}B_{\boldsymbol{k}}}{\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2}+\omega_{\boldsymbol{k}}}\right). \tag{3·5}$$

We then arrive at the result,

$$H=\sum_{\boldsymbol{K}}\left\{\frac{K^2}{2}+\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_{\boldsymbol{k}}^2}\right\}A^*_{\boldsymbol{K}}A_{\boldsymbol{K}}$$

$$+\sum_{\boldsymbol{K},\boldsymbol{K}'}\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_{\boldsymbol{k}}^2}A^*_{\boldsymbol{K}'-\boldsymbol{k}}A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}}A_{\boldsymbol{K}'}$$

$$+\sum_{\boldsymbol{k}}\frac{\omega_{\boldsymbol{k}}}{2}(B^*_{\boldsymbol{k}}B_{\boldsymbol{k}}+B_{\boldsymbol{k}}B^*_{\boldsymbol{k}})$$

$$+\sum_{\boldsymbol{k}}\left\{\frac{\Omega_{\boldsymbol{k}}^2-\omega_{\boldsymbol{k}}^2}{4\omega_{\boldsymbol{k}}}-\frac{C_{\boldsymbol{k}}^2}{4M\omega_{\boldsymbol{k}}}\sum_{\boldsymbol{K}}\frac{\boldsymbol{k}^2}{\frac{k^4}{4}-(\boldsymbol{K}\cdot\boldsymbol{k}-\omega_{\boldsymbol{k}})^2}A^*_{\boldsymbol{K}}A_{\boldsymbol{K}}\right\}\cdot$$

$$\cdot(B^*_{\boldsymbol{k}}B_{\boldsymbol{k}}+B_{\boldsymbol{k}}B^*_{\boldsymbol{k}}+B_{\boldsymbol{k}}B_{-\boldsymbol{k}}+B^*_{\boldsymbol{k}}B^*_{-\boldsymbol{k}})$$

$$+\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k},\boldsymbol{k}'\neq\boldsymbol{k}}\frac{C_{\boldsymbol{k}}C_{\boldsymbol{k}'}}{4M\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}\left\{\frac{1}{\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2}-\omega_{\boldsymbol{k}}}-\frac{1}{(\boldsymbol{K}+\boldsymbol{k}')\cdot\boldsymbol{k}+\frac{k^2}{2}-\omega_{\boldsymbol{k}}}\right\}\cdot$$

$$\cdot A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}-\boldsymbol{k}'}(B_{\boldsymbol{k}}B_{\boldsymbol{k}'}+B_{\boldsymbol{k}}B^*_{-\boldsymbol{k}'})$$

$$+\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k},\boldsymbol{k}'\neq-\boldsymbol{k}}\left\{\frac{C_{\boldsymbol{k}}C_{\boldsymbol{k}'}}{4M\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}\left\{\frac{1}{\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2}+\omega_{\boldsymbol{k}}}-\frac{1}{(\boldsymbol{K}+\boldsymbol{k}')\cdot\boldsymbol{k}'+\frac{k^2}{2}+\omega_{\boldsymbol{k}}}\right\}\right.$$

$$\left.A^*_{\boldsymbol{K}+\boldsymbol{k}}A_{\boldsymbol{K}-\boldsymbol{k}'}(B^*_{-\boldsymbol{k}}B_{\boldsymbol{k}'}+B^*_{-\boldsymbol{k}}B^*_{-\boldsymbol{k}'})+\cdots\cdots. \tag{3·6}$$

The terms under the sixth line representing the multiple collision between electrons and lattice

waves are quite small of the same order as those which was neglected on the way of deriving the Hamiltonian (2.2), and therefore these also are to be neglected. After neglecting this we determine the parameter ω_k by the equation,*

$$\frac{\Omega_k^2-\omega_k^2}{4\omega_k}-\frac{C_k^2}{4M\omega_k}\sum_K \frac{k^2}{\frac{k^4}{4}-(\boldsymbol{K}\cdot\boldsymbol{k}-\omega_k)^2}A_K^*\,A_K=0. \tag{3·7}$$

Then we arrive at the final form of hamiltonian

$$H=\sum_K\left\{\frac{K^2}{2}+\sum_k\frac{C_k^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_k^2}\right\}A_K^*\,A_K$$

$$+\sum_{K,\,k}\sum_{K'\neq k}\frac{C_k^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_k^2}\cdot A_{K'-k}^*\,A_{K+k}^*\,A_K\,A_{K'} \tag{3·8}$$

$$+\sum_k\frac{\omega_k}{2}(B_k^*\,B_k+B_k\,B_k^*).$$

The terms in the third line are capable to be regarded reasonably as the Hamiltonian of the actual lattice waves, whose frequencies ω_k have the relation (3·7) to the virtual frequencies Ω_k. Thus eq. (3·8) contains no more non-realistic quantity such as Ω_k. The terms in the second line give the immediate interaction energy among electrons themselves, and there contain no operators representing the states of the lattice waves. It is our conclusion that Fröhlich, who adopted the actual frequency in the Hamiltonian of the lattice waves, was not allowed to use the immediate interaction, of the form (2·3), between the electron assembly and the lattice wave assembly, instead he had to adopt the interaction among electrons intermediated by the lattice waves such as the terms in the second line of eq. (3·8).

§ 4. The interpretation of Fröhlich's theory

The Hamiltonian (3·8) is separated into the part from the electron assembly and that from the lattice waves. We have require to study only the former part.

If we now take into account the spin suffices ρ, ρ', we can rewrite this as follows,

$$H_e=\sum_{K,\,\rho}\left\{\frac{K^2}{2}+\sum_k\frac{C_k^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_k^2}\right\}A_{K,\,\rho}^*\,A_{K,\,\rho} \tag{4·1}$$

$$+\sum_{K,\,\rho}\sum_{K'\rho'}\sum_k\frac{C_k^2}{2M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_k^2}A_{K-k,\rho'}^*\,A_{K+k,\,\rho}^*\,A_{K,\,\rho}\,A_{K',\rho'}.$$

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* Therefore ω_k depends on the state of the electron assembly. Strictly speaking, ω_k is not a classical quantity but a quantum one. The error, however, committed in regarding this as a classical quantity would be negligibly small.

If we carry out the perturbation due to the perturbing term in this second line, we obtain the exchange energy, in the first approximation and then the correlation energy due to the polarization from the higher approximations, quite similar to the case of the usual Coulomb interaction. If we stop at the first approximation and neglect the higher terms, we have the approximate value,*

$$E=\sum_{\boldsymbol{K}}\left\{K^2+\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}^2}{M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\omega_{\boldsymbol{k}}^2}\right\}f(\boldsymbol{K}), \tag{4·2}$$

where $f(\boldsymbol{K})$ represents the number of electrons having the wave number $\boldsymbol{K}$.

On the other hand, in manipulating the Hamiltonian of the form (2,3′), Fröhlich obtained the perturbing energy eigenvalue in the second approximation, as follows**

$$E_F=-\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}^2}{M\Omega_{\boldsymbol{k}}}\frac{1}{\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2}+\Omega_{\boldsymbol{k}}}f(\boldsymbol{K})$$

$$-\sum_{\boldsymbol{K}}\sum_{\boldsymbol{k}}\frac{C_{\boldsymbol{k}}^2}{M}\frac{1}{(\boldsymbol{K}\cdot\boldsymbol{k}+\frac{k^2}{2})^2-\Omega_{\boldsymbol{k}}^2}f(\boldsymbol{K}+\boldsymbol{k})f(\boldsymbol{K}). \tag{4·3}$$

The second term in this expression is coincident with that in (4·2), if $\Omega_{\boldsymbol{k}}$ is replaced by $\omega_{\boldsymbol{k}}$. The difference between the virtual frequency, $\Omega_{\boldsymbol{k}}$, and the actual $\omega_{\boldsymbol{k}}$ is important. Fröhlich appeared to give no detailed considerations as to the quantity $\Omega_{\boldsymbol{k}}$ in the expression (2·3′), and, owing to the ambiguity arising from this indifference, his theory incurred the censure by Wentzel and others. If the theory is inspected on the line of our consideration, it is free from objections of this kind, as we shall see in the following paragraph.

§ 5. The one-dimensional problem

According to Tomonaga's theory[4] the electron assembly in the one dimensional space can be substituted by an equivalent phonon assembly. Wentzel[2] studied the one-dimensional problem by utilizing this circumstances. In this problem the Hamiltonian of the system***

** This perturbation is equivalent to the calculation that we put $\omega_{\boldsymbol{k}}=\Omega_{\boldsymbol{k}}$ in our canonical transformation. Then the left-hand side of (3·7) can never vanish. This state of affairs presumably gives rise to important disadvantages, and makes the theory confronted with the objection as pointed out by Wentzel.

*** A renewed notation will be adopted in this paragraphs viz.,

in the preceding paragraphs	K_F/k	$\omega_{\boldsymbol{k}}$	$\Omega_{\boldsymbol{k}}$	$C_{\boldsymbol{k}}^2/\Omega_{\boldsymbol{k}}$
in this paragraph	$\Omega_{\boldsymbol{k}}$	$\omega_{\boldsymbol{k}}'$	$\omega_{\boldsymbol{k}}$	$C_{\boldsymbol{k}}^2\, kL/(\pi M)$

where L is the length of the linear box. Please do not confuse especially the new and the old meaning of $\Omega_{\boldsymbol{k}}$!

* Because there exists no lattice wave having the zero wave number, $k=0$, the direct integration, which appears in the case of Coulomb interaction, does not appear.

$$H=\frac{1}{2}\sum_k(P_k^* P_k+\Omega_k^2 Q_k^* Q_k)$$

$$+\frac{1}{2}\sum_k(p_k^* p_k+\omega_k^2 q_k^* q_k)$$

$$+\frac{1}{2}\sum_k C_k(Q_k^* q_k+Q_k q_k^*) \tag{5·1}$$

can be transformed to

$$H=\frac{1}{2}\sum_k(P_k'^* P_k'+\Omega_k'^2 Q_k'^* Q_k')+\frac{1}{2}\sum_k(p_k'^* p_k'+\omega_k'^2 q_k'^* q_k'), \tag{5·2}$$

by means of the canonical transformation,

$$Q_k=e^{-iS} Q_k' e^{iS},\ q_k=e^{-iS} q_k' e^{iS},$$
$$P_k=e^{-iS} P_k' e^{iS},\ p_k=e^{-iS} p_k' e^{iS}, \tag{5·3}$$
$$S=\sum_k{}' \sin^{-1}\sqrt{\frac{1}{2}\left\{1-\frac{\Omega_k^2-\omega_k^2}{\sqrt{(\Omega_k^2-\omega_k^2)+4C_k^2}}\right\}}\cdot(P_k' q_k'+P_k'^* q_k'^*-Q_k' p_k'-Q_k'^* p_k'),$$

where we used the notations,

$$2\Omega_k'^2=\Omega_k^2+\omega_k^2+\sqrt{(\Omega_k^2-\omega_k^2)^2+4C_k^2},$$
$$2\omega_k'^2=\Omega_k^2+\omega_k^2-\sqrt{(\Omega_k^2-\omega_k^2)^2+4C_k^2}. \tag{5·4}$$

We next treat this problem by means of our method. If we execute the canonical transformation of Bohm-Pines' type,

$$H=\mathcal{H}_0-i[S,\ \mathcal{H}_0]+\mathcal{H}_1-\frac{1}{2}[S,\ [S,\ \mathcal{H}_0]]-i[S,\ \mathcal{H}_1]+\mathcal{H}_2+\cdots,$$

$$\mathcal{H}_0=\frac{1}{2}\sum_k(P_k'^* P_k'+\Omega_k^2 Q_k'^* Q_k')+\frac{1}{2}\sum_k(p_k'^* p_k'+\omega_k^2 q_k'^* q_k'),$$

$$\mathcal{H}_1=\frac{1}{2}\sum_k C_k(Q_k'^* q_k'+Q_k' q_k'^*), \tag{5·5}$$

$$\mathcal{H}_2=\frac{1}{2}\sum_k(\omega_k^2-\omega_k'^2)\ q_k'^* q_k',$$

$$S=\sum_k\frac{C_k}{\Omega_k^2-\omega_k^2}(P_k' q_k'+P_k'^* q_k'^*-Q_k' p_k'-Q_k'^* p_k'^*),$$

we obtain,

$$H=\frac{1}{2}\sum_k(P_k'^* P_k'+\Omega_k^2 Q_k'^* Q_k')+\frac{1}{2}\sum(p_k'^* p_k'+\omega_k'^2 q_k'^* q_k')$$

$$+\frac{1}{2}\sum_k\frac{C_k^2}{\Omega_k^2-\omega_k'^2}Q_k'^* Q_k'+\frac{1}{2}\sum_k\left(\omega_k^2-\omega_k^2-\frac{C_k^2}{\Omega_k^2-\omega_k^2}\right)q_k'^* q_k'+\cdots \tag{5·6}$$

We determine the parameter ω_k' by the equation,

$$\omega_k^2-\omega_k'^2-\frac{C_k^2}{\Omega_k^2-\omega_k'^2}=0, \tag{5·7}$$

and neglect the commutators of higher order, then we attain to the expression,

$$\begin{aligned} H=&\frac{1}{2}\sum_k(P_k'^* P_k'+\Omega_k^2 Q_k'^* Q_k')\\ &+\frac{1}{2}\sum_k(p_k'^* p_k'+\omega_k'^2 q_k'^* q_k')\\ &+\frac{1}{2}\sum_k\frac{C_k^2}{\Omega_k^2-\omega_k'^2}Q_k'^* Q_k'. \end{aligned} \tag{5·8}$$

We can see again in eq(5·8) that the term of the interaction energy appears in the form of the immediate interaction among electrons themselves, $\frac{1}{2}\sum_k C_k^2(\Omega_k^2-\omega_k'^2)^{-1}\cdot Q_k'^* Q_k'$. If we use the quantity

$$\Omega_k'^2=\Omega_k^2+\frac{C_k^2}{\Omega_k^2-\omega_k^2}, \tag{5·9}$$

we can rewrite (5·8) as

$$H=\frac{1}{2}\sum_k(P_k'^* P_k'+\Omega_k'^2 Q_k'^* Q_k')+\frac{1}{2}\sum_k(p_k'^* p_k'+\omega_k'^2 q_k'^* q_k'). \tag{5·8'}$$

The eigenvalues of this Hamiltonian are just*

$$E=\sum_k\Omega_k'(N_k+\frac{1}{2})+\sum_k\omega_k'(n_k+\frac{1}{2}),\quad (N_k, n_k=0, 1, 2, 3,\cdots). \tag{5·10}$$

Here the values of ω_k' and Ω_k' determined by eq.(5·7) and (5·9) are both quite equal to those determined by (5·4). Thus we ought to say that our method gives quite exact consequence in the one-dimensional case. In the three-dimensional case however, we cannot solve exactly the eigenvalue problem of the kind of (5·8), and cannot but rely on the perturbational calculation. For the purpose of inspecting the adequacy of this perturbation, we shall carry out this calculation in the one dimensional case by regarding the last term in eq. (5·8) as a perturbation, and we shall compare the result with the exact (5·10). In the first approximation the value of energy is

$$E=\sum_k\left\{\Omega_k+\frac{C_k}{2\Omega_k(\Omega_k^2-\omega_k'^2)}\right\}\left(N_k+\frac{1}{2}\right)+\sum_k\omega_k'\left(n_k+\frac{1}{2}\right). \tag{5·11}$$

On the other hand the perturbation calculation starting from the Hamiltonian (5·1), where the third term is regarded as a perturbation, leads to the following expression of energy values in the second approximation

* In the one-dimensional case we can also derive the result (5·10) by means of the direct application of Tomonaga's theory to (3·8), where ω_k does not actually depend on the state of the electron assembly on the contrary to the appearance of the expression (3·7).

$$E=\sum_k\left\{\Omega_k+\frac{C_k^2}{2\Omega_k(\Omega_k^2-\omega_k^2)}\right\}\left(N_k+\frac{1}{2}\right)$$
$$+\sum_k\left\{\omega_k-\frac{C_k^2}{2\omega_k(\Omega_k^2-\omega_k^2)}\right\}\left(n_k+\frac{1}{2}\right). \qquad (5\cdot12)$$

The energy values given by (5·12) are good approximation only when $C_k/(\Omega_k\,\omega_k)\ll1$, as Wentzel pointed out, and when the coupling is so strong that $C_k/(\Omega_k\,\omega_k)\gtrsim1$, the perturbation does fail. On the contrary the consequence (5·11) of our perturbation starting from (5·8) remains to be a quite good approximation even in the case of strong coupling, $C_k/(\omega_k\,\Omega_k)\sim1$. Explicitly speaking, the error of our result is only of the order of $\Omega_k\,N_k(\omega_k/\Omega_k)^4$ $(\approx10^{-10}\,\Omega_k\,N_k)$.

It seems to be possible to predict, based upon the consequences of this paragraph, the following two points:

(a) Our approximate Hamiltonian obtained as the result of applying Bohm-Pines' transformation is fairly near the truth (in the one-dimensional case, actually quite exact).

(b) The perturbation calculation starting from our above-mentioned approximate Hamiltonian gives a tolerably good approximation even when the coupling is pretty strong.

These facts give fairly strong support to our method. Actually Fröhlich's calculation, if the interaction between electrons and lattice waves is treated as a perturbation as it stands, would fall into the catastrophe except in the case of the weak coupling as Wentzel pointed out. If we, however, give to Fröhlich's theory the more appropriate interpretation that not the self-energy of electrons resulting from the second order approximation of the perturbation but the exchange energy due to the direct interaction among electrons resulting from Bohm-Pines' transformation does make a contribution to the phenomena of superconductivity, the important objection in the way of Fröhlich's theory is cleared away. It seems that Wentzel's arguments on the instability of the crystal lattice arised from a confusion of the actual and the virtual frequencies.

§ 6. Conclusion

We itemize the main points of our preceding arguments:

1) When we write the total Hamiltonian of the system consisting of the electron assembly and the lattice wave assembly in the form

$$H_e(x)+H_L(q)+H_{int}\,(x,q), \qquad (6\cdot1)$$

we must remember clearly that $H_L(q)$ represents the Hamiltonian of the virtual lattice waves immersed in the static sea of a fixed electron distribution and not of the actual lattice waves.

2) In the result of applying Bohm-Pines' transformation the effects of the interaction term $H_{int}(x,q)$ are firstly to displace the energy spectrum of a simple electron, secondly to vary the frequencies of the lattice waves and lastly to give rise to the term representing the direct interaction energy among electrons. The final form of the transformed Hamiltonian is the sum of three parts, viz.,

$$H_e'(x') + H_L'(q') + H'_{int}(x'). \qquad (6\cdot2)$$

Each term, $H_e'(x')$, $H_L'(q')$ and $H'_{int}(x')$ correspond respectively to the Hamiltonian of the electron assembly with the modified energy spectrum of one electron, to the Hamiltonian of the actual lattice waves and to that of the immediate interaction among electrons. It seems that Fröhlich adopted, on the one hand, the Hamiltonian, $H'_L(q)$ of the actual lattice waves, and he adopted on the other hand, $H_{int}(x, q)$ as the interaction term. This appears us to contain a self-contradiction.

3) We may regard that the exchange energy due to our interaction term accounts for the phenomena of superconductivity. The expression of this energy is formally coincident with the self-energy of electrons in the phonon fields of the virtual lattice vibrations.

4) We have applied our method tentatively to the one-dimensional case, and obtained the consequence identical with that obtained by the strict treatment which is possible in this case. The perturbation treatment of the Hamiltonian (6.1), distinguished from that of the Hamiltonian (6·1) gives a extremely good approximation even in the case of strong coupling in the one-dimensional case. Even though we cannot affirm that this state of affairs remains to be valid also in the three-dimensional case, we are probably able to expect that our method is tolerably effective also in the three-dimensional case.

After all it is fairly to say that Fröhlich's theory is in its main line to be regarded as reasonable, if interpreted properly.

We should like to express our cordial appreciations to Prof. K. Husimi and our colleagues in this institute for their kind suggestions and valuable discussions.

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Note added in proof :

After the calculation in this paper had been done early in the last year (read at the Sendai meeting of the Physical Society of Japan, May, 1952), the publication was so much delayed.

Dr. S. Nakajima recently told us that H. Fröhlich also had carried out the renormalization calculation* similar to ours in § 3 in our present paper.

In Fröhlich's calculation, the damping term was also treated which was ignored by us.

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*) H. Fröhlich, Proc. Roy. Soc., **A215** (1952) 291.

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Miscellanea in Elementary Quantum Mechanics, II

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It is a rather curious fact that the exact quantal treatment of forced oscillation has only recently been discovered to be feasible. There may be a lot of general arguments[1] which make the possiblity of such a treatment very convincing, but I myself find the ground for it in an elementary observation that the wave equation of forced harmonic oscillation, including the case of subharmonic resonance, admits a solution of Gaussian type, *i.e.* of the form: an exponential of a quadratic. On this basis the complete solution and the related problems are analyzed here.

II. The forced harmonic oscillation

§ 1. Solutions of Gaussian type

Let the wave equation be

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{m}{2}\omega(t)^2x^2\psi-f(t)x\psi, \tag{1·1}$$

where f, the impressed force, may be an arbitrary function of time variable t. Moreover, we admit the circular frequency ω also to be variable in time. It is evident, by mere inspection, that the equation can be satisfied by a function of the form:

$$\psi=\exp\left[\frac{i}{2\hbar}\{a(t)x^2+2b(t)x+c(t)\}\right]. \tag{1·2}$$

Inserting this expression into eq. (1·1) we get quadratic expression on both sides, and we have enough degrees of freedom (three coefficients a, b, c) to be able to equate the coefficients of the different power of x separately. We get in fact

$$\begin{cases}\dfrac{1}{m}\dfrac{da}{dt}=-\dfrac{a^2}{m^2}-\omega^2, & (1·3a)\\[2ex] \dfrac{db}{dt}=f-\dfrac{a}{m}b, & (1·3b)\\[2ex] \dfrac{dc}{dt}=i\hbar\dfrac{a}{m}-\dfrac{1}{m}b^2. & (1·3c)\end{cases}$$

If the first equation is solved, it is a simple matter to treat the second, which is linear, and the third equation.

The first equation (1·3a) is of Riccati type, and it is a well known fact that this equation is an outcome of the classical equation of *force free* oscillation :

$$\frac{d^2X}{dt^2}=-\omega(t)^2X. \tag{1·4}$$

In fact the ratio of momentum to coordinate

$$a=m\dot{X}/X \tag{1·5}$$

satisfies the Riccati equation (1·3a). We may assume the classical equation has already been solved. Then the solution of the second equation (1·3b) can be explicitly written down :

$$b=\frac{1}{X}(\text{const}+\int^t fX\,dt), \tag{1·6}$$

and the third equation gives

$$c=i\hbar\log X-\frac{1}{m}\int^t b^2dt. \tag{1·7}$$

The imaginary part gives rise to the amplitude.

The corresponding circumstance one finds also in the classical theory. If we try as usual to solve (1·1) by putting

$$\psi=A\exp(iS/\hbar)\quad(A,\ S:\ \text{real}), \tag{1·8}$$

we get by separating the real and the imaginary parts :

$$-\frac{\partial S}{\partial t}=\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{2}m\omega(t)^2x^2-f(t)x-\frac{2\hbar^2}{2m}\frac{1}{A}\frac{\partial^2A}{\partial x^2},$$

$$-m\frac{\partial A}{\partial t}=\frac{\partial S}{\partial x}\frac{\partial A}{\partial x}+\frac{1}{2}\frac{\partial^2S}{\partial x^2}A.$$

If it is assumed that A is at most a linear function of x, then the so-called quantal potential $\propto\frac{1}{A}\frac{\partial^2A}{\partial x^2}$ vanishes and the first equation becomes exactly equal to the classical Hamilton-Jacobi equation, which is again soluble by a quadratic function of x. One gets a similar set of ordinary differential equations as (1·3). The second equation is compatible with the assumption that A is linear and S is quadratic in x. Actually we have in the foregoing discussion taken A to be dependent only on t.

§ 2. The reduction of the problem to the case $f\equiv 0$

Before proceeding it is convenient to reduce our problem to the simpler case of vanishing f. The following canonical transformation has kindly been supplied by my colleague Mr. T. Taniuti, Kobe University.

The coordinate of the particle may be referred to a moving origin ; let the new

coordinate be

$$x_1 = x - \xi(t), \tag{2·1}$$

where $\xi(t)$ is to be specified later. In terms of x_1 the wave equation (1·1) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\dot{\xi}\frac{\partial\psi}{\partial t_1} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2} + \frac{1}{2}m\omega^2\cdot(x_1+\xi)^2\psi - f\cdot(x_1+\xi)\psi,$$

which is transformed, by putting

$$\psi = \exp(im\dot{\xi}x_1/\hbar)\varphi, \tag{2·2}$$

into the following:

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x_1^2} + \frac{1}{2}m\omega^2 x_1^2\varphi + \{m\ddot{\xi} + m\omega^2\xi - f\}x_1\varphi$$
$$+\left\{-\frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}m\omega^2\xi^2 - f\xi\right\}\varphi. \tag{2·3}$$

The first bracket vanishes if we choose ξ so that

$$m\ddot{\xi} = f - m\omega^2\xi \tag{2·4}$$

i. e. the classical equation. The second bracket, which is a function of t only, may be eliminated by introducing a further phase factor:

$$\varphi = \chi\cdot\exp\left(\frac{i}{\hbar}\int_0^t\left\{\frac{1}{2}m\dot{\xi}^2 - \frac{1}{2}m\omega^2\xi^2 + f\xi\right\}dt\right). \tag{2·5}$$

$\chi(x,t)$ will then satisfy the wave equation of force-free oscillator. Thus our original wave function assumes the following form:

$$\psi(x,t) = \chi(x-\xi(t),t)\exp\left\{im\dot{\xi}(x-\xi)/\hbar + \frac{i}{\hbar}\int_0^t L\,dt\right\} \tag{2·6}$$

where L is the classical Lagrangian:

$$L = \frac{1}{2}m\dot{\xi}^2 - \frac{1}{2}m\omega^2\xi^2 + f\xi \tag{2·7}$$

ξ may, without loss of generality, be assumed to be that solution of the classical equation of motion, which initially vanishes together with its derivative $\dot{\xi}$. Then the extra phase factor in (2·6) vanishes at the initial epoch.

§ 3. The treatment of the case $f=0$

The solution obtained in § 1 is only a special solution of the wave equation. However, we can obtain from it a fundamental solution, the so-called transformation function, by suitable choice of integration constants. The transformation function U is defined by

$$\lim_{t\to t_0+0} U(x_1, t \mid x_0, t_0) = \delta(x-x_0), \tag{3·1}$$

and is given by

$$U_f(x, t \mid x_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\{im(x-x_0)^2/2\hbar(t-t_0)\} \tag{3·2}$$

for a free particle. In this section we try to construct the transformation function for the case $f \equiv 0$ from the solution of Gaussian type of § 1, by imitating U_f for $t \sim t_0$.

To imitate the behavior of the coefficient of x^2, it is necessary to choose X as that solution of (1·4) which is initially vanishing as $t-t_0$; we set

$$X=0, \quad \dot{X}=1 \quad \text{for} \quad t=t_0. \tag{3·3}$$

Next, by comparing the coefficients of x, we have to set the integration constant of b in (1·6) in such a way that

$$b=-mx_0/X. \tag{3·4}$$

Finally we deal with the integral $\int dt/X^2$ in (1·7). Let Y be a second solution of the classical equation (1·4) with the initial values

$$Y=1, \quad \dot{Y}=0 \quad \text{for} \quad t=t_0. \tag{3·5}$$

Then, as well known, we have

$$\dot{X}Y-X\dot{Y}=1, \quad Y/X=-\int dt/X^2. \tag{3·6}$$

Thus we may put

$$c=i\hbar \log X + mx_0^2 Y/X + \text{const.} \tag{3·7}$$

Summarizing the above results we finally obtain the required function:

$$U(x, t|x_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar X}} \exp\left\{\frac{im}{2\hbar X}(\dot{X}x^2-2xx_0+Yx_0^2)\right\}. \tag{3·8}$$

One has to note that $\dot{X}=+O(t-t_0)^2$, $Y=1+O(t-t_0)^2$ as $t\to t_0$.

The exponent of (3·8) is essentially the Hamilton principal function, as noted at the end of § 1, and, as such, may be obtained most easily. In fact we have for a classical motion $x(t)$

$$S=\int_{t_0}^{t}\left\{\frac{m}{2}\dot{x}^2-\frac{m}{2}\omega^2x^2\right\}dt=\frac{m}{2}(x\dot{x}-x_0\dot{x}_0)$$

and we have to set

$$\left.\begin{aligned} x&=x_0Y+\dot{x}_0X \\ \dot{x}&=x_0\dot{Y}+\dot{x}_0\dot{X} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \dot{x}_0&=(x-x_0Y)/X \\ \dot{x}&=(x\dot{X}-x_0)/X \end{aligned}\right\},$$

so that

$$S=\frac{m}{2X}\{\dot{X}x^2-2xx_0+Yx_0^2\}.$$

There is an apparent dissymmetry between the coefficients of x^2 and x_0^2. To clarify this point we observe the combination :

$$X(t,t')\equiv X(t)\,Y(t')-X(t')\,Y(t). \tag{3·9}$$

It, as function of the first argument t, satisfies the classical equation (1·4) as much as $X(t)$, $Y(t)$ do, and has the initial values : $X(t',t')=0$, $\dfrac{\partial X(t',t')}{\partial t}=1$. Since it is defined completely by these properties, $X(t,t')$ does not depend on t_0. If $t'=t_0$, it reduces to $X(t)$:

$$X(t,t_0)=X(t). \tag{3·10}$$

Likewise one sees that

$$\left(\frac{\partial X(t,t')}{\partial t'}\right)_{t'=t_0}=-Y(t).$$

Thus we have

$$\dot{X}(t)=\frac{\partial X(t,t_0)}{\partial t},\quad Y(t)=-\frac{\partial X(t,t_0)}{\partial t_0} \tag{3·11}$$

and a symmetry really exists between the coefficients of x^2 and x_0^2. (3·9) may now be written

$$X(t,t')=-X(t,t_0)\frac{\partial X(t',t_0)}{\partial t_0}+X(t',t_0)\frac{\partial X(t,t_0)}{\partial t_0}. \tag{3·12}$$

It holds irrespective of the order of time epochs t, t', t_0. This law of composition will be essential if we want to verify directly the law of composition for the transformation functien (3·8) :

$$\int_{-\infty}^{\infty}U(x,t|x_1,t_1)\,U(x_1,t_1|x_0,t_0)\,dx_1=U(x,t|x_0,t_0). \tag{3·13}$$

It is further easy to verify the unitary character of U :

$$\int_{-\infty}^{\infty}U(x,t|x'_0,t_0)^*U(x,t|x_0,t_0)\,dx=\delta(x_0-x_0'). \tag{3·14}$$

It is usual on the basis of this relation to set

$$U(x,t\mid x_0',t_0)^*=U^{-1}(x_0',t_0\mid x,t), \tag{3·15}$$

in the sense of matrix theory. But in view of (3·13), which should hold for any time order t, t_1, t_0, it is unnecessary to introduce the inverse matrix $U^{-1}(x_0,t_0\mid x,t)$ besides $U(x_0,t_0\mid x,t)$ with $t_0<t$. Hence we may write

$$U(x,t\mid x_0,t_0)^*=U(x_0,t_0\mid x,t). \tag{3·16}$$

This leads to a symmetry of the matrix elements of U. Let us introduce a complete system of orthogonal functions, say the eigenfunctions of the undisturbed harmonic oscillator. The matrix elements of U with respect to this system will show, by virtue of (3·16),

$$U(t, t_0)^*_{nm} = U(t_0, t)_{mn} \tag{3·17}$$

or, taking the squares of absolute values,

$$|U(t, t_0)_{nm}|^2 = |U(t_0, t)_{mn}|^2. \tag{3·18}$$

This symmetry may be interpreted in terms of *transition probabilities* (This might be a misnomer, since the term transition probability appears usually in a special, technical sense.) Let us suppose the circular frequency ω changes in time, starting from a normal value ω_0 at time t_0 and ending again in this value at a later time t but otherwise in an arbitrary manner. Then, through this disturbance, the oscillator will make transitions from its original state. The transition probability from the m-th to the n-th state is then

$$P_{n \leftarrow m}(t, t_0) = |U(t, t_0)_{nm}|^2.$$

Now we consider a second oscillator of a similar structure, but exposed to another disturbance due to the change of ω *reversed* in time course. In this case we designate the transition probabilities by $P(t_0, t)$. Then (3·18) teaches us the symmetry:

$$P_{n \leftarrow m}(t, t_0) = P_{m \leftarrow n}(t_0, t). \tag{3·19}$$

This is a quite general rule, resulting from (3·13) and (3·14) alone. If the Hamiltonian does not involve the time variable explicitly, or if, in any case, there is no physical difference between future and past, then the time order would be irrelevant and we expect the familiar symmetry of transition probabilities: $P_{m \leftarrow n} = P_{n \leftarrow m}$ to be valid for one and the same course of events. A set of conditions for the validity of this symmetry will be the following: (1) the Hamiltonian $H(t)$ is a *real* expression and is symmetric in its temporal behavior: $H(-t) = H(t)$; (2) the complete orthogonal system $\{\psi_n(x)\}$ is all *real*. From the first assumption follows that, if $\psi(t)$ is a solution of the Schrödinger equation, then $\psi^*(-t)$ is another solution; especially, if $\psi(t)$ starts from a *real* function ψ_n, $\psi(0) = \psi_n$, then we have identically $\psi_n^*(-t) = \psi_n(t)$. Now we envisage the transition probability $P_{n \leftarrow n}(t) = |(\psi_m, \psi_n(t))|^2 = |(\psi_m, \psi_n^*(-t))|^2 = |(\psi_m, \psi_n(-t))|^2 = P_{m \leftarrow n}(-t)$, where we have used the reality of ψ_m. This equality reveals the expected time reversal symmetry. Combined with the general rule (3·19), this leads to the symmetry of transition probabilities

$$P_{n \to m}(t) = P_{m \to n}(t) \tag{3·20}$$

as anticipated. I have intensionally repeated the probably well-known arguments for the symmetry of the transition probabilities in order to clarify the conditions for their validity.

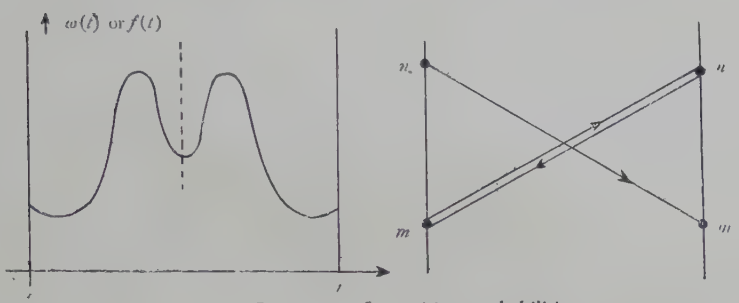

Fig. 1 Symmetry of transition probabilities

We shall learn still another symmetry of transition probabilities in our special case of harmonic oscillators, quite outside the reign of the above theory.

§ 4. The determination of the eigenfunctions for the stationary case

Since the knowledge of the transformation function should yield all the relevant informations concerning the behaviors of the mechanical system, it should also supply the knowledge of the eigenfunctions of the Hamiltonian in the time-independent case $\omega=\text{const}=\omega_0$ $f=0$. In this case X and Y of the preceding section are simply sine and cosine; to be explicit, we have

$$X=\frac{\sin\omega_0(t-t_0)}{\omega_0},\quad \dot{X}=Y=\cos\omega_0(t-t_0), \tag{4·1}$$

and the law of composition (3·12) is merely the addition theorem of trigonometrical functions. Since our transformation function U, (3·8), depends on t only through X, $\dot{X}$, Y, it would be a periodic function of period $2\pi/\omega_0$. Actually the square root $X^{-1/2}$ introduces complexity. In order to determine the sign changes of the square root, we remind us of the analyticity of U over the negative imaginary domain of t, by virtue of parabolic differential equations. We suppose the time variable t has a negative imaginary part and set $\exp(-i\omega_0 t)=z$, $|z|<1$. Then

$$U=\sqrt{\frac{m\omega_0}{\pi\hbar}}\sqrt{\frac{z}{1-z^2}}\exp\left\{-\frac{m\omega_0}{2\hbar(1-z^2)}((1+z^2)(x^2+x_0^2)-4zxx_0)\right\}. \tag{4·2}$$

As t increases, z describes a circle with a radius <1 about the origin and should go through the two Riemann surfaces of $\sqrt{z}$ alternately. Thus $\exp(i\omega_0 t/2)U$ will have a period $2\pi/\omega_0$.

Now from the bilinear expansion

$$U(x,t|x_0,0)=\sum_n \psi_n(x)\psi_n^*(x_0)\exp(-iE_n t/\hbar), \tag{4·3}$$

we can in general get the knowledge of the eigenvalues and the eigenfunctions by taking time averages

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\exp(iEt/\hbar)U(x,t|x_0,0)dt.$$

In our case U is essentially periodic, and we have to regard (4·3) as a Fourier expansion. Each and every eigenvalue E_n should be of the form

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega_0,$$

where n is a certain integer, zero or natural numbers, since we know *a priori* $E_n>0$. We can determine the eigenfunction product $\psi_n(x)\psi^*{}_n(x_0)$ as a Fourier coefficient. This amounts to determine the coefficients of different powers of z in the power series expansion of (4·2), apart from the factor $\sqrt{z}$. In this way we get the well-known generating function for the oscillator eigenfunctions:

$$\frac{1}{\sqrt{\pi}(1-z^2)}\exp\left\{-\frac{(1+z^2)(\bar{x}^2+\bar{x}_0^2)-4zx x_0}{2(1-z^2)}\right\}=\sum_{n=0}^{\infty}z^n\psi_n(x)\psi_n^*(x_0)\qquad(4\cdot4)$$

where $\bar{x}=x/\sqrt{\hbar/m\omega_0}$ etc. We should not like to persue the calculation since the results are too well-known. But we shall indicate the way the more usual linear generating function for Hermite polynomials is related to the above bilinear generating function. Let us write z/x_0 for z and make $x_0\to\infty$ in the resulting equation. The left hand side becomes

$$\frac{1}{\sqrt{\pi}}\exp\left(-\frac{2z^2+x^2-4zx}{2}\right)\cdot\exp\left(-\frac{x_0^2}{2}\right).$$

Thus we may infer

$$\psi_n^*(x_0)\quad N_n^*2^n x_0^n\cdot\exp(-x_0^2/2),\qquad(4\cdot5)$$

for large x_0. The normalization factor N_n may then be determined to $(\sqrt{\pi}\,n!\,2^n)^{-1/2}$, by letting x increase without limit. In this way we get the familiar generating function

$$\exp\left\{-\frac{1}{2}(2z^2+x^2-4zx)\right\}=\sum_{n=1}^{\infty}\sqrt{\frac{\sqrt{\pi}\,2^n}{n!}}z^n\psi_n(x).\qquad(4\cdot6)$$

§ 5. The transition probabilities for the case $f=0$

Now we want to calculate transition probabilities in the sense explained at the end of § 3. For the sake of greater simplicity we shall adopt natural units: $\hbar=1$, $m=1$, and $\omega_0=1$, where ω_0 is that normal frequency for which the eigenfunctions of the preceding section have been defined. In evaluating the transition probabilities

$$P_{mn}=\left|\iint\psi_m^*(x)U(x,t\mid x_0,t_0)\psi_n(x_0)dxdx_0\right|^2,\qquad(5\cdot1)$$

we shall use the method of generating functions. The integral in

$$P(u,v)=\sum u^n v^m P_{mn}$$
$$=\frac{1}{\pi\sqrt{1-u^2}\ \sqrt{1-v^2}}\iiint U(x',t\mid x_0',t_0)^*U(x,t\mid x_0,t_0)$$
$$\times\exp\left\{-\frac{(1+u^2)(x_0^2+x_0'^2)-4ux_0x_0'}{2(1-u^2)}-\frac{(1+v^2)(x^2+x'^2)-4vxx'}{2(1-v^2)}\right\}dx\,dx'dx_0dx_0' \tag{5·2}$$

is a four-fold Gaussian integral and may be evaluated by means of the well-known formula :

$$\int\cdots\int\exp\left\{-\frac{1}{2}\sum_{rs}a_{rs}x_rx_s\right\}dx_1\cdots dx_n=\frac{(2\pi)^{n/2}}{(\det(a_{rs}))^{1/2}}. \tag{5·3}$$

The result runs astonishingly simple :

$$P(u,v)=2^{1/2}\{Q(1-u^2)(1-v^2)+(1+u^2)(1+v^2)-4uv\}^{-1/2}, \tag{5·4}$$

where

$$Q=\frac{1}{2}\{X^2+\dot{X}^2+Y^2+\dot{Y}^2\}. \tag{5·5}$$

This result may be checked in various aspects : first, we have

$$P(u,1)=(1-u)^{-1} \tag{5·6}$$

equivalent to the law of total probability $\sum_m P_{mn}=1$; second, if there is no disturbance at all, $\omega=\omega_0=1$, then we have to put $X=-\dot{Y}=\sin t$, $\dot{X}=Y=\cos t$, and

$$Q=1 \tag{5·7}$$

so that

$$P(u,v)_{Q=1}=(1-uv)^{-1} \tag{5·8}$$

equivalent to $P_{mn}=\delta_{mn}$, *i. e.* no transition ; thirdly we have to observe

$$P(-u,-v)=P(u,v) \tag{5·9}$$

which means :

$$P_{mn}=0 \quad \text{if } m,\ n \text{ are of different parity.} \tag{5·10}$$

This should have been expected on account of the strict selection rule

$$m=n, \quad \text{or } n\pm 2$$

in the words of perturbation theory.

Further an inspection reveals the symmetry

$$P(u,v)=P(v,u) \tag{5·11}$$

which implies the symmetry of transition probabilities :

$$P_{nn}=P_{mn}.$$

This peculiar symmetry lies outsides the scope of the general arguments at the end of

§ 3 and will be discussed later.

In spite of its apparent simplicity, the generating function (5·4) is not manageable if we want to expand it in powers of u and v. Explicit expressions for P_{mn} will be given in the following section. Here we confine ourselves to simpler consequences. First we want to get the average quantum number $\langle m\rangle_n$ of the final states m for a given initial state n. For this purpose we construct

$$\sum_n u^n \sum_m m P_{mn} = \left.\frac{\partial P(u,v)}{\partial v}\right|_{v=1} = 2^{-1}(1-u)^{-2}\{Q(1+u)-(1-u)\}$$

and get immediately

$$\langle m\rangle_n = \sum m P_{mn} = (n+1/2)Q - 1/2, \tag{5·12}$$

or, in terms of energy,

$$\langle E_m\rangle_n = Q\cdot E_n. \tag{5·13}$$

Since $\langle m\rangle \geq 0$, it is a necessary condition for the validity of (5·12), that

$$Q \geq 1. \tag{5·14}$$

This inequality will be discussed at the end of this section.

In the same manner we can proceed to evaluate the average $\langle m(m-1)\rangle_n$, from which we can derive the dispersion

$$(\Delta m)^2_n = \langle m^2\rangle_n - \langle m\rangle^2_n = \frac{1}{2}(Q^2-1)(n^2+n+1) \tag{5·15}$$

or, in terms of energy,

$$(\Delta E_m)^2_n = \langle E^2_n\rangle - \langle E_m\rangle^2_n = \frac{1}{2}(Q^2-1)(E^2_n+3/4). \tag{5·16}$$

For large excitation $Q \gg 1$, the dispersion is of the same order of magnitude as the square of the average energy and one cannot expect any concentration of the probability distribution in the sense of the law of large numbers. Thus the distribution will be very flat and have no conspicuous features, as may be inferred from the special cases of P_{m0}, P_{m1}. These are relatively simply obtained from

$$P(0,v) = 2^{1/2}\{(Q+1)-(Q-1)v^2\}^{-1/2},$$

and

$$\left(\frac{\partial P}{\partial u}\right)_{u=0} = 2^{3/2}\{(Q+1)-(Q-1)v^2\}^{-3/2}v.$$

We have

$$P_{m0} = \left(\frac{2}{Q+1}\right)^{1/2}\binom{-1/2}{m/2}\left(\frac{Q-1}{Q+1}\right)^{m/2} \qquad (m=0,2,4,\cdots), \tag{5·17}$$

$$P_{m1} = \left(\frac{2}{Q+1}\right)^{3/2}\binom{-3/2}{(m-1)/2}\left(\frac{Q-1}{Q+1}\right)^{(m-1)/2} \quad (m=1,3,5,\cdots) \tag{5·18}$$

Since the classical phenomenon $Q \geq 1$ might give an impression of apparent irreversibility of a purely mechanical system, it would not be superfluous here to give an analysis of the inequality.

A simple proof of that inequality has kindly been afforded by Mr. Y. Suzuki of this institute. It suffices to write down just one line:

$$(X+\dot{Y})^2+(\dot{X}-Y)^2=X^2+\dot{X}^2+Y^2+\dot{Y}^2-2(\dot{X}Y-X\dot{Y})\geq 0. \quad (5\cdot 19)$$

The inequality really has a statistical bearing. Let Z be a solution of the equation of motion

$$\ddot{Z}=-\omega(t)^2 Z$$

with initial values Z_0, $\dot{Z}_0$. Then we have

$$Z=\dot{Z}_0 X+Z_0 Y.$$

If we build the "total energy" corresponding to the motion Z,

$$E=\frac{1}{2}(\dot{Z}^2+Z^2)=\frac{1}{2}\{\dot{Z}_0^2\dot{X}^2+Z_0^2\dot{Y}^2+\dot{Z}_0^2X^2+Z_0^2Y^2+2\dot{Z}_0Z_0(XY+\dot{X}\dot{Y})\} \quad (5\cdot 20)$$

and take average over the initial values consistent with the given initial energy $E_0=1/2\cdot(\dot{Z}_0^2+Z_0^2)$, then we have obviously

$$\langle E\rangle=\left\langle\frac{1}{2}(\dot{Z}^2+Z^2)\right\rangle=Q\left\langle\frac{1}{2}(\dot{Z}_0^2+Z_0^2)\right\rangle=Q\cdot E_0. \quad (5\cdot 21)$$

Physically interpreted, $Q \geq 1$ means thus the following:

"Let the string of a pendulum, while in oscillation, be shortened or lengthened in an arbitrary manner and let it return to its initial length after a time. Then we expect, *on the average*, to find the energy of the pendulum increased."

Eq. (5·21) is an exact classical counterpart of the quantal one (5·13). There is also the classical counterpart of the dispersion formula (5·16). If we make use of

$$\langle Z_0^4\rangle=\langle\dot{Z}_0^4\rangle=3E_0^2/2,\quad \langle Z_0^2\dot{Z}_0^2\rangle=E_0^2/2$$

and the identity

$$(\dot{X}^2+X^2)(\dot{Y}^2+Y^2)=(\dot{X}\dot{Y}+XY)^2+(\dot{X}Y-X\dot{Y})^2,$$

we get, in just the same way we derived (5·21),

$$\langle E^2\rangle-\langle E\rangle^2=\frac{1}{2}(Q^2-1)\cdot E_0^2. \quad (5\cdot 22)$$

In comparing the classical (5·21) and the quantal (5·16) it is to be noted that the mean final energy vanishes in the classical case if the initial energy is zero, whereas in the quantal case the mean final energy never vanishes on account of the zero point motion.

From the expression (5·20) of the final energy, which may be written in the form

$$E/E_0=Q+\sqrt{Q^2-1}\ \cos(2\theta+\text{const.}) \tag{5·23}$$

with $Z_0=\sqrt{2E_0}\cos\theta$, $\dot{Z}_0=\sqrt{2E_0}\sin\theta$, the full distribution for the final energy can easily be derived:

$$\frac{d\theta}{2\pi}=\frac{dE}{4\pi\sqrt{E_0^2(Q^2-1)-(E-E_0Q)^2}},\quad Q-\sqrt{Q^2-1}\leq\frac{E}{E_0}\leq Q+\sqrt{Q^2-1}\ . \tag{5·24}$$

As one sees, there is a fundamental difference between the classical distribution (5·24) and the quantal one (5·17) or (5·18) if one starts from relatively smaller values of the initial energy, inspite of their coincidence in mean and dispersion.

§ 5. Adiabatic theorem

We can verify the adiabatic theorem for our quantum oscillator on the basis of the same theorem within the classical theory. Let the angular frequency $\omega(t)$ of our oscillator changes infinitely slowly from an initial value ω_0 to a final value ω_1. It is a well known fact of the classical mechanics that the quantity

$$(\dot{X}^2+\omega^2(t)X^2)/\omega(t) \tag{5a·1}$$

is an adiabatic invariant, where $X(t)$ is any solution of the classical equation. If $X(t)$ and $Y(t)$ are defined as in the preceding section, we have then the two adiabatic invariants (5a·1)$=\omega_0^{-1}$ and

$$(\dot{Y}^2+\omega^2Y^2)/\omega(t)=\omega_0. \tag{5a·2}$$

If we explicitly introduce the angular frequency ω in our oscillator eigenfunctions, the generating function is modified to

$$\sqrt{\frac{\omega}{\pi(1-z^2)}}\exp\left\{-\frac{(1+z^2)(x^2+x'^2)-2zxx'}{2(1-z^2)}\ \omega\right\}. \tag{5a·3}$$

In proving the adiabatic theorem, it is necessary to adapt the eigenfunctions to the instantaneous value of the angular frequency. Thus we have to ask the transition from the state $\omega_0^{1/2}\psi_n(\omega_0^{1/2}x)$ to the state $\omega_1^{1/2}\psi_m(\omega_1^{1/2}x)$. The transition probabilities defined in this sense have kindly been calculated by Mr. T. Dodo according to the method of § 5. The generating function for them is this:

$$P(u,v)=\sum u^nv^mP_{mn}=2^{1/2}\{Q^*(1-u^2)(1-v^2)+(1+u^2)(1+v^2)-4uv\}^{-1/2} \tag{5a·4}$$

where

$$2Q^*=\{\omega_0^2(\omega_1^2X^2+\dot{X}^2)+(\omega_1^2Y^2+\dot{Y}^2)\}/\omega_0\omega_1. \tag{5a·5}$$

For a adiabatic process we have (5a·1) and (5a·3) and hence we get $Q^*=1$, $P(u,v)$

$=(1-uv)^{-1}$, $P_{mn}=\delta_{mn}$, proving the adiabatic theorem for the quantum oscillator.

§ 6. The matrix elements of the transformation function

We have probably been too much hasty in deriving the transition probabilities. We have lost sight in finding the nature and origin of the unexpected symmetry of transition probabilities. We now turn back to get an explicit expressions for the matrix elements

$$U_{mn}(t, t_0)=\int\int \psi_m^*(x)U(x, t \mid x_0, t_0)\psi_n(x_0)\,dx\,dx_0. \tag{6·1}$$

Again we shall make use of generating functions, this time, the linear generating function (4·5). We then have to evaluate

$$\begin{aligned} U(u, v)&=\sum_{n,m}\left\{\frac{\pi 2^{n+m}}{n!\,m!}\right\}^{1/2} u^n v^m U_{mn}(t, t_0) \\ &=\frac{e^{-(u^2+v^2)}}{\sqrt{2\pi i X}}\iint dx\,dx_0 \exp\left[-\frac{1}{2}\left\{\left(1-\frac{i\dot{X}}{X}\right)x^2+\frac{2ixx_0}{X}\right.\right. \\ &\qquad \left.\left.+\left(1-\frac{iY}{X}\right)x_0^2-4vx-4ux_0\right\}\right]. \end{aligned} \tag{6·2}$$

The integral may be computed by the formula

$$\iint dx\,dy \exp\left\{-\frac{1}{2}(ax^2+2bxy+cy^2-2px-2qy)\right\}=\frac{2\pi^2}{\sqrt{ac-b^2}}\exp\left\{\frac{aq^2-2bpq+cp^2}{2(ac-b^2)}\right\}.$$

The result runs

$$U(u, v)=\sqrt{\frac{2\pi}{i\sigma}}\exp\left\{\frac{\rho u^2-4iuv+\rho^* v^2}{\sigma}\right\}, \tag{6·3}$$

where

$$\left.\begin{aligned} \rho&=X-i\dot{X}+iY+\dot{Y}, \qquad |\rho|^2=2(Q-1), \\ \sigma&=X-i\dot{X}-iY-\dot{Y}, \qquad |\sigma|^2=2(Q+1). \end{aligned}\right\} \tag{6·4}$$

In this case the expansion in powers of u and v is immediate.

$$U_{mn}(t, t_0)=\left\{\frac{n!\,m!\,\rho^n\rho^{*m}}{2^{n+m-l}i\sigma^{n+m+1}}\right\}^{1/2}\sum_{l\geq 0}^{\min(m,n)}\frac{\{-2i\sqrt{2/(Q-1)}\}^l}{l!\,[(n-l)/2]!\,[(m-l)/2]!}. \tag{6·5}$$

where l runs over even numbers only, if m, n are even, and over odd numbers only, if m, n are odd. Explicitly,

$$U_{2\mu,2\nu}=\left\{\frac{(2\nu)!\,(2\mu)!\,\rho^{2\nu}\rho^{*2\mu}}{2^{2\nu+2\mu-1}i\sigma^{2\nu+2\mu+1}}\right\}^{1/2}\sum_{\lambda=0,1,\cdots}^{\min(\mu,\nu)}\frac{\{-2^3/(Q-1)\}^\lambda}{(2\lambda)!\,(\nu-\lambda)!\,(\mu-\lambda)!}. \tag{6·6}$$

$$\begin{aligned} U_{2\mu+1,2\nu+1}=&\left\{\frac{(2\nu+1)!\,(2\mu+1)!\,\rho^{2\nu+1}\rho^{*2\mu+1}}{2^{2\nu+2\mu+1}i\sigma^{2\nu+2\mu+3}}\right\}^{1/2}(-2i)\sqrt{\frac{2}{Q-1}} \\ &\sum_{\lambda=0,1,\cdots}^{\min(\mu,\nu)}\frac{\{-2^3/(Q-1)\}^\lambda}{(2\lambda+1)!\,(\nu-\lambda)!\,(\mu-\lambda)!}. \end{aligned} \tag{6·7}$$

The corresponding transition probabilities become now

$$P_{2\mu,2\nu}=\left(\frac{2}{Q+1}\right)^{1/2}\left(\frac{Q-1}{Q+1}\right)^{\mu+\nu}\frac{(2\nu)!\,(2\mu)!}{2^{2\nu+2\mu}}\left\{\sum_{\lambda=0}^{\min(\mu,\nu)}{}'\frac{\{-2^3/(Q-1)\}^{\lambda}}{(2\lambda)!\,(\nu-\lambda)!\,(\mu-\lambda)!}\right\}^2, \tag{6·8}$$

$$P_{2\mu+1,2\nu+1}=\left(\frac{2}{Q+1}\right)^{3/2}\left(\frac{Q-1}{Q+1}\right)^{\mu+\nu}\frac{(2\nu+1)!\,(2\mu+1)!}{2^{2\nu+2\mu}}\left\{\sum_{\lambda=0}^{\min(\mu,\nu)}{}'\frac{\{-2^3/(Q-1)\}^{\lambda}}{(2\lambda+1)!\,(\nu-\lambda)!\,(\mu-\lambda)!}\right\}^2. \tag{6·9}$$

The symmetry property of transition probabilities is seen to originate in a rather intricate structure of U_{mn}. The matrix U_{mn} is neither hermitic nor antihermitic, but consists of three parts: the hermitic part $(\rho^n\rho^{*m})^{1/2}$, the symmetric but complex part $(\sigma^{n+m+1})^{1/2}$ and the remaining real (or purely imaginary) symmetric part. U_{mn} should be of course unitary; a direct proof for this will require the following properties of the parameters ρ and σ. In view of (3·11) we have

$$\left.\begin{aligned}\rho&=\left(1-i\frac{\partial}{\partial t}\right)\left(1-i\frac{\partial}{\partial t_0}\right)X(t,t_0),\\ \sigma&=\left(1-i\frac{\partial}{\partial t}\right)\left(1+i\frac{\partial}{\partial t_0}\right)X(t,t_0),\end{aligned}\right\} \tag{6·10}$$

so that

$$\left.\begin{aligned}\rho(t,t_0)&=-\rho(t_0,t),\\ \sigma(t,t_0)&=-\sigma(t_0,t),\end{aligned}\right\} \tag{6·11}$$

from which we can derive the remarkable symmetry

$$Q(t,t_0)=Q(t_0,t). \tag{6·12}$$

The unitary character of (6·5) is guaranteed if we can verify the universal symmetry (3·17), which appears to hold good by virtue of these properties (6·11), (6·13), except for ambiguity of sign. This disagreeable point may have its origin in our rather uncareful treatment of square root (cf. § 4).

§ 7. The transformation function in the general case; ω, f both variable

According to the result obtained in § 2, the two cases of free ($f=0$) and forced ($f\neq 0$) oscillations are intimately connected. Once a solution of the free case has been obtained, the corresponding solution under arbitrary external disturbances f can immediately be constructed by means of the simple formula (2·6). This procedure may in particular be applied to the transformation function and we get

$$U(x, t|x_0, t_0)=\frac{1}{\sqrt{2\pi i X}}\exp\left\{\frac{i}{2X}(\dot{X}(x-\xi)^2-2(x-\xi)x_0+Yx_0^2)\right.$$
$$\left.+i\dot{\xi}x+i\int L dt-i\xi\dot{\xi}\right\}, \qquad (7\cdot1)$$

where the last two terms in the exponential are functions of t only and we have

$$\int_{t_0}^{t} L dt-\xi\dot{\xi}=\int_{t_0}^{t}\left\{\frac{1}{2}\dot{\xi}^2-\frac{\omega^2}{2}\xi^2+f\xi\right\}dt-\xi\dot{\xi}=\frac{1}{2}\int_{t_0}^{t} f\xi dt-\frac{1}{2}\xi\dot{\xi}, \qquad (7\cdot2)$$

on account of the equation of motion

$$\ddot{\xi}=-\omega^2\xi+f$$

and the initial values

$$\xi(t_0)=0,\ \dot{\xi}(t_0)=0.$$

ξ may be explicitly written down as

$$\xi(t)=\int_{t_0}^{t} X(t, t')f(t')\,dt' \qquad (7\cdot3)$$

in terms of the function $X(t, t')$ defined by (3·9). Then the integral appearing in (7·2) is quadratic in f:

$$\int_{t_0}^{t} f\xi dt=\int_{t_0}^{t} f(t')\,dt'\int_{t_0}^{t'} X(t', t'')f(t'')dt''$$
$$=\int_{t_0}^{t} f(t'')\,dt''\int_{t''}^{t} X(t', t'')f(t')\,dt'=-\int_{t}^{t_0} f(t'')\,dt''\int_{t}^{t''} X(t'', t')f(t')\,dt'.$$

As one sees, it alternates its sign on the interchange of t and t_0. It will be appropriate to introduce the initial time epoch t_0 in the arguments of ξ and to write $\xi(t, t_0)$:

$$\xi(t, t_0)=\int_{t_0}^{t} X(t, t')f(t')\,dt'. \qquad (7\cdot3a)$$

Then

$$\xi(t_0, t)=\int_{t}^{t_0} X(t_0, t')f(t')\,dt'$$

represents a reversed motion. And the above relation may be expressed as:

$$\int_{t_0}^{t}\xi(t', t_0)f(t')\,dt'=-\int_{t}^{t_0}\xi(t', t)f(t')\,dt'. \qquad (7\cdot4)$$

Upon expansion of $(x-\xi)^2$ in (7·1), we are led to treat the following expression:

$$\dot{X}\xi-X\dot{\xi}=\int_{t_0}^{t}\left\{\frac{\partial X(t, t_0)}{\partial t}X(t, t')-\frac{\partial X(t, t')}{\partial t}X(t, t_0)\right\}f(t')\,dt'$$
$$=\int_{t_0}^{t} X(t_0, t')f(t')\,dt'=-\xi(t_0, t). \qquad (7\cdot5)$$

In this way we arrive at the following expression for our transformation function:

$$U(x,t|x_0,t)=\frac{e^{tT}}{\sqrt{2\pi iX(t,t_0)}}\exp\frac{i}{2X}\{\dot{X}x^2-2xx_0+Yx_0^2+2\xi(t_0,t)x+2\xi(t,t_0)x_0\}, \tag{7·6}$$

where

$$T(t,t_0)=-\frac{\xi(t,t_0)\xi(t_0,t)}{2X(t,t_0)}+\frac{1}{2}\int_{t_0}^{t}f(t')\xi(t',t_0)\,dt'=-T(t_0,t). \tag{7·7}$$

Now we proceed to compute the matrix elements of U in quite an analogous way as in § 6. The generating function for them is

$$U(u,v)=\sum\left\{\frac{\pi 2^{n+m}}{n!\,m!}\right\}^{1/4}u^nv^mU_{mn}(t,t_0)=\frac{e^{-u^2-v^2+tT}}{\sqrt{2\pi iX}}\iint dx\,dx_0$$

$$\exp-\frac{1}{2}\left\{\left(1-\frac{i\dot{X}}{X}\right)x^2+\frac{2ixx_0}{X}+\left(1-\frac{iY}{X}\right)x_0^2-2\left(2v+\frac{i\eta}{X}\right)x-2\left(2u+\frac{i\xi}{X}\right)x_0\right\}$$

$$=\sqrt{\frac{2\pi}{i\sigma}}e^{tT}\exp\left\{-\frac{\xi^2+\eta^2-i(\xi\dot{\xi}-\eta\eta')}{2\sigma X}\right\}\cdot\times$$

$$\exp\left\{\frac{\rho u^2-4iuv+\rho^*v^2+2i(\xi-i\dot{\xi})u+2i(\eta+i\eta')v}{\sigma}\right\}, \tag{7·8}$$

where

$$\eta\equiv\xi(t_0,t),\quad \eta'=\partial\xi(t_0,t)/\partial t_0 \tag{7·9}$$

and where we have utilized the relation (7·5), *i.e.*

$$\dot{X}\xi+\eta=X\dot{\xi} \tag{7·10}$$

and the inverse relation

$$Y\eta+\dot{\xi}=-X\eta' \tag{7·11}$$

so that we have also

$$Y\eta^2+2\eta\dot{\xi}+\dot{X}\dot{\xi}^2=X(\dot{\xi}\dot{\xi}-\eta\eta'). \tag{7·12}$$

Next we want to construct a direct generating function for the transition probabilities, with aid of (5·2),

$$P(u,v)\equiv\sum_{m,n=0}^{\infty}u^nv^mP_{mn}$$

$$=2^{1/2}\{Q(1-u^2)(1-v^2)+(1+u^2)(1+v^2)-4uv\}^{-1/2}$$

$$\times\exp\left[-\frac{(1-u^2)(1-v^2)\left\{(\eta^2+\eta'^2)\frac{1-v}{1+v}+(\dot{\xi}^2+\dot{\xi}^2)\frac{1-u}{1+u}\right\}}{2\{Q(1-u^2)(1-v^2)+(1+u^2)(1+v^2)-4uv\}}\right. \tag{7·13}$$

The possibility of a symmetry of P_{mn} depends on the equality or inequality of $(\dot{\xi}^2+\dot{\xi}^2)$

and $(\eta^2+\eta'^2)$.

§ 8. The special case: $\omega=\text{const.}=1$

The results of the foregoing section will be greatly simplified in the elementary case $\omega=\text{const.}=1$. We have to set

$$X(t, t_0)=\sin(t-t_0),\ \rho=\rho^*=0,\ \sigma=2i\exp(it-it_0),$$

$$\xi-i\dot{\xi}=-ie^{it}\int_{t_0}^{t} e^{-it'}f(t')dt',$$

$$\eta+i\eta'=-ie^{-it_0}\int_{t_0}^{t} e^{it'}f(t')dt',$$

$$\frac{2i}{\sigma}(\xi-i\dot{\xi})=\eta-i\eta',\quad \frac{2i}{\sigma}(\eta+i\eta')=\xi+i\dot{\xi},$$

$$\xi^2+\dot{\xi}^2=\eta^2+\eta'^2\equiv 2V,$$

$$(\eta-i\eta')(\xi+i\dot{\xi})=-2e^{i(t_0-t)}W$$

$$2V=\int_{t_0}^{t}\int_{t_0}^{t}\cos(t'-t'')f(t')f(t'')dt'dt''=\frac{1}{X}(\dot{\xi}\eta-\xi\eta').$$

It is a remarkable characteristic of a classical harmonic oscillator that the work V done against the oscillator initially at rest is the same for the reversed motion as for the direct motion.

The last factor of (7·8) now becomes

$$\exp\{2e^{i(t_0-t)}uv+(\eta-i\eta')u+(\xi+i\dot{\xi})v\}$$

$$=\exp\left\{-\frac{1}{W}(\eta-i\eta')u(\xi+i\dot{\xi})v+(\eta-i\eta)u+(\xi+i\dot{\xi})v\right\},$$

which can be expanded in powers of u and v according to the formula

$$\exp(\alpha+\beta-\alpha\beta/W)=\sum_{m,n=0}^{\infty}C(m, n\mid W)\frac{\alpha^m}{m!}\frac{\beta^n}{n!} \tag{8·1}$$

where C's are the well-known Charlier polynomials of the probability theory:

$$C(m, n\mid W)\equiv\sum_{l=0}^{\min(m,n)}\frac{m!\,n!}{l!(m-l)!\,(n-l)!}(-W)^{-l}. \tag{8·2}$$

It is a curiosity that Poisson's law of small numbers, to which the Charlier polynomials belong, appears side by side with Laplace's law of large numbers, to which the Hermite polynomials belong, in the quantum mechanics of a harmonic oscillator.

Summing up the above results we now write down the matrix elements of the transformation function:

$$U_{mn}(t, t_0)=\frac{e^{i(t_0-t)/2+iV/2}}{(2^{m+n}m!\,n!)^{1/2}}e^{-W/2}C(m, n \mid W)(\xi+\xi i)^m(\eta-i\eta')^n, \tag{8·3}$$

where

$$V=\int_{t_0}^{t}\int_{t_0}^{t'}\sin(t'-t'')f(t')f(t'')dt'dt''. \tag{8·4}$$

The transition probabilities are

$$P_{mn}(t, t_0)=\frac{e^{-W}W^{m+n}}{m!\,n!}(C(m, n|W))^2 \tag{8·5}$$

which are again symmetric with respect to m, n, since the Charlier polynomials are symmetric. If $n=0$, $C(m, 0 \mid W)$ reduces to unity, and we have the Poisson distribution

$$P_{m0}=\frac{e^{-W}W^m}{m!}. \tag{8·6}$$

The Charlier polynomials may be defined by another generating function

$$G(m, \alpha \mid W)\equiv(1-\alpha/W)^m e^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha^n}{n}C(m, n \mid W). \tag{8·7}$$

We construct

$$\sum_{m=0}^{\infty}\frac{e^{-W}W^m}{m!}G(m, \alpha \mid W)G(m, \beta \mid W)=e^{\alpha\beta/W}, \tag{8·8}$$

from which we can deduce

$$\sum_{m=0}^{\infty}\frac{e^{-W}W^m}{m!}C(m, n \mid W)C(m, n' \mid W)=\delta_{nn'}\cdot n!\,W^{-n}. \tag{8·9}$$

This proves the law of total probability : $\sum_m P_{mn}=1$. If we construct

$$\sum_{m=0}^{\infty}m\frac{e^{-W}W^m}{m!}G(m, \alpha \mid W)G(m, \beta \mid W)=\{W-\alpha-\beta+\alpha\beta/W\}e^{\alpha\beta/W},$$

we can deduce the average excitation :

$$\langle m\rangle=\sum_{m=0}^{\infty}mP_{mn}=n+W. \tag{8·10}$$

In the same way we arrive at

$$\langle \Delta m^2\rangle=\langle m^2\rangle-\langle m\rangle^2=(2n+1)W. \tag{8·11}$$

In terms of energy these relations run :

$$\langle E_m\rangle=E_n+W,\ \langle \Delta E_m\rangle=2WE_n. \tag{8·12}$$

Classically we expect the following distribution for the final energy E

$$\frac{dE}{2\pi\sqrt{4E_0W-(E-E_0-W)^2}}$$

so that

$$\langle E\rangle = E_0 + W, \quad \langle \Delta E^2\rangle = 2WE_0$$

in accordance with (8·12).

It will be remarked that the transition amplitude (8·3) can also be expressed as

$$U_{mn}(t, t_0) = \frac{e^{i(t_0-t)(n+1/2)+iW/2}}{(2^{n+m}m!\,n!)^{1/2}}(-1)^n e^{-W/2} C(m, n \mid W)(\xi+i\dot{\xi})^m(\xi-i\dot{\xi})^n. \tag{8·13}$$

The phase factor $\exp i(t_0-t)(n+1/2)$ exists also for an undisturbed motion, and may properly be ignored. The second phase factor $\exp(iW/2)$ and the sign $(-1)^n$ prevent U_{mn} to be a hermitian matrix, and make difficult an interpretation of the symmetry of the transition probabilities obtained above.

§ 9. The perturbation theory

In order to get an idea of how the symmetry of our transition probabilities comes out, we revert to the usual theory of perturbation. Thanks to the recent advancement in this theory, as far as the oscillator problem concerns, we can compute the terms of any order, and we may expect the series can be summed in the closed form we have obtained by the direct way.

In the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2\right)\psi - fx\psi, \tag{9·1}$$

we expand the wave function $\psi(x, t)$ in terms of the eigenfunctions of the unperturbed Hamiltonian in the bracket

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n(t) e^{-it(n+1/2)}\psi_n(x) \tag{9·2}$$

and obtain for the equations governing the "variation of constants"

$$i\frac{dc_m}{dt} = -\sum^{\infty} c_n f e^{i(m-n)t}\int \psi_m x \psi_n dx,$$

or, in matrix notation

$$i\frac{dc}{dt} = H(t)c, \tag{9·3}$$

the elements of the matrix $H(t)$ being

$$H(t)_{mn} = -f(t)e^{i(m-n)t}x_{mn}, \tag{9·4}$$

$$x_{mn} = \sqrt{\frac{n+1}{2}} \text{ for } m = n+1, = \sqrt{\frac{n}{2}} \text{ for } m = n-1, \ = 0 \text{ otherwise.}$$

The differential equation becomes on formal integration

$$c(t)=c(0)+\frac{1}{i}\int_0^t H(t')c(t')\,dt'$$

and the method of iteration yields a formal solution in the form of infinite series

$$c(t)=\left\{1+\frac{1}{i}\int_0^t H(t')dt'+\frac{1}{i^2}\int_0^t H(t')dt'\int_0^{t'} H(t'')\,dt''+\cdots\right\}c(0)\equiv S(t,0)c(0). \tag{9·5}$$

The k-th term of the operator $S(t,0)$ on the right-hand side may be written in the following form

$$\frac{(-i)^k}{k!}\int_0^t\cdots\int^t dt'dt''\cdots dt^{(k)}T(H(t')H(t'')\cdots H(t^{(k)})), \tag{9·6}$$

where T is the chronologically ordered product of Wick[3] ;

$$x(t)_{mn}=e^{i(m-n)t}x_{mn} \tag{9·7}$$

are the elements of the operator x in the interaction representation. They satisfy the commutation relations

$$[x(t),x(t_0)]=\frac{1}{i}X(t,t_0)=\frac{1}{i}\sin(t-t_0). \tag{9·8}$$

As usual, we separate it into creation and annihilation operators :

$$x(t)=b(t)+b^+(t) \tag{9·9}$$

with

$$[b(t),b^+(t')]=\frac{1}{2}e^{i(t'-t)} \tag{9·10}$$

and define the normal product

$$N\{x(t')x(t'')\cdots x(t^{(k)})\}.$$

Then it holds Wick's theorem[3]

$$T\{x(t')\cdots x(t^{(k)})\}=\sum N\{x(t^{(j_1)})\cdots x(t^{(j_l)})\}\sum(x(t^{(\alpha)})x(t^{(\alpha')}))\cdots(x(t^{(\omega)})x(t^{(\omega')})), \tag{9·11}$$

where

$$\underbrace{t^{(j_1)},\cdots,t^{(j_l)}}_{l}\quad+\quad\underbrace{t^{(\alpha)},t^{(\alpha')},\cdots,t^{(\omega)},t^{(\omega')}}_{2p}\quad=k$$

is a rearrangement of $t',\cdots,t^{(k)}$ and the summation is over essentially different expressions. Further

$$(x(t)x(t'))=\frac{1}{2}\{\cos(t-t')-i\,\varepsilon(t-t')\sin(t-t')\} \tag{9·12}$$

with

$$\varepsilon(t-t')=+1 \text{ for } t>t',\ =-1 \text{ for } t>t'.$$

In the integration of (9·6) we first meet with the integral (see (8·4))

$$\int_0^t\int_0^t (x(t')x(t''))f(t')f(t'')\,dt'\,dt''=W-iV. \tag{9·13}$$

For a given set $t^{(j_1)}\cdots t^{(j_l)}$ and given p, there arise from (9·11) $(2p)!/p!\,2^p$ terms of the same magnitude $(W-iV)^p$, and there are $\binom{k}{l}$ ways of choosing $t^{(j_1)}\cdots t^{(j_l)}$. Thus (9·6) becomes

$$\frac{(-i)^k}{k!}\sum(-1)^l\frac{k!}{(k-l)!\,l!}\int_0^t\cdots\int N\{x(t')\cdots x(t^{(l)})\}\,f(t')\cdots f(t^{(l)})\,dt'\cdots dt^{(l)}$$

$$\frac{(2p)!}{2^p\,p!}(W-iV)^p, \tag{9·14}$$

which goes, on summation over $k=l+2p$, into

$$\sum_{l=0}^{\infty}\frac{i^l}{l!}e^{-W/2+iV/2}\int_0^t\cdots\int N\{x(t')\cdots x(t^{(l)})\}\cdot f(t')\cdots f(t^{(l)})\,dt'\cdots dt^{(l)}. \tag{9·15}$$

We first note

$$\left.\begin{aligned}\int_0^t b(t')f(t')\,dt'&=b(0)\int_0^t e^{-it'}f(t')\,dt'=b(0)\,(+i)e^{-it}(\xi-i\dot\xi),\\ \int_0^t b^+(t')f(t')\,dt'&=b^+(0)\int_0^t e^{it'}f(t')\,dt=b(0)\,(-i)e^{it}(\xi+i\dot\xi),\end{aligned}\right\} \tag{9·16}$$

so that (9·15) becomes

$$e^{-W/2+iV/2}\cdot\sum_{l=0}^{\infty}\frac{1}{l!}N\{(-b(0)e^{-it}(\xi-i\dot\xi)+b^+(0\;e^{it}(\xi+i\dot\xi))^l\}. \tag{9·17}$$

Let us seek the m-n-element of this S-matrix. A product of the form

$$b^+(0)^\mu b(0)^\nu \quad\text{with}\quad n-\nu=m-\mu=\lambda$$

will contribute to the matrix element

$$\sqrt{\frac{m}{2}}\sqrt{\frac{m-1}{2}}\cdots\sqrt{\frac{m-\mu+1}{1}}\sqrt{\frac{n-\nu+1}{2}}\cdots\sqrt{\frac{n}{2}}$$

$$=\sqrt{\frac{m!\,m!}{2^{\mu+\nu}(m-\mu)!\,(n-\nu)!}}=\left(\frac{m!\,n!}{2^{m+n-2\lambda}}\right)^{1/2}\frac{1}{\lambda!}.$$

We have then

$$S_{mn}=e^{-iW/2+iV/2}e^{i(m-n)t}\sum_{\lambda=0}^{\min(m,n)}\frac{1}{(m+n-2\lambda)!}$$

$$\frac{(m+n-2\lambda)!(\xi+i\dot\xi)^{m-\lambda}(-1)^{n-\lambda}(\xi-i\dot\xi)^{n-\lambda}}{(m-\lambda)!\,(n-\lambda)!}\left(\frac{m!\,n!}{2^{m+n}}\right)^{1/2}\frac{2^\lambda}{\lambda!}$$

$$=\frac{e^{-W/2+iV/2}\cdot e^{i(m-n)t}}{\sqrt{2^{m+n}m!\,n!}}(-1)^n(\xi+i\dot\xi)^m(\xi-i\dot\xi)^n\sum_{\lambda=0}^{\min(m,n)}\frac{m!\,n!(-2/W)^\lambda}{\lambda!(m-\lambda)!(n-\lambda)!}, \tag{9·18}$$

in complete agreement with our previous result (8·13). Viewed in the light of the perturbation theory, the common factor $\exp(-W/2+iV/2)$ comes out to be connected with the damping phenomena. The unexpected symmetry of the transition probabilities seems to depend essentially on the *almost* hermitian character of the normal product; in our case the presence of the sign $(-1)^n$ spoils the exact hermitian character of the normal product. Let us write the factor in the form: $(-1)^n H_{mn}$, where H_{mn} is hermitian. It may be written, as usual, in the form of a sum of a hermitian and an antihermitian part:

$$(-1)^n H_{mn}=\frac{1}{2}\{(-1)^n+(-1)^m\}H_{mn}+\frac{1}{2}\{(-1)^n-(-1)^m\}H_{mn}.$$

If m, n are of the same parity, the hermitian part only survives, and if, on the other hand, m, n are of the different parity, the hermitian part drops out. Thus the symmetry of the transition probabilities originates from this mixed hermitian character of our normal product. The either-or character may be traced to the selection rule $n\rightarrow n+1$. This requires an even number of operators b, b^+ for the transitions between the states of the same parity and an odd number for the different parity. This state of affairs has been anticipated by Mr. M. Ôtuka.

In the calculation of this paragraph I have been much assisted by Dr. R. Utiyama, who also will extend the perturbation theory to the case ω variable. I wonder whether this sort of calculation would lead to a reasnable interpretation of the symmetry of transition probabilities.

§ 10. Concluding remarks

There are numerous problems which can be tested on our elementary example of the forced harmonic oscillator. The adiabatic theorem, the asymptotic equivalence of the classical and quantal treatments, and others. These problems will be taken up elsewhere. There are also a number of open questions, which have not been analyzed in the text. For example, the characteristic difference between (6·8) and (6·9) which are proportional to $Q^{-1/2}$ and $Q^{-3/2}$ respectively for large Q, remains quite incomprehensible to me.

I owe much to my colleagues of this institute for their interests and advices on this work. Some of their names apppear in the text. Mr. H. Arita independently calculated the results of § 5 by directly solving the Heisenberg equation of motion.

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The Heisenberg Theory of Ferromagnetism of The Crystal constituted by the Atoms with Spin One

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The Heisenberg theory of ferromagnetism of the crystal whose atoms have the spin one is developed. It is proved that the hamiltonian of the crystal constituted by the atoms having the spin s larger than $\frac{1}{2}$ can be expressed by $-2J\sum \boldsymbol{s}_i \cdot \boldsymbol{s}_j$ in terms of the spin vector $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, of the nearest neighboring atoms in the same way as in the case $s=1/2$, as far as the Hund rule prevails in each constituent atom in the molecular fields. The Curie points of a few lattice types are obtained in the stage of the first, the second and the third approximation. These are all higher than those corresponding to the case $s=1/2$. Especially in the second approximation the Curie point T_c is given by the formula, $T_c=11\,J/[3k(1-\sqrt{1-\frac{11}{2z}})]$ in terms of the exchange integral J between the nearest neighboring atoms, the coordination number z and the Boltzmann constant k. The condition for appearing of ferromagnetism in this approximation is that $z=6$.

§ 1. Introduction

If we know the mean value of every power of the energy of the crystal over all the states having any given value S of the total spin quantum number, we can establish the partition function of the crystal system by making use of Thiele's semi-invariant theory. When we know only the means of the energy powers lower than and equal to the n-th, we are able to discuss up to the n-th approximation. We have developed the calculation for the crystal constituted by the atoms having the spin one up to the third approximation.

As the preliminary, it is proved in § 2 that the hamiltonian of the crystal system can be expressed in the same form of Dirac's vector model as in the case $s=1/2$ also in the general cases $s \geq 1$, so far as the Hund rule is applicable in each atom in the crystal. We explain the formal aspect of the theory in § 3; and based on this results, work out the calculations in § 4, where the Curie points of a few crystals distinguished by their lattice types are evaluated in the first, the second and the third approximation, and the relation of the reciprocal susceptibility to the temperature is traced above the Curie point in each case.

2. The Hamiltonian of the crystal where $s \geq 1$

The hamiltonian of the crystal, constituted by atoms bearing more than one electron out of the closed shell, may be essentially expressed

$$H=-2\sum_{(i,\,k)}\sum_{\mu,\nu}J_{ik}^{(\mu\,\nu)}\,\boldsymbol{s}_i^{(\mu)}\cdot\boldsymbol{s}_k^{(\nu)}. \tag{1}$$

The suffices of the Roman letters i, k refer to the atoms, and those of the Greek letters μ, ν to the outer-shell electrons of each atom, $\boldsymbol{s}_i^{(\mu)}$ represents the spin vector of the μ-th outer-shell electron in the i-th atom, and $J_{ik}^{(\mu,\,\nu)}$ the exchange integral between the μ-th electron in the i-th atom and the ν-th electron in the k-th atom. The summation of μ, ν are to be extended over all outer-shell electrons belonging respectively to the i-th and the k-th atom, and those of i, k are to be taken over all pairs of nearest neighboring atoms. We further, by the way, consider a more general quantity

$$X=\sum_{(i\,\,k)}\sum_{\mu,\,\nu}x_{ik}^{(\mu\nu)}\cdot X_{ik}(\boldsymbol{s}_i^{(\mu)},\ \boldsymbol{s}_k^{(\cdot)}), \tag{2}$$

where $x_{ik}^{(\mu\nu)}$ is a scalar factor and X_{ik} $(\boldsymbol{s}_i^{(\mu)},\ \boldsymbol{s}_k^{(\nu)})$ is a scalar function of $\boldsymbol{s}_i^{(\mu)}$, $\boldsymbol{s}_k^{(\nu)}$. An atom isolated from others will be found in the state of the highest multiplicity, if Hund's rule is applicable. Even when this is brought into the crystal, this atom will remain to be found in this state so far as the disturbance due to the action of the neighboring atoms is not so strong. Assuming this condition granted, we hereafter discuss the hamiltonian of eq. (1) or more generally the quantity of the type (2).

We assume the number of outer-shell electrons in the i-th atom to be n_i, and the number of atoms in the whole crystal to be A. The summation,

$$N=\sum_{i=1}^{A}n_i, \tag{3}$$

is the total number of outer-shell electrons in the entire crystal. We consider the symmetric group $\mathfrak{S}(N)$ corresponding to all these electrons and its subgroup

$$\mathfrak{S}'(N)=\mathfrak{S}_1(n_1)\times\mathfrak{S}_2(n_2)\times\cdots\times\mathfrak{S}_i(n_i)\times\cdots\times\mathfrak{S}_A(n_A), \tag{4}$$

where $\mathfrak{S}_i(n_i)$ is the symmetric group corresponding to the n_i electrons in the i-th atom. If the above assumed condition prevails, it suffices to consider only that part of representations of $\mathfrak{S}(N)$ which give rise to the identity representations of $\mathfrak{S}'(N)$ on being regarded as those of this subalgebra. From the view point of the group-algebra this is nothing but to consider only what H. Weyl[2] called quantities of the character $(\varepsilon,\ \varepsilon)$ for the idempotent ε corresponding to the identity representation of $\mathfrak{S}'(N)$,

$$\varepsilon=\sum_{\pi\,\in\,\mathfrak{S}'(N)}^{A}\pi/\Big[\prod_{i=1}(n_i!)\Big]. \tag{5}$$

That part of X given by this character is

$$\begin{aligned}\varepsilon X\varepsilon&=\sum_{(i\,\,k)}\sum_{\mu,\,\nu}x_{ik}^{(\mu\,\nu)}\ \varepsilon\,X_{ik}(\boldsymbol{s}_i^{(\mu)},\ \boldsymbol{s}_k^{(\nu)})\ \varepsilon\\&=\sum_{(i\,\,k)}\sum_{\mu,\,\nu}x_{ik}^{(\mu\,\nu)}\ \varepsilon\,\pi_i^{(\mu\,\rho)}\ \pi_k^{(\nu\,\sigma)}\,X_{ik}(\boldsymbol{s}_i^{(\rho)},\ \boldsymbol{s}_k^{(\sigma)})\ \pi_i^{(\mu\,\rho)}\,\pi_k^{(\nu\,\sigma)}\,\varepsilon,\end{aligned}$$

where $\pi_i^{(\mu\,\rho)}$ and $\pi_k^{(\nu\,\sigma)}$ represent the interchange between the μ-th and the ρ-th electron in the i-th atom and that between the ν-th and σ-th electron in the k-th atom respectively; $\pi_i^{(\mu\,\rho)}\in\mathfrak{S}_i(n_i)\subset\mathfrak{S}'(N)$, $\pi_k^{(\nu\,\sigma)}\in\mathfrak{S}_k(n_k)\subset\mathfrak{S}'(N)$. By using the evident relation

$$\varepsilon\pi_i^{(\mu\rho)}\varepsilon=\varepsilon\pi_i^{(\mu\ \rho)}=\pi_i^{(\mu\ \rho)}\varepsilon=\varepsilon, \tag{6}$$

we can rewrite the above expression, viz.

$$\begin{aligned}\varepsilon X\varepsilon&=\sum_{(i\,k)}\Big(\sum_{\mu,\,\nu}x_{ik}^{(\mu\ \nu)}\Big)\varepsilon X_{ik}\,(\boldsymbol{s}_i^{(\rho)},\,\boldsymbol{s}_k^{(\sigma)})\,\varepsilon\\&=\varepsilon\Big[\sum_{(i\,k)}\sum_{\rho,\,\sigma}x_{ik}\,X_{ik}(\boldsymbol{s}_i^{(\rho)},\,\boldsymbol{s}_k^{(\sigma)})\Big]\,\varepsilon=\varepsilon\,X'\,\varepsilon,\end{aligned} \tag{7}$$

where

$$X'=\sum_{(i\,k)}x_{ik}\sum_{(\mu,\,\nu)}X_{ik}(\boldsymbol{s}_i^{(\mu)},\,\boldsymbol{s}_k^{(\nu)}), \tag{2a}$$

and

$$x_{ik}=\sum_{\mu,\,\nu}x_{ik}^{(\mu\ \nu)}/(n_i\,n_k). \tag{2b}$$

In words, that part of given by eq. (2), of the character $(\varepsilon,\,\varepsilon)$ is just equal to the part of the same character of X' given by (2a). Thus we can conclude that discussions of a quantity of the form of X' leads to quite the same consequences as that of the form of X, so far as Hund's rule is applicable.* The hamiltonian of the crystal can be treated as a special case, where we are allowed to substitute H in (1) by the quantity,

$$H'=-2\sum_{(i,\,k)}J_{ik}\sum_{\mu,\,\nu}\boldsymbol{s}_i^{(\mu)}\cdot\boldsymbol{s}_k^{(\nu)}=-2\sum_{(i,\,k)}J_{ik}\,\boldsymbol{s}_i\cdot\boldsymbol{s}_k, \tag{1a}$$

where

$$J_{ik}=\sum_{\mu,\,\nu}J_{ik}^{(\mu\ \nu)}/n^2, \tag{1b}$$

and

$$\boldsymbol{s}_i=\sum_{\mu}\boldsymbol{s}_i^{(\mu)} \tag{1c}$$

represents the total spin of the i-th atom. We have here assumed that the constituent atoms are all equivalent to one another and that they have therefore the same number of electrons, every $n_i=n$. In the case that J_{ik}'s take a single value represented by the letter J throughout all pairs of nearest neighboring atoms, (2a) can be re-expressed thus :*

$$H''=-2J\sum_{(i,\,k)}\boldsymbol{s}_i\cdot\boldsymbol{s}_k. \tag{8}$$

In the following we use this form of hamiltonian[4] exclusively.

3. The formal aspect of the theory

We assume the total number of sites of the shell exterior to the closed core to be $2p$, and we hereafter represent the total number of constituent atoms in the crystal by N, irrespective of the previous meaning. If the Hund rule is applicable and the state of the highest multiplicity is the most stable, the state of this multiplicity,

$$\boldsymbol{s}_i^2=s(s+1), \tag{9}$$

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* Professor T. Matsubara, in his informal paper[3], tried to extend our theorem, and proved that eq. (8) is applicable also in the general cases that the condition of Hund's rule does not hold. There are found, however, a few delicate errors in his proof, and this extension seems to be impossible, as more clearly seen by the algebraic considerations.

is prevalent over all constituent atoms of the crystal, where s is equal either to $n/2$ or to $p-n/2$ according to the cases: $n \leq p$, or $n > p$. In this case the total number of states of the crystal system, belonging to the total spin S and its z-component M, are given by the following expressions $\phi(S;\ N)$ and $\varphi(S;\ N)$ respectively.

$$\varphi(M;\ N) = \textit{the coefficient of } z^{M} \textit{ or } z^{-M} \textit{ in the expansion of } (z^{s}+z^{s-1}+\cdots+z^{-s+1}+z^{-s})^{N}, \tag{10}$$

$$\phi(S;\ N) = \varphi(S;\ N) - \varphi(S+1;\ N). \tag{11}$$

From the view point of the representation theory of algebras there exist the $\varphi(M;\ N)$-dimensional reducible representation R_M' and the $\phi(S;\ N)$-dimensional irreducible representation R_s of the algebra engendered by all quantities of the form,

$$P_{ij} = \boldsymbol{s}_i \cdot \boldsymbol{s}_j, \tag{12}$$

for each given value of M and S. There holds a relation that the one is a direct sum of those of the other kind, to the effect

$$R_M' = R_M + R_{M+1} + \cdots\cdots + R_{Ns-1} + R_{Ns}. \tag{13}$$

We express the traces of a quantity Q of this algebra in the representations R_s and R_M' respectively by $\chi_s(Q)$ and $\chi_M'(Q)$, which can be expressed by certain combinations of $\varphi(M;\ N)$'s. In the case, $s=1$, these expressions turn out to be the following (Appendix I)

$$\begin{aligned}
&\chi_M'(1) = \varphi(M;\ N),\\
&\chi_M'(P_{ij}) = (a_2-2)\ \varphi(M;\ N-2),\\
&\chi_M'(P_{ij}^2) = (a_2+2a_1+6)\ \varphi(M;\ N-2),\\
&\chi_M'(P_{ij}\,P_{ik}) = (a_3-a_1)\ \varphi(M;\ N-3),\\
&\chi_M'(P_{ij}\,P_{kl}) = (a_4-4a_2+6)\ \varphi(M;\ N-4),\\
&\chi_M'(P_{ij}^3) = (a_2-8)\ \varphi(M;\ N-2),\\
&\chi_M'(P_{ij}^2\,P_{ik}) = (a_3+a_2-a_1-2)\ \varphi(M;\ N-3),\\
&\chi_M'(P_{ij}^2 P_{kl}) = (a_4+2a_3+4a_2-2a_1-10)\ \varphi(M;\ N-3),\\
&\chi_M'(P_{ij}\,P_{ik}\,P_{il}) = (a_4-4a_2+6)\ \varphi(M;\ N-4),\\
&\chi_M'(P_{ij}\,P_{jk}\,P_{kl}) = (a_3+3a_1)\ \varphi(M;\ N-3),\\
&\chi_M'(P_{ij}\,P_{ik}\,P_{lm}) = (a_5-3a_3+2a_1)\ \varphi(M;\ N-5),\\
&\chi_M'(P_{ij}\,P_{kl}\,P_{mn}) = (a_6-6a_4+15a_2-20)\ \varphi(M;\ N-6);
\end{aligned}$$

where the different letters appearing as suffices of P always represent different numbers and the operators a_q introduced for the sake of abbreviation are defined by

$$a_p = a^{q} + a^{-q}, \tag{14'}$$

where the operator a is defined by

$$a\varphi(M;\cdots) = \varphi(M+1;\cdots). \tag{14''}$$

According to the relation (13) we also obtain.

$$\chi_s(Q) = \chi'_s(Q) - \chi_s'(Q). \tag{15}$$

The mean value of Q over all states of the crystal having the resultant spin S is expressed by

$$\langle Q\rangle_s=\chi_s(Q)/\phi(s), \tag{16}$$

where $\phi(S)$ is an abbreviation for $\phi(S\,;\,N)$ and this abbreviation will be retained for the following. The averages of H, H^2 and H^3 are expressed by those of P_{ij}'s and their products, as follows,

$$\langle H\rangle_s\equiv W(s)=-JNz\langle P_{ij}\rangle_s, \tag{17}$$

$$\langle H^2\rangle=J^2\cdot[2Nz\cdot\langle P_{ij}^2\rangle_s+4Nz(z-1)\cdot\langle P_{ij}P_{ik}\rangle_s + Nz(Nz-4z+2)\cdot\langle P_{ij}P_{kl}\rangle_s], \tag{18}$$

$$\begin{aligned}\langle H^3\rangle_s=&-J^3[4Nz\cdot\langle P_{ij}^3\rangle_s+24Nz(z-1)\cdot\langle P_{ij}^2P_{ik}\rangle_s\\ &+6Nz(Nz-4z+2)\langle P_{ij}^2P_{kl}\rangle_s+8Nz(z-1)\cdot\langle P_{ij}P_{ik}P_{il}\rangle_s\\ &+\left\{\begin{array}{c}0\\16Nz\\32Nz\end{array}\right\}\cdot\langle P_{ij}P_{jk}P_{ki}\rangle_s+\left\{\begin{array}{c}24Nz(z^2-2z+1)\\24Nz(z^2-2z-1)\\24Nz(z^2-2z-3)\end{array}\right\}\cdot\langle P_{ij}P_{ik}P_{kl}\rangle_s\\ &+\left\{\begin{array}{l}12Nz(Nz^2-6z^2+10z-4)\\12Nz(Nz^2-Nz-6z^2+6z)\\12Nz(Nz^2-Nz-6z^2+10z+4)\end{array}\right\}\cdot\langle P_{ij}P_{ik}P_{lm}\rangle_s\\ &+Nz(N^2z^2-6Nz^2+6Nz+40z^2-48z-16)\cdot\langle P_{ij}P_{kl}P_{mn}\rangle_s].\end{aligned} \tag{19}$$

In the expression (19) for the average of H^3 which explicitly depends on the lattice type of the crystal, the first, the second and the third lines of the terms in the brackets $\{\ \}$ are to be adopted respectively corresponding to the following three cases (a), (b) and (c).

(a) No pair of the atoms nearest-neighboring to a certain atom is nearest-neighboring as in the simple-cubic and the body-centred cubic lattice. Among the possible cases where this condition does not hold, we only mention the following two cases,

(b) the hexagonal layer lattice,

(c) the face-centred cubic or the close-packed hexagonal lattice. The square and the cubic fluctuations,

$$\Delta_2(S)=\langle(H-W(S))^2\rangle_s=\langle H^2\rangle_s-W(S)^2, \tag{20}$$

$$\Delta_3(S)=\langle(H-W(S))^3\rangle_s=\langle H^3\rangle_s-3\langle H^2\rangle_s\,W(S)^2+2W(S)^3, \tag{21}$$

are called the semi-invariants by Thiele, according to whose theory in the approximation including up to the third moment of the energy, the partition function of the crystal system is (Appendix II)

$$\begin{aligned}Z&=\frac{kT}{2\beta\mathcal{H}}\int\phi(S)\cdot\exp\left[\frac{2\beta S\mathcal{H}-W(S)}{kT}+\frac{\Delta_2(S)}{2k^2T^2}-\frac{\Delta_3(S)}{6k^3T^3}\right]dS\\ &=\frac{kT}{2\beta\mathcal{H}}\int\exp\,[f(S)]\,dS,\end{aligned} \tag{22}$$

$$f(S)=\frac{2\beta S\mathcal{H}-W(S)}{2kT}+\frac{\Delta_2(S)}{2k^2T^2}-\frac{\Delta_3(S)}{6k^3T^3}+\lg\phi(S), \tag{22'}$$

where β is the Bohr magneton and $\mathcal{H}$ is the applied field. We can evaluate asymptotically by the method of the steepest decent, viz.

$$Z=\frac{kT}{2\beta\mathcal{H}}\sqrt{\frac{\pi}{f''(S^+)}}\cdot\exp\,[f(S^+)]. \tag{23}$$

The magnetization I may be evaluated similarly.

$$I=kT\,\partial Z/\partial\mathcal{H}/Z=2\beta S^+. \tag{24}$$

The saddle point S^+ is determined by the equation,

$$f'(S^+)=0. \tag{25}$$

4. Calculations and results

We can evaluate $\varphi(M;\ N)$ in (6) by making use of the contour integral on the complex plane,

$$\varphi(M;\ N)=\frac{1}{2\pi_i}\phi\Big(\Big(\sum_{t=-s}^{+s}z_t\Big)^N\cdot z^{M-1}\,dz, \tag{26}$$

where the integration contour is along an arbitrary closed curve around the origin of the z-plane[5].

In the limit $N,\ M\to\infty$, we can evaluate this asymptotically by the method of the steepest decent, and the results for the case, $s=1$, is

$$\varphi(M;\ N)=\frac{3N}{2^{M+1}\pi L^{1/2}(L-N)^{1/2}}\cdot\frac{(L+N)^N\cdot(L-N)^M N^N}{(N-M)^{N-M}\cdot(N+M)^{N+M}}, \tag{27}$$

where

$$L=\sqrt{4N^2-3M^2}. \tag{27'}$$

In the case that $N,\ M\gg 1$ and $\Delta N,\ \Delta M\simeq 1$ this gives the formula,

$$\varphi(M+\Delta M;\ N+\Delta N)=\varphi(M;\ N)\cdot\Big[\frac{N(L+N)}{N^2-M^2}\Big]^{\Delta N}\Big[\frac{L-M}{2(N+M)}\Big]^{\Delta M} \tag{28}$$
$$\cdot[1+a_1(\Delta M,\ \Delta N,\ M,\ N)+a_2(\Delta M,\ \Delta N,\ M,\ N),$$

where a_1 and a_2 are infinitesimals respectively of the order N^{-1} and N^{-2} as follows,

$$a_1(\Delta M,\ \Delta N,\ M,\ N)=\{2N(N^2-M^2)L^2\}^{-1}\{(-4N^4+3M^4-NML)\ \Delta N$$
$$+(7N^3M-6NM^3+N^2ML)\Delta M+(-4N^2M^2+3M^2-NM^2L)\cdot(\Delta N)^2$$
$$+(8N^3M-6NM^3+2N^2ML)\cdot\Delta N\cdot\Delta M+(-4N^4+3N^2M^2-N^3L)\cdot(\Delta M)^2\}, \tag{28'}$$
$$a_2(\Delta M,\ \Delta N,\ M,\ N)=\{12N^2L^4(N^2-M^2)^2\}^{-1}[\{48N^8+84N^6M^2-288N^4M^4$$
$$+189N^2M^6-27M^8+(36N^5M^2-30N^3M^4)L\}(\Delta N)^2+\{-336N^7M$$
$$+576N^5M^3-252N^3M^5+(-48N^6M+18N^4M^3+18N^2M^5)L\}\Delta N\cdot\Delta M$$
$$+\{84N^8-69N^6M^2-63N^4M^4+54N^2M^6+(12N^7+12N^5M^2-18N^3M^4)L\}\cdot(\Delta M)^2$$

$$+\{96N^6M^2-196N^4M^4+102N^2M^6-18M^8+(24N^5M^2-20N^3M^4)L\}(\Delta N)^3$$
$$+\{-192N^7M+288N^5M^3-108N^3M^5+(-48N^6M+18N^4M^3+18N^2M^5)L\}$$
$$\times(\Delta N)^2\cdot\Delta M$$
$$+\{96N^8-48N^6M^2-90N^4M^4+54N^2M^6+(24N^7+24N^5M^2-36N^3M^4)L\}$$
$$\Delta N\cdot(\Delta M)^2$$
$$+\{-64N^7M+96N^5M^3-36N^3M^5+(-22N^6M+18N^4M^3)L\}(\Delta M)^3]. \quad (28'')$$

From (14), (15), (16), and (28) we obtain,

$$\langle P_{ij}\rangle_s=S^2N^{-2}(-2N^2+NS+S^2)N^{-3},$$
$$\langle P_{ij}^2\rangle_s=1/3\cdot(8N^2-3S^2-2NL')N^{-2},$$
$$\langle P_{ij}\,P_{kl}\rangle_s=1/3\cdot(4N-L')S^2N^{-3},$$
$$\langle P_{ij}\,P_{kl}\rangle_s=S^4N^{-4}+2/3\cdot(-14N^2S+3NS^2+9S^3+2NSL')SN,$$
$$\langle P_{ij}^3\rangle_s=1/3\cdot(-10N^2+9S^2+4NL')N^{-2},$$
$$\langle P_{ij}^2\,P_{ik}\rangle_s=S^2N^{-2}, \quad (29)$$
$$\langle P_{ij}^2\,P_{kl}\rangle_s=1/9\cdot(24N^2S^2-9S^4-6NS^2L')N^{-4},$$
$$\langle P_{ij}\,P_{ik}\,P_{il}\rangle_s=S^4N^{-4}$$
$$\langle P_{ij}\,P_{ik}\,P_{kl}\rangle_s=1/9\cdot(40N^3-15NS^2-16N^2L')N^{-3},$$
$$\langle P_{ij}\,P_{ik}\,P_{lm}\rangle_s=1/3(4N-L')S^4N^{-5},$$
$$\langle P_{ij}\,P_{kl}\,P_{mn}\rangle_s=S^6N^{-6},$$

where we use an abbreviation,

$$L'=\sqrt{4N^2-3S^2}. \quad (29')$$

From these and eq. (17), (18) and (19) we attain to the expressions,

$$W(S)=-zJN^{-1}S^2, \quad (30)$$

$$\Delta_2(S)=2/3\cdot ZJ^2N^{-3}(N^2-S^2)(8N^2-2NL'), \quad (31)$$

$$\Delta_3(S)=4/9\cdot ZJ^3N^{-5}\cdot\begin{cases}[30N^6-165N^4S^2+189N^2S^4-(12N^5-66N^3S^2+36NS^4)\cdot L'], & (31a)\\ [130N^6+135N^4S^2+24N^2S^4+36S^6+(52N^5-30N^3S^2)L'], & (31b)\\ [-290N^6+435N^4S^2-207N^2S^4+36S^6 & \\ \quad+(116N^3S^2-162N^3S^2+36NS^4)L']. & (31c)\end{cases}$$

Using the abbreviation,

$$\sigma=I/2N\beta=S^+/N, \quad \gamma=J/kT, \quad (32)$$

we obtain the following simultaneous equation derived from (24), (25) and (22′), into which (30) and (31) are substituted,

$$\sigma=\beta_1(\xi)\equiv 3/2\cdot\mathrm{cth}\ (3/2\cdot\xi)-1/2\cdot\mathrm{cth}\ (1/2\cdot\xi), \tag{33}$$

$$\zeta=A(\sigma_0\,\mathcal{H},\ T)\equiv 2\gamma\left(\frac{\beta\mathcal{H}}{J}+z\right)\sigma+\frac{2}{3}\gamma^2 z\,\sigma\left(-11+6\sigma^2+\frac{11-8\sigma^2}{\sqrt{4-3\sigma^2}}\right)+\frac{2}{27}\gamma^3 z\cdot$$

$$\left\{\begin{aligned}&[330\sigma-612\sigma^3-216\sigma^5-(132\sigma-144)\sqrt{4-3\sigma^2}-\frac{36\sigma-198\sigma^3+108\sigma^5}{\sqrt{4-3\sigma^2}}, && (33'a)\\&[-270\sigma-96\sigma^3-216\sigma^5+60\sigma\sqrt{4-3\sigma^2}+\frac{156\sigma-90\sigma^2}{\sqrt{4-3\sigma^2}}, && (33'b)\\&[-870\sigma+828\sigma^3-216\sigma^5+(252\sigma-144\sigma^3)\sqrt{4-3\sigma^2}+\frac{348\sigma-378\sigma^3+108\sigma^5}{\sqrt{4-3\sigma^2}}. && (33'c)\end{aligned}\right.$$

These equations have a non-zero solution for σ even in the vanishing applied field, so far as the temperature is lower than a critical value, T_c, determined by the equation,

$$\left[\frac{\sigma B_1(\xi)}{d\xi}\right]_{\xi=0}\cdot\left[\frac{\partial A(\sigma,\,o,\,T_c)}{\partial\sigma}\right]_{\sigma=0}=1. \tag{34}$$

Again with the abbreviation,

$$\gamma_c=\frac{J}{kT_c}, \tag{35}$$

We can explicitly write down eq. (34) respectively for the before-mentioned three cases (a) (b) (c) by using either eq. (33′a) or (33′b) or (33′c),

$$3/2z-2\gamma_c+11/3\cdot\gamma_c^2-32/9\cdot\gamma_c^3=0, \tag{36a}$$

$$3/2z-2\gamma_c+11/3\cdot\gamma_c^2-48/9\cdot\gamma_c^3=0, \tag{36b}$$

$$3/2z-2\gamma_c+11/3\cdot\gamma_c^2+128/9\cdot\gamma_c^3=0. \tag{35c}$$

The value of T_c, the Curie temperature of ferromagnetism, can be determined in the first, the second and the third approximation, according to whether the terms of up to the first, the second or the third degree in eq. (36abc) are retained. Below the Curie point the crystal has the spontaneous magnetization determined from a root of the simultaneous equation (33), (33′) for σ with $\mathcal{H}=0$. Especially in the second approximation, T_c is determined uniquely without giving more detailed structure of the lattice other than the coordination number, z, viz.,

$$T_c=11J/3k(1-\sqrt{1-11/2z}). \tag{37}$$

This formula gives the criterion for appearance of ferromagnetism,

$$z\geq 6. \tag{37′}$$

According to this result, even the crystals of hexagonal layer and simple cubic lattice being

unable to be ferromagnetic in the case, $s=1/2$, can be ferromagnetic in our case, $s=1$. In the third approximation the latter remains to be ferromagnetic, but the former is found to be non-ferromagnetic. The values of γ_c in different approximations are shown in Table 1, which give the Curie points all higher than in the corresponding cases of $s=1/2$.

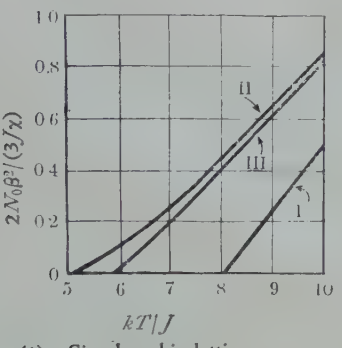
Lattice type	First approx.	Second approx.	Third approx.
Hexagonal layer	0.125	0.194	none
Simple cubic	0.125	0.194	0.168
Body-centred cubic	0.094	0.120	0.115
Face-centred cubic Close-packed hexagonal	0.053	0.074	0.079

Table 1. Value of $\gamma_c=J/kT_c$.

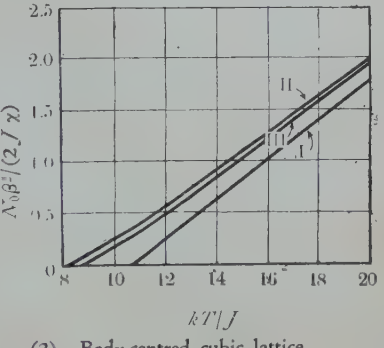
Above the Curie point we obtain the following relation between the magnetic susceptibility, χ of the crystal and the temperature by using eq. (33), (33′).

$$4N_0\beta^2/ZJ\chi=3/2Z\cdot kT/J-2+11/\varepsilon\cdot J/kT+\left\{\begin{matrix}-32/9\\+48/9\\+128/9\end{matrix}\right\}\cdot(J/kT)^2. \qquad \begin{matrix}(38a)\\(38b)\\(38c)\end{matrix}$$

In Fig. 1. $4N_0\beta^2/ZJ\chi$ is plotted as a function of J/kT.



(1) Simple cubic lattice.



(2) Body-centred cubic lattice.

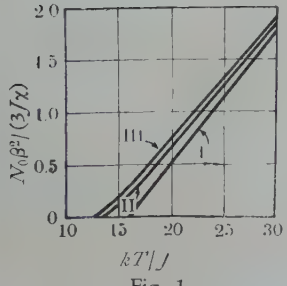


Fig. 1.

(3) Face-centred cubic lattice. The curves I, II and III are those respectively according to the first, the second and the third approximation.

§ 5. Conclusions

We proved that we can express the hamiltonian of the crystal, constituted of the atoms having more than one outer-shell electrons, in the form of Dirac's vector model by using the total spins of these constituent atoms, so far as the condition of Hund's rule is prevailing in these atoms. In spite of the fact that we have no established theory with respect to the cause of the interaction of vector-model type among the magnetic atoms in the antiferromagnetic crystal, we may probably apply this proof also to the antiferromagnetic crystal constituted of the magnetic atom, where the condition of Hund's rule is prevailing.

Based upon this vector model, we worked out the calculation of the Heisenberg theory of ferromagnetism up to the third approximation in the case that the constituent atoms of the crystal have two outer-shell electrons or holes. The results are fairly reasonable.

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Appendix I

The matrix elements of the components s_{ix}, s_{iy} and s_{iz} of $\boldsymbol{s}_i$, in the scheme where the z component s_{iz} is diagonal, are the following.

$$\begin{gathered}(m_i|s_{ix}|m_i-1)=1/\sqrt{2}\,,\quad (m_i|s_{iy}|m_i-1)=-i/\sqrt{2}\,,\\ (m_i|s_{ix}|m_i+1)=1/\sqrt{2}\,,\quad (m_i|s_{iy}|m_i+1)=i/\sqrt{2}\,,\\ (m_i|s_{iz}|m_i)=m_i,\quad (m_i=1,0,-1);\end{gathered}\tag{I.1}$$

or

$$s_{ix}=\begin{pmatrix}0 & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 0\end{pmatrix},\quad s_{iy}=\begin{pmatrix}1 & -i/\sqrt{2} & 0\\ i/\sqrt{2} & 0 & -i/\sqrt{2}\\ 0 & i/\sqrt{2} & 0\end{pmatrix},$$

$$s_{iz}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.\tag{I·1′}$$

From these we can know the matrix elements of $P_{ij}=\boldsymbol{s}_i\cdot\boldsymbol{s}_j$ by means of the relation,

$$\begin{aligned}(m_i, m_j|\,P_{ij}|m_i', m_j') &= (m_i|s_{ix}|m_i')\,(m_j|s_{ix}|m_j')\\ &+(m_i|s_{iy}|m_i')\,(m_j|s_{jy}|m_j')+(m_i|s_{iz}|m_i')\,(m_j|s_{iz}|m_j').\end{aligned}\tag{I·2}$$

Furthermore we can evaluate the matrix elements of the quantity which is a product of a few members of P_{ij}'s; for example, $P_{12}\,P_{13}\,P_{23}$.

We consider a representative of these products, we express this as Q and assume that the total of different suffices of P_{ij} factors consituting Q are $i, j, \cdots\cdots, k$ and the total number of these is ΔN (For example, in the case, $Q=P_{12}\,P_{13}\,P_{23}$, these suffices are 1, 2, 3, and $\Delta N=3$). We can evaluate the trace $\chi'_M(Q)$ of Q in the representation, R_M', by means of the formula,

$$\chi'_M(Q)=\sum_{m_i, m_j, \cdots, m_k}(m_i, m_j, \cdots, m_k|Q|m_i, m_j, \cdots, m_k)\cdot\varphi(M-m_i-m_j-\cdots-m_k;\ N-\Delta N).\tag{1·3}$$

For example, in the case $Q=P_{ij}$,

$$\begin{aligned}\chi'_M(Q)&=(1,1|P_{ij}|1,1)\cdot\varphi(M-2;\ N-2)\\ &+2(1,-1\,|P_{ij}|1,-1)\cdot\varphi(M;\ N-2)+(-1,-1|P_{ij}|-1,-1)\varphi(M+2;\ N-2).\end{aligned}$$

All expressions in (14) are obtained in similar ways by means of the formula (I·3).

Appendix II

We represent a set of quamtum numbers, uniquely determining the energy eigenvalue E of the crystal system, other than the total spin quantum number, S, symbolically by a single letter ρ; thus $E=E(S, \rho)$. The partition function Z of the system may be formulated as

$$Z=\iint\sum_{M=-S}^{+S}\exp\left[-\{2M\beta\mathcal{H}+E(S, \rho)\}/(kT)\right]\cdot\phi(S)g(S, \rho)dSd\rho, \qquad (\text{II}\cdot 1)$$

where $\phi(S)$ and $g(S, \rho)$ are distribution functions of state; $\phi(S)$ being the total number of states having a given eigen-value of the total spin S, out of which a fraction $g(S, \rho)$ have a given set of eigenvalues ρ; $g(S, \rho)$ is normalized in the sense that

$$\int g(S, \rho)d\rho=1.$$

The summation over M in (II·1) may be replaced by the integration, and Z is transformed into

$$Z=\frac{kT}{2\beta\mathcal{H}}\iint\exp\left[\{2S\beta\mathcal{H}-E(S, \rho)\}/(kT)\right]\phi(S)g(S, \rho)dSd\rho.$$

We may introduce here the conception of the semi-invariants into the integration over ρ (in general the multiple integration), and then Z becomes

$$Z=\frac{kT}{2\beta\mathcal{H}}\int\phi(S)\cdot\exp\left[\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\left(-\frac{1}{kT}\right)^n\right]dS, \qquad (\text{II } 2)$$

where the semi-invariants λ_n are defined by the equation.

$$\int\exp\left[E(S, \rho)\cdot t\right]g(S, \rho)d\rho=\exp\left[\sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\cdot t^n\right]. \qquad (\text{II}\cdot 2')$$

If we approximately stop at the third term in the series in the argument of the exponential function of (II, 2), we obtain the expression (22) and (22′) in the text, where Eq. (II, 2) tells us that

$$\lambda_1=\int E(S, \rho)\ g(S, \rho)d\rho=W(S),$$
$$\lambda_2=\int[E(S, \rho)-\lambda_1]^2g(S, \rho)d\rho=\Delta_2(S),$$
$$\lambda_3=\int[E(S, \rho)-\lambda_1]^3g(S, \rho)d\rho=\Delta_3(S).$$

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Models of V-particles

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Possible models of V-particles, mainly concerned with V^0 (V_1^0)-particles, are investigated in a systematic way on the basis of their characteristic features, one being experimental evidences and the other theoretical requirements such as the stability of nuclei and the charge independence. The latters play a role for ruling out a proposed decay mode, $V^0 \to p + \pi^- + \nu$, and the single production of hot V^0-particles, respectively. Since the pair production of V^0-particles seems unfavourable according to recent experimental evidences, we are led to propose a model as considerably plausible that is based upon the classification of elementary particles due to the generalization of the Pais' even-odd rule. Introducing a pair of short lived X-particles yet unobserved, the production and decay of V^0-particles take place through the following interactions:

$$N+N \underset{(g_{even})}{\longrightarrow} X+X, \quad X \underset{(g_{int})}{\longrightarrow} \begin{cases} V^0+\cdots & \\ N+\cdots & V^0 \underset{(g_{odd})}{\longrightarrow} p+\pi^-. \end{cases}$$

The magnitude of coupling constant g_{int} lies between both strong g_{even} and weak g_{odd} and the decay of X into $N+\cdots$ is slightly faster than into $V^0+\cdots$. Several modifications of this model are shown to be permitted.

§ 1. Introduction

Since the discovery of Rochester and Butler several hundreds of the so-called V-particles have been observed in cloud chamber photographs and their existence is now indisputable.

Their established properties together with the most preferable interpretations are summarized in what follows.[1)–36)]

a. Classification of V-particles

There are two kinds of V-particles, charged and neutral. They are further subdivided into more precise species according to their characteristic decay modes:

$$V^0 \to p + \pi^-, \qquad (V^0 = V_1^0)^{4)} \tag{1·1}$$

$$v^0 \to \pi^+ + \pi^-, \qquad (v^0 = V_2^0)^{4)} \tag{1·2}$$

$$v^\pm \to \text{meson}^\pm + ?^0 + (?^0). \tag{1·3}$$

Charged V-particles including S-particles[22)25)] are believed to be identical with κ- or χ-particles[37)–42)] found in nuclear emulsion and presumed to decay as

$$\kappa^\pm \to \mu^\pm + ?^0 + ?^0, \tag{1·4}$$

$$\chi^\pm \to \pi^\pm + ?^0. \tag{1·5}$$

b. Abundance of V-particles

(1) V-particles are observed with penetrating showers. The energies of the incident particles which are responsible for the production of such showers are of the order of 10 Bev, depending upon the counter selection systems usually employed in cloud chamber experiments.

(2) They are observed as frequently as $\gtrsim 1\%$ of penetrating showers. The frequency of all V-particles including unobserved ones is roughly estimated to be about 10% of pions, in reference to the number of $\pi-\mu$ decays and electron-positron pairs observed in the same cloud chamber[4] and the assumed lifetime of the V-particle.

(3) Their relative frequencies are tentatively given by

$$V^0/v^0 \sim \begin{cases} 1.6 \pm 0.5, & \text{(Manchester)}^{4)} \\ 2, & \text{(Pasadena)}^{10)} \end{cases} \tag{1·6}$$

$$\text{charged/neutral} \sim 0.15. \quad \text{(Manchester)}^{5)} \tag{1·7}$$

c. Remarks on the decay scheme of the V^0-particle

(1) It is very likely that the V^0-particle undergoes the two body decay as (1·1) on account of the coplanarity and the balance and distribution of the transverse momenta of decay products.

It can not be concluded, however, that the Q-value is unique, because of the large experimental errors, i.e. Manchester and M. I. T. groups assert the unique Q-value, while Pasadena group opposes to it.

$$Q = 46 \pm 6 \text{ MeV}, \quad \text{(Manchester)}^{4)}$$

$$Q = \begin{cases} 35 \pm 3 \text{MeV}, \\ 75 \pm 5 \text{MeV}, \end{cases} \quad \text{(Pasadena)}^{14)} \tag{1·8}$$

$$Q = 38 \pm 2 \text{MeV}. \quad \text{(M. I. T.)}^{26)}$$

(2) There seems to be no third neutral decay product which gives rise to electron-showers in the lead plates.[4)36)43)]

(3) The negative decay product with mass around meson is π-meson with good certainty, since there are examples identified as $\pi-\mu$ decays.[4)10)42)]

(4) There is no identified anti-proton among decay products of V^0-particles.

d. Remarks on the decay schemes of the v^0- and $v^\pm$-particles

(1) It is not so sure as in the case of V^0-particles that the v^0-particle undergoes a two body decay and that the secondary mesons are π-mesons. Cf. (1·2).

(2) It is not likely that all of $v^\pm$-particles suffer two body decay from the observed distribution of the transverse momenta of charged decay products.[6)]

(3) The $v^\pm$-particles observed so far are lighter than nucleons. See, in this connection, *a.* and *e.*

e. The masses of the V^0-particles

The exact mass values of these particles are yet unsettled. They are roughly given by

$$V^0 \sim 2200\ m_e,$$
$$v^0 \sim 800 - 1000\ m_e, \qquad (1\cdot 9)$$
$$v^{\pm} \sim 1000 - 1500 m_e.$$

f. The lifetimes of the V-particles

$$V^0,\ v^0 \sim 10^{-9} - 10^{-10}\ \text{sec.}$$
$$v^{\pm} \sim 10^{-8} - 10^{-9}\ \text{sec.}$$

We need further experimental elucidation of these phenomena especially for v^0- and $v^{\pm}$-particles, in order to understand the complete nature of these particles, but we shall, for the time being, try to look for possible model schemes as closely as possible based on the above information. On account of meager experimental material for v^0- and $v^{\pm}$-particles our discussions are mainly confined to V^0-particles.

First of all, we should stress the apparent contradiction between the long lifetimes and the copious production of these particles that can never be understood on a simple detailed balance consideration

Suppose an interaction

$$H = i f \bar{\psi}_V \gamma_5 \psi_p \cdot \varphi^* + h.\ c., \qquad (1\cdot 10)$$

where ψ_V, ψ_p and φ represent the wave functions for V^0-particle, proton and pion respectively, which causes the decay of a V^0-particle into a proton and a negative pion, then the lifetime τ is given by

$$\frac{1}{\tau} = \frac{f^2}{4\pi} \cdot \frac{1}{M}\, p(\sqrt{m^2+p^2} - m), \qquad (1\cdot 11)$$

where M and m are the masses of V^0 and proton respectively, and p the momentum of the decay products in the centre of mass system.

With $M = 2200 m_e$ and $\tau = 10^{-10}$ sec.,

$$f^2/4\pi \sim 10^{-11}. \qquad \text{(ps-coupling)} \qquad (1\cdot 12)$$

For pseudovector coupling

$$H = i \frac{g}{\mu} \bar{\psi}_V \gamma_5 \gamma_\mu \psi_p \frac{\partial \varphi^*}{\partial x_\mu} + h.c., \qquad (1\cdot 13)$$

where μ is the pion rest mass, we have only to replace f in the lifetime formula (1·11) by

$$f \to \frac{M+m}{\mu} g,$$

in virtue of the equivalence theorem, and we have for this time

$$g^2/4\pi \sim 10^{-13}. \qquad \text{(pv-coupling)} \qquad (1\cdot 14)$$

At any rate, the interaction

$$VN\pi \tag{1·15}$$

turns out to be very weak compared to the pion-nucleon interaction

$$NN\pi, (f^2/4\pi \sim 10 \text{ or } g^2/4\pi \sim 10^{-1}) \tag{1·16}$$

and it is obvious that the V-particles should be produced through other strong interactions.

Careless introduction of strong interactions, however, brings about rapid decays of V-particles in quite short lifetimes of the order of 10^{-21} -10^{-22} sec. Thus it is undoubtedly one of the central problems in studying the models for these particles how to reconcile their abundance with their long lifetimes.

There is another important subject to be concerned with, the stability of nucleons being a necessary condition for any kind of models for these unstable particles, as far as these are elementary particles.

We shall begin our investigation with the discussion of this condition since it is instructive as well as necessary to get insight into the nature of the V^0-particle.

§ 2. The stability of the nucleons*

We know empirically that the nucleon is stable, i.e. it never decays into lighter particles, otherwise the matter would not be stable.

This stability condition is intimately connected with the problem of heavy unstable particles. For instance, suppose an interaction like $VN\mu$ (V^0: elementary Bose particle) leading to

$$V^0 \to N + \mu, \tag{2·1}$$

then nucleons may decay through this interaction as

$$N + N \to N + V^0 + \bar{\mu} \to \bar{\mu} + \bar{\mu} \qquad (\bar{\mu}: \text{antiparticle of } \mu) \tag{2·2}$$

in contradiction to the stability condition.[45]

Realizing the importance of this condition, we are to express this condition mathematically. It may be necessary and sufficient for a moment to assume that the number of nucleons or more strictly the difference between the number of nucleons and that of anti-nucleons is conserved:

$$n(N) = \int \psi_N^* \, \psi_N \, d^3x = i \int \bar{\psi}_N \gamma_\mu \, \psi_N \, d\sigma_\mu = \text{const.}, \tag{2·3}$$

ψ_N is the quantized wave function of a nucleon.

This was the stability condition for nuclei before the discovery of V-particles. At present, however, it is obvious that $n(N)$ is not a strict quantum number in reference to the V^0-particle decay, and we should improve the expression (2·3) so as to include such particles, whether they are Fermions or Bosons, that are transformable into nucleons.

Thus the expression (2·3) must be generalized into

$$n = n(N) + n(V^0) + \cdots$$

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*) Similar discussions were made by several authors, especially by Oneda.[44]

$$=\sum_{\text{Fermions}}{}'(\pm)i\int\bar{\psi}_s\gamma_\mu\psi_s\,d\sigma_\mu+\sum_{\text{Bosons}}{}'(\pm)\int(\phi_s\frac{\partial\phi_s^*}{\partial x_\mu}-\frac{\partial\phi_s}{\partial x_\mu}\phi_s^*)\,d\sigma_\mu=\text{const.},\qquad(2\cdot4)$$

where the $(\pm)$ signs are determined according to the particle-antiparticle assignment. The whole of the members in $(2\cdot4)$ is called hereafter as the nucleon family. Then the above formula represents the conservation of the number of the nucleon family.

We shall next examine characteristic features of this family. First remember the requirement that the theory of quantized fields is invariant as a whole under charge conjugation. Since n is a strict quantum number, it must undergo a definite transformation under charge conjugation, i.e. even or odd*. The first term $\bar{\psi}_N\gamma_\mu\psi_N$ in $(2\cdot4)$ is odd, and hence we have :

(A) The quantity n is odd under charge conjugation.

We cannot construct such an odd quantity from the wave functions of neutral Bosons** or Majorana particles, and therefore they cannot belong to the nucleon family.

(B) The members of the nucleon family cannot be lighter than the nucleon. Otherwise the nucleons would ultimately be transformed into lighter members in contradiction to the stability condition.

We shall show an application of these results to the investigation of the decay scheme of the V^0-particle.

In the V^0-decay, there remains experimentally a possibility of the third neutral decay product :

$$V^0\to p+\pi^-+?^0.\qquad(2\cdot5)$$

It is known that the third particle would not seriously violate the coplanarity and distribution of transverse momenta of the charged decay products within a rather limited experimental accuracy if it were much lighter than nucleon.[30] Hence the third particle obviously cannot belong to the nucleon family in virtue of (B), and the conservation of the nucleon family requires the V^0-particle to be a member of the nucleon family. (We shall simply denote as N. F. hereafter.) With reference to (A), we see that *the V^0-particle should be a Fermion provided that it is elementary.*

The condition (*c.* 2) excludes the possibility that the third particle is identical with π^0 or γ, and the following mode is especially of interest both experimentally and theoretically :

$$V^0\to p+\pi^-+\nu.\qquad(\nu:\text{neutrino})\qquad(2\cdot6)$$

If it were really the case, the V^0-particle could not be a Fermion and should be a composite substance. This is not very likely on various reasons. The above argumentation

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*) If a conservative quantity is decomposed as the sum of even and odd parts, then both parts are separately conserved.

**) Strictly speaking, the term "neutral Bosons" should be replaced by "neutral Bosons described by real wave functions", however we shall discard such neutral Bosons that are described by complex wave functions in this paper.

gives a theoretical evidence in favour of the two-body decay scheme of the V^0-particle.

Also it must be remarked that the experimental fact (*c.* 4) can be interpreted in terms of the nucleon family conservation.[46]

§ 3. The classification of the models

As has been emphasized in § 1, it is the most important object of study how to reconcile the large yield of V-particle production with their remarkably long lifetimes.

Although the decay schemes of V-particles are comparatively well known, we have as yet only scarce information on their production mechanisms. Thus various assumptions on the production mechanisms lead us to a wide variety of possible models of interaction schemes, which are further multiplied by the minute settlement of decay schemes.[45] We need, therefore, to choose a suitable means of classification, in order to avoid too enormous number of models to be dealt with.

In this respect, it is worth while to remember that V-particles are produced through strong interactions comparable with the $NN\pi$ coupling, but decay through very weak interactions. Cf. § 1. Hence it seems favourable to classify possible models in reference only to their production mechanisms regardless of their decay mechanisms, since the very weak interactions responsible for their decay can, if necessary, be modified independently of other processes.

We shall divide models according to whether V^0-particles are produced directly or indirectly from nuclear interactions. The following two sections are devoted to the full investigation of the above two possibilities.

In this paper, we shall further infer that all V-particles are elementary* but not composite, since the structural theories in general suffer from the high possibilities of the radiative decay unless artificial selection rules as well as ambiguous cut-off procedures are unwillingly employed.

§ 4. Models for the direct production of V-particles

If the V^0-particle is produced directly from nuclear interactions, it should have a strong nuclear interaction (Cf. *b.* 2.), and we must solve the problem how to stabilize the V^0-particle against rapid decays through this strong nuclear interaction.

For this purpose we further subdivide such models into the following two cases:

(I) They are produced singly.

(II) They are produced in pairs.

Theoretical argumentations for the above two cases are completely dissimilar in nature.

The former is naive and preferred experimentally**, while the latter is of theoretical interest and instructive for further investigations as we shall see in the next section. The most conclusive discrimination between the above two will be furnished by the determina-

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*) Interactions are, of course, considered within the frame of the canonical formalism.

**) If they are always produced in pairs, the probability that two V^0's are observed without missing one of them is expected to be much larger than the observed ones.[16)21)]

tion of the threshold for the production reaction as has been stressed by Pais.[47] In this connection, the production of V^0-particles at very low energies is especially interesting.

I. Models for single production

If V^0 is hot, i.e. has strong nuclear interaction, and produced singly*, the production mechanism will be

$$\left.\begin{matrix}\pi\\N\end{matrix}\right\} + N \rightarrow V^0 + a + \cdots. \tag{4·1}$$

The particles $a, \cdots$should be hot, since they interact with nucleons through a strong interaction (4·1), and may be identical with nucleon, pion or possibly photon.

Then a V^0 would decay through the interaction (4·1) as

$$V^0 \rightarrow N + \left\{\begin{matrix}\pi\\N\end{matrix}\right. + \bar{a}\cdots\cdots, \tag{4·2}$$

where $\bar{a}, \cdots$are the antiparticles of $a, \cdots$.

The transition (4·2) is in most cases forbidden energetically, but V^0 can ultimately decay through (4·2) as a virtual step :

$$V^0 \rightarrow N + \left\{\begin{matrix}\pi\\N\end{matrix}\right. + \bar{a} + \cdots \rightarrow N + \pi \text{ and/or } \gamma. \tag{4·3}$$

No absolute selection rule is operative in the above process since Fermions are involved. Then we are forced to prepare special devices to stabilize V^0.

In general, selection rules are intimately connected with the conservative quantities which we shall inquire for a system involving Fermions. Those are the total angular momentum J and the total isotopic spin I of the system, and we shall give two kinds of models utilizing J and I.

The V^0-particle with a very large spin

As remarked above the V^0-particle would, for such Q-values as listed in (*c*. 1), decay very rapidly as

$$V^0 \rightarrow p + \pi^-,$$

if the orbital angular momentum of the final state is small.

The decay probability may be so much depressed for very high angular momentum states, say 7 or 8 as suggested by many people, that the lifetime of V^0 turns out to be sufficiently longer to account for the striking stability of the V^0-particle. Let us, therefore, consider a model in which a very large spin of V^0 is assumed.

Since V^0 is supposed as elementary in this paper, the wave function may be expressed in a spinor-tensor mixed form as

$$\psi_{\alpha\beta\cdots\cdots}. \tag{4·4}$$

The decay of V^0 can be attributed to such an interaction as

*) By singly, we mean that not only another V^0 but also v^0 and $v^\pm$ are not accompanied with the one V^0.

$$if\ \bar{\psi}_{\alpha\beta\ldots}\gamma_5\psi\cdot(\partial_\alpha\partial_\beta\cdots\phi^*)+h.c., \tag{4·5}$$

where ψ is a nucleon wave function and the coupling constant f is sufficiently large to account for the production rate.

To our regret such a highly singular interaction is far out of the scope of existing localizable field theories, and cannot be regarded as an elementary interaction, i.e. we should derive an effectively equivalent interaction from the combination of other elementary interactions.

For instance, introduce a series of auxiliary elementary particles being members of N.F.

$$N_1, N_2, \cdots\cdots, N_s, \tag{4·6}$$

where N_s has a spin $s+\frac{1}{2}$.
Also introduce a series of elementary interactions

$$V^0 N_s \pi,\ N_s N_{s-1}\pi, \cdots,\ N_1 N_0 \pi. \quad (N_0=N) \tag{4·7}$$

A typical form of such interactions is given by

$$\bar{\psi}_{\alpha\beta\ldots\ \lambda\mu}\gamma_5\psi_{\alpha\beta\ldots\lambda}\cdot\partial_\mu\varphi. \tag{4·8}$$

Then they may effectively lead to the coupling (4.5).
We have, however to settle a restriction on the masses of members of (4·6) in order to avoid such possible rapid decays of V^0 as

$$V^0\to N_t+\pi \text{ or } \gamma, \qquad (t\geq 1) \tag{4·9}$$

which prevent V^0 to actualize the observed decay.

For this purpose, it is necessary and sufficient to settle the relation*):

$$m(N_t)>m(V^0), \quad (t=1,\ 2,\cdots s) \tag{4·10}$$

$m(N_t)$ is the rest mass of particle N_t.

It is worth noticing that such an arbitrary choice of the masses is allowed only for elementary particle theory but not for structural theories.

The V^0-particle with a very large isotopic spin

For a system composed of nucleons and pions, the isotopic spin I is a very good quantum number.**)
If V^0 is an elementary particle having a large isotopic spin, then the decay

$$V^0\to p+\pi^- \tag{4·11}$$

becomes highly forbidden, since the system $p+\pi^-$ can assume only two values $I=1/2$ and 3/2, which are very low compared to that of V^0.

In order to realize the observed decay, therefore, we need to take account of the electromagnetic interaction that violates the charge independence. Every time the electromagnetic interaction takes part in, the isotopic spin changes by

*) However it is not guaranteed that V^0 will surely decay into a proton and a negative pion.

**) The conservation of the isotopic spin is not assumed for the weak interaction which gives rise to the decay of a V^0 into $p+\pi^-$.

$$\Delta I = 0, \pm 1, \qquad (4\cdot12)$$

and $\alpha = "1/137"$ is multiplied to the square of the absolute value of the transition matrix element, until the decay is realized. Hence the transition probability will become sufficiently small provided that the isotopic spin of V^0 is fairly large.

There is, however, a fatal objection against this model.[48] If V^0 had really a large isotopic spin, high charge states would exist and be observable, which contradicts to our experience. Cf. (*d.* 3).

II. Models for pair production

As we have seen above, the models for single production are more or less unsatisfactory, if not quite unacceptable.

Hence several authors have proposed models in which unstable particles are produced always in pairs*). This model is now studied with some generalizations.

By suitable means to be stated later, we first divide hot elementary particles**) among several groups: $G_0, G_1, \cdots$. $(4\cdot13)$

To group G_0 belong nucleons, pions, and photons, i.e. familiar hot particles, and we let V-particles belong to group G_1 if they were really hot. To other groups belong, if any, other hot particles yet unobserved possibly due to their short lives.

Next, let the number of particles of group G_a for a given state be

$$n(G_a), \qquad (4\cdot14)$$

and also define the following quantity

$$\varepsilon(G_a) = (-1)^{n(G_a)}. \qquad (4\cdot15)$$

Now we introduce the following assumption:

"*For strong interactions responsible for the production of hot particles,* $\varepsilon(G_1)$, $\varepsilon(G_2), \cdots$ *are separately conserved.*"

Of course, $\varepsilon(G_0)$ is not conserved.

Since V-particles are produced in pion-nucleus or nucleon-nucleus impact, $n(G_a)$ $(a = 1, 2, \cdots)$ in the initial state are equal to zero, so that $\varepsilon(G_a) = 1$. Hence, on the above assumption, ε's should be equal to 1 in the final state too.

Thus V-particles are produced in pairs.

If, on the contrary, they were produced always in pairs, it results that ε's should be conserved by a suitable grouping of hot elementary particles.

As for the decays, the situation stands quite clear. For instance, the decay of V^0 into $p + \pi^-$ is forbidden since $\varepsilon(G_1)$ changes from -1 to $+1$. We see that strong interactions do not contribute to the decay processes and we have only to introduce such very weak interactions as $(1\cdot12)$ or $(1\cdot14)$ in order to realize the observed decays.

We shall from now on call the strong interactions for production as even and the very weak interactions for decays as odd, and denote these coupling constants as g_{even} and

*) We mean that V^0 is produced always accompanied with another V^0 or $\bar{V}^0$. See, for instance, ref. 47.

**) Photons are considered to be hot in this case.

g_{odd} respectively. They represent only the order of magnitude and indiscriminately be used for various interactions, as has been utilized by Pais.

Their rough magnitudes are given by

$$\frac{g^2_{even}}{4\pi}\sim 1, \qquad \frac{g^2_{odd}}{4\pi}\sim 10^{-12}. \tag{4·16}$$

These values are of course susceptible to fluctuations by as large a factor as $\sim 10^2$, depending upon interactions.*)

Pais' even-odd rule

Pais divided hot particles between two groups, i.e. particles with even suffices and those with odd suffices, the former is equivalent to our G_0 and the latter to our G_1.

In his notation, hot particles are denoted as

$$\begin{aligned} N &= N_0 \quad N' = N_1 \cdots, \\ \pi &= \pi_0, \quad \tau = \pi_1, \cdots, \end{aligned} \tag{4·17}$$

and the even-odd rule is expressed as follows:

"The coupling $N_i\ N_j\ \pi_k$ is strong or weak according as $i+j+k=$even or odd." This rule is obviously equivalent to the statement that $\varepsilon(G_1)$**) is conserved for strong interactions, and we see that our present treatment is a generalization of the even-odd rule.

We should add here some remarks concerning the decay processes. The decay of an unstable particle into lighter unstable particles is not forbidden by the even-odd rule, and we must present special devices for each case. Pais were able to forbid the unfavourable process

$$N_1 \to N_0 + \pi_1 \tag{4·18}$$

by the mass relation

$$m(N_1) < m(N_0) + m(\pi_1). \quad \text{Cf. } (1·9)$$

As the number of the members of the unstable particles increases, however, such a device becomes increasingly difficult. For instance, how to take the decay of a χ particle into π and τ mesons***),

$$\chi^{\pm} \to \pi^{\pm} + \tau^0, \tag{4·19}$$

into Pais' scheme, provided that the above process is experimentally accessible? In view of this difficulty, the present generalization is not trivial, but more than two groups even and odd may be needed in future.

§ 5. Models for the indirect production of V-particles

In the previous section, we reached the conclusion that no model can satisfactorily

*) For instance, $f^2/4\pi\sim 10$ for $NN\pi$, and $e^2/4\pi\sim 1/137$ are indiscriminately represented by the same symbol $g^2_{even}/4\pi$.

**) It may be better to replace $\varepsilon(G_1)$ by $\varepsilon(G_{odd})$.

***) We assume tentatively that the suffices of χ and τ are odd.

account for the single production of $l^{\prime 0}$, provided that they are produced directly.

Hence we are inclined to think of that $l^{\prime 0}$ particles may be produced indirectly via other hot particles.

The long lives of $l^{\prime}$-particles suggest us an idea that they may be cold but not hot. If it were really the case, their large abundance might be explained just in a similar way to "*the two meson theory*." Along these lines of reasoning, we shall examine such models in what follows. The most essential point in these models is to assume the existence of unknown particles of short lives. However, the particles that we know at present by direct observations are restricted to such ones whose lives are longer than $\sim 10^{-12}$ sec., besides indirectly observed π^0, and we have no *a priori* reason to rule out the existence of unstable particles with shorter lives.*)

There is no tax imposed upon the elementary particles.

We shall, therefore, assume the existence of short-lived parents of V^0-particle.

So far such models have been buried by arguments as follows. The parent particle X is produced by a reaction like

$$N+N\rightarrow X+N, \quad \text{(strong)} \tag{5·1}$$

then X decays to produce V^0:

$$X\rightarrow V^0+\cdots. \tag{5·2}$$

The coupling (5·2) should be strong, otherwise X will rapidly decay through (5·1), before decaying into V^0, as

$$X\rightarrow N+N+\bar{N}\rightarrow N+\pi \text{ or } \gamma. \tag{5·3}$$

If the coupling (5·2) were unfavourably strong, then $l^{\prime 0}$ will decay rapidly through (5·1) and (5·2) as

$$V^0\rightarrow X+\cdots\rightarrow N+\pi \text{ or } \gamma \tag{5·4}$$

in contradiction to our experience.

We can, however, remedy the above model by a slight modification. The difficulty of the above model arises in the instability of X, which we must improve.

Now remember the discussions of § 4, II, then one will readily be able to stabilize X, i.e. we have only to replace (5·1) by

$$N+N\rightarrow X+X. \quad \text{(strong)} \tag{5·5}$$

Since X is stable, we can introduce a weak interaction

$$X\rightarrow V^0+\cdots \quad \text{(weak)} \tag{5·6}$$

without being bothered by the competing process (5·3). If (5·6) is the only decay mode of X, however, we again face to the pair production of V-particles.

Hence we improve (5·6) as

*) On the contrary, the existence of such very short-lived particles (and consequently very hot) as nucleon isobars may be recognized indirectly.

$$X \to \begin{matrix} N + \cdots (g_N) \\ V^0 + \cdots (g_V) \end{matrix}, \qquad \text{(weak)} \tag{5·7}$$

where the two modes are competing.
We assume further that the former mode is several times more probable than the latter, then in most cases V^0 appears singly:

$$N + N \to X + X \to V^0 + N + \cdots, \tag{5·8}$$

and the pair production

$$N + N \to X + X \to V^0 + V^0 + \cdots \tag{5·9}$$

are rarely observed.

The condition imposed upon the magnitudes of the coupling constants g_N and g_V are stated as follows.

(1) In order that V^0 is produced singly in most cases

$$g_N^2/4\pi > g_V^2/4\pi. \tag{5·10a}$$

(2) The decay of V^0 through (5·7) as

$$V^0 \to X + \cdots \to N + \cdots$$

should be sufficiently slow. This condition is roughly expressed in the order of magnitude theory by

$$(g_N^2/4\pi)(g_V^2/4\pi) \lesssim g_{odd}^2/4\pi. \tag{5·10b}$$

(3) Since X is not observed and the coplanarity of V^0-decay holds in most cases*), the lifetime of X must satisfy the inequality

$$\tau_X < 10^{-12} \text{ sec.} \tag{5·10c}$$

Of these the first condition (5·10a) is not so serious and we may regard g_N and g_V as being of the same order in our crude discussion, so that we shall indiscriminately denote them as g_{int} just as g_{even} and g_{odd}.

The meaning of the suffix *int.* will stand clear soon.
For the next condition (5·10b), we now read

$$(g_{int}^2/4\pi)^2 \lesssim g_{odd}^2/4\pi \sim 10^{-12}$$

or

$$g_{int}^2/4\pi \lesssim 10^{-6}. \tag{5·11}$$**)

The last condition (5·10c) requires

$$g_{int}^2/4\pi \gtrsim 10^{-10}. \tag{5·12}$$

We see that the two inequalities (5·11) and (5·12) are consistent with each other, and that such a model can assert itself as possible. That the magnitude of g_{int} falls between

*) The rather rare non coplanar cases may be attributed to the appearance of N in the intermediate stage.

**) Rigorously speaking, this condition becomes looser if we take into account the matrix elements.

g_{even} and g_{odd} is the reason for the suffix *int.* being the abbreviation of "intermediate."

The above model can further be generalized as to include other unstable particles than V^0.

Following the method described in the previous section, we divide the hot elementary particles into several groups

$$G_0,\ G_1,\cdots\cdots. \tag{5·13}$$

V^0 belongs to none of them, since it is considered as being cold in the present section, while X belongs to G_1. As for other unstable particles we are not sure yet whether they are cold or hot, but hot members, if any, including parent particles should belong to some of the above groups except G_0. And then *we assume again that* ε*'s are very good quatum numbers. Observed hot particles decay with the coupling constant* g_{odd}, *while unobserved hot parent particles decay**) *with* g_{int}.

Next we shall prove that the decay of an observed unstable particle through a parent is sufficiently slow.

The process must at least be the second order in g_{int}, as the unobserved parent is created and then destructed, i.e.

observed V's$\xrightarrow{g_{int}}$unobserved parents$\xrightarrow{g_{int}}$members of G_0,

since, however, we know the inequality

$$(g_{int}^2/4\pi)^2 \lesssim g_{odd}^2/4\pi,$$

the stability is guaranteed.

This model permits a number of modifications. One of them is as follows. The process (5·5) may be replaced by

$$N+N\rightarrow X_1+X_2, \qquad (g_{even}) \tag{5·14}$$

$$\begin{cases} X_1\rightarrow V^0+\cdots, \\ X_2\rightarrow N+\cdots. \end{cases} \qquad (g_{int}) \tag{5·15}$$

In this model, V^0 is always produced singly, but not in competition as was the case for (5·7). Further the threshold of V^0 production can be lowered**), if necessary, by assuming

$$\begin{aligned} &X_1\rightarrow V^0+\gamma, \\ &X_2\rightarrow N+\gamma. \end{aligned} \tag{5·16}$$

If X_1 were much heavier than X_2, X_1 might decay into $N+\cdots$ as

---

*) They must decay into the members of G_0 as well as into V-particles in order that V's are produced singly.

**) In the former model (5·5), both X's had to be heavier than V^0, while in the above model X_2 is not necessarily heavier than V^0.

$$X_1 \underset{g_{even}}{\rightarrow} X_2 + V + V \underset{g_{int}}{\rightarrow} V + \gamma + \cdots, \tag{5·17}$$

but it is not hard to let X_1 decay into V^0 faster than (5·17).

Another example similar to Pais' even-odd scheme is also possible in our model.

$$NN\pi,\ XNx,\ XX\pi, \qquad (g_{even}) \tag{5·18}$$

$$X,\ x \rightarrow \text{members of } G_0 \text{ and cold } V\text{'s}, \quad (g_{int}) \tag{5·19}$$

where X is a Fermion parent and x a Boson parent.

§ 6. Charge independence and V-particles

As seen from the arguments of the previous sections, V^0 may be produced indirectly rather than directly. There is another reason to support this conclusion though it is not conclusive.

In connection with the recent investigations concerning pion reactions, the importance of the charge independence becomes widely recognized. At low energies available at present in laboratories, it seems likely that the charge independence holds, whereas we know very little about its validity at high energies. It seems quite probable that the charge independence holds near the threshold for the V-production, since the threshold energy is not far from that available in laboratories.

If it were unfortunately not the case, it would seriously disturb virtually the charge independence even at considerably low energies, since the interaction is comparably strong with the pion-nucleon interaction*). The lower the threshold, the more likely this assumption becomes. We may, therefore, assume the charge independence at such energies tentatively in this section.

The so-called charge independence means that the theory is invariant with respect to rotations in isotopic spin space provided that we take only very hot interactions into account.**)

The infinitesimal rotation operators in isotopic space are given by

$$\boldsymbol{I} = \frac{1}{2} i \int \bar{\psi}_N \boldsymbol{\tau} \gamma_\mu \psi_N \, d\sigma_\mu + \int \boldsymbol{\phi} \times \frac{\partial \boldsymbol{\phi}}{\partial x_\mu} \, d\sigma_\mu, \tag{6·1}$$

the thick letters denote vectors in isotopic space.

The representation is chosen as

$$\tau_3 p = p, \qquad \tau_3 n = -n. \tag{6·2}$$

If there were other very hot particles, it may be better to add terms like

$$\frac{1}{2} i \int \bar{\Psi} \boldsymbol{T} \gamma_\mu \Psi \, d\sigma_\mu \tag{6·3}$$

*) Cf. (*b*. 2). Also if we take account of the alternative decay mode $V^0 \rightarrow n + \pi^0$, the production yield should be larger.

**) Since electromagnetic interactions (hot in our nomenclature) are excluded, we used the term "very hot".

to $\boldsymbol{I}$ for Fermions and similar ones for Bosons. We shall call the members in the expression $\boldsymbol{I}$ by a collective name "group C."

For convenience, we assign zero isotopic spin to particles which are not the members of group C, regardless of their charge, since they do not undergo transformations in the isotopic space.

An important aspect of the isotopic spin is that Fermions can assume half integral values as well as integral values, while Bosons can assume only integral values.

Since there seems to be no charged counter particle to $l^{\tau 0}$, we have to assign zero isotopic spin to $V^{\tau 0}$. Cf. $(d.\ 3)$.

Suppose that $l^{\tau 0}$ is produced singly and that there is no other member of N. F. participating in this process but nucleons and $V^{\tau 0}$, then

$$n = n(N) + n(V^{\tau 0}) = \text{const.} \tag{6·4}$$

If $V^{\tau 0}$ is produced singly*), then we have from (6·4)

$$\Delta n(V^{\tau 0}) = +1, \text{ and } \Delta n(N) = -1. \tag{6·5}$$

Now consider the corresponding change of isotopic spin

$$\begin{aligned} \Delta I_3 &= \Delta I(N)_3 + \Delta I(\text{Bosons})_3 \\ &= \pm 1/2 + \text{integer} \neq 0, \text{ i.e. not conserved.} \end{aligned} \tag{6·6}$$

Thus the above assumption that $l^{\tau 0}$ is singly produced is not acceptable on the basis of the hypothesis of charge independence, if $V^{\tau 0}$ were produced directly. This is another reason to support the models for indirect production, though the assumptions made in this section are not conclusively grounded**).

§ 7. Conclusions

We have so far investigated the consistency of various models based upon experimentally established features of $V^{\tau 0}$-particles, i.e. (1) their long lives, (2) their large abundance, and (3) the evidence that they are produced singly in most cases.

Our main conclusion is that these unstable particles, at least $V^{\tau 0}$, seem to be produced indirectly via short-lived parent particles.

As for their decay mechanisms, we have only to introduce very weak interactions causing observed decays, which obviously do not affect other processes as fully argued in the present paper.

Once the interaction scheme is established, Fukuda-Miyamoto's theorem[40] as well as various absolute selection rules will play an important rôle, but notice that competitions should be discussed only after the settlement of model schemes like "even-odd rule". For instance, if light ν' were cold, Pais' argument on the competition between $\pi_1^{\pm} \to \mu^{\pm} + \nu$ and $\pi_1 \to 2\pi_0$ must completely be altered. Since the application of various selection rules to the observed

*) The other ν's may be accompanied with.

**) A Similar argument rules out the composite theory of $V^{\tau 0}$-particle.

decay processes seems to be rather trivial at present, we do not further touch upon their decay mechanisms.

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Note added in proof.

To section 4: It is not certain that the interaction like (4.8) can be treated within the scope of the present field theory.

To section 6: Recently Leighton *et al.* suggested the existence of a heavy V^+-particle.
(York, Leighton, and Bjornerud, Phys. Rev. **90** (1953), 167.)
So long as their abundance is much less than that of V^0, however, the argument in § 6 is not altered.

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Difficulty of Divergence of the Perturbation Method in the Quantum Field Theory

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The behaviour of the S-matrices is investigated in the cases of the systems with Lagrangians of (1.2) and (1.9). The series-representation of S in powers of the coupling constant diverges though each term of this series is finite by virtue of the non-local form factor. The divergence of this series results from the fact that the number of graphs corresponding to the n-th term increases rapidly with the increase in n. The number of graphs is estimated by taking advantage of the "Green-function"

Introduction

The recent development of the theory of S-matrix in the quantum field theory is quite brilliant, especially it is quite noticeable that every term S_n of the S-matrix is proved to be finite when S is expanded as a power series of the coupling constant e such as

$$S=\sum_n e^n S_n,$$

and the procedure of renormalization has been used. As Dyson[1] has pointed out, however, the series introduced above seems not to converge in general though renormalization has been performed.

Accordingly it is worth-while to make a study of the convergence of this series in more detail. For example, let us consider in the quantum electrodynamics a particular scattering problem with specified initial and final states. The corresponding matrix element of S is a sum of contributions from various irreducible graphs G. A particular contribution from a particular G is to be written down as an integral over momentum variables according to the well known rule. The integrand is a product of $S'_{F1}(p_i)$, $D'_{F1}(p_i)$, $I'_{\mu 1}(p_i \cdot p'_i)$ and free operators. Here S'_{F1}, D'_{F1}, $I'_{\mu 1}$, correspond to internal electron-lines, and vertices in G respectively, and include every contribution from radiative interactions to the internal lines or vertices of G. These functions themselves are represented as power series of the charge constant e. These series as well as the infinite sum of contributions from all the irreducible graph G's are presupposed to be convergent provided that the renormalization has been performed. If the power series S'_{F1} does not converge, then the element of S-matrix being represented as a power series must be divergent as a matter of course.

Accordingly it is necessary to investigate the convergence of the series S'_{F1} etc. before we examine the more general problem, that is, the convergence of any matrix element of S. S'_{F1} can be represented as

$$S'_{F1}=\sum_{n=0}^{\infty} c^n S_{F1}^{(n)},$$

where each $S_{F1}^{(n)}$ is finite in virtue of the renormalization procedure, that is to say, the infinity of $S_{F1}^{(n)}$ arising from the interaction is compensated by an explicit infinite constant. The finite difference $S_{F1}^{(n)}$ between these infinities, however in practice, can not be given explicitly. This situation makes it very difficult to prove the convergence of S'_{F1}. In order to avoid this difficulty we shall introduce a non-local interaction. Since it is quite hard to introduce a non-local interaction in quantum electrodynamics on account of the requirement of the gauge invariance, let us consider, for simplicity, a system of the neutral scalar meson field with non-local and non-linear self-interaction. Even in this simple case, it is also troublesome to construct the unitary S-matrix. In fact the S-matrix we obtained in this paper seems not to be unitary as will be shown later. Though this non-unitary character of S is a great defect, however, the conclusion of this paper may be not affected by this defect as will be explained in more detail in § 3. The discussions concerning the S-matrix in case of the non-local interaction will be given elsewhere by the present authors in the near future.

The result obtained in this paper gives one of the conditions restricting the so-called "form-factor" of the non-local interaction. Since this conclusion essentially depends on the scalar character of the field considered, it is doubtful to expect that the similar conclusion will be obtained in case of Fermion fields, especially in the quantum electrodynamics. According to our opinion, however, the S-matrix in the latter case may be also divergent by virtue of the number of graphs.

§ 1. Construction of S-matrix

For the sake of simplicity, let us consider a non-linear neutral scalar meson field $\varphi(x)$ with the Lagrangian density

$$L(x)=-\frac{1}{2}(\partial_\mu\varphi\cdot\partial_\mu\varphi+M^2\varphi^2)+\frac{g}{3}\varphi^3+J\cdot\varphi. \qquad (1\cdot1)$$

Here $J(x)$ is a c-number function describing the external source.

The S-mairix of this system has only two types of the primitive divergent graph which are shown in Fig. 1. In order to avoid this divergence we take the Lagrangian density as

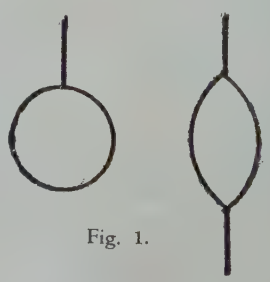

Fig. 1.

$$L(x)=-\frac{1}{2}(\partial_\mu\varphi\cdot\partial_\mu\varphi+M^2\varphi^2)+J(x)\varphi(x)$$
$$+\frac{g}{3}\int U(x,x',x'')\,T(\varphi(x),\varphi(x'),\varphi(x''))\,dx'dx'' \qquad (1\cdot2)$$

in place of $(1\cdot1)$, where T means Wick's chronological operator.

The function U in $(1\cdot2)$ is the so-called "form factor" and symmetric function of $(x-x')(x'-x'')(x''-x)$ with the property

$$U(x,x',x'')\begin{cases}\neq 0: & \text{only if each of } x-x',\ x'-x'',\ x''-x \text{ is space-like,}\\ =0: & \text{otherwise.}\end{cases}$$

Further U is supposed to satisfy Bloch's conditions,[2] that is to say, the contribution from the integral in $(1\cdot2)$ only arises from the spatial vicinity of the point x.

The S-matrix corresponding to $(1\cdot2)$ can be obtained in the following way by using Schwinger's[3] new method of quantization. Let us start from the case of the free meson field $\varphi_0(x)$. If the infinitesimal disturbance

$$\frac{1}{3}gT(\varphi_0(x_1)\varphi_0(x_1')\varphi_0(x_1''))U(x_1x_1'x_1'')\,dx_1dx_1'dx_1''$$

is inserted into this free system at any particular set of points $(x_1x_1'x_1'')$, the field operator $\varphi_1(x)$ after the disturbance will be infinitesimally modified, i. e.

$$\varphi_1(x)\begin{cases}=\varphi_0(x) & t<\mathrm{Min}(t_1t_1't_1'')\\ \neq\varphi_0(x) & t>\mathrm{Max}(t_1t_1't_1'').\end{cases}$$

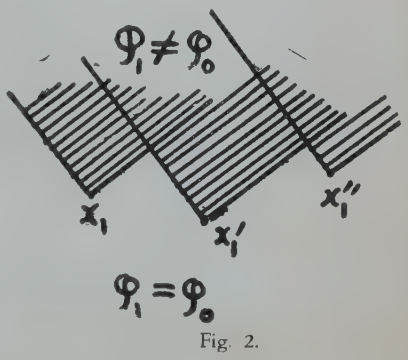

Fig. 2.

The variation of the transformation function corresponding to the above disturbance is given by

$$\delta_1\Big(\varphi_0(x)=\varphi',\sigma_1=+\infty\mid\varphi_0=\varphi'',\sigma_2=-\infty\Big)$$
$$=i\Big(\varphi_0=\varphi',+\infty\ \Big|\ \frac{g}{3}T(1)U_1d\omega_1\ \Big|\ \varphi_0=\varphi'',-\infty\Big) \qquad (1\cdot3)^{4}$$

with the abbreviation

$$T(1)=T(\varphi_0(x_1)\varphi_0(x_1')\varphi_0(x_1'')),$$
$$U_1=U(x_1x_1'x_1''),$$
$$d\omega_1=dx_1dx_1'dx_1''.$$

Next suppose that the second disturbance $\frac{1}{3}gT(2)U_2d\omega_2$ is inserted into the system of φ_1-field, and denote the meson field after the disturbance by $\varphi_2(x)$. The variation of the transformation function in this case is given by

$$\delta_2(\varphi_1=\varphi' : \sigma_1=+\infty \mid \varphi_0=\varphi'' : \sigma_2=-\infty)$$
$$=(\varphi_2=\varphi' : +\infty \mid \varphi_0=\varphi'' : -\infty)-(\varphi_1=\varphi' : +\infty \mid \varphi_0=\varphi'' : -\infty)$$
$$=i\Big(\varphi_1=\varphi' : +\infty \mid \frac{g}{3} T(2)U_2 d\omega_2 \mid \varphi_0=\varphi'' : -\infty\Big). \tag{1·4}$$

Continuing this procedure step by step, we get a sequence of functions $\varphi_0, \varphi_1 \cdots$ with the limit $\varphi=\lim \varphi_n$, and that of functionals $(\varphi_0=\varphi' \mid \varphi_0=\varphi'')$, $(\varphi_1=\varphi' \mid \varphi_0=\varphi'')$, $\cdots (\varphi_n=\varphi' \mid \varphi_0=\varphi'')\cdots$. The S-matrix is defined by

$$\lim_{n\to\infty}(\varphi_n=\varphi' : +\infty \mid \varphi_0=\varphi'' : -\infty)=(\varphi_0=\varphi' : +\infty \mid S_U \mid \varphi_0=\varphi'' : -\infty). \tag{1·5}$$

The left hand side of (1·5) can be obtained by solving the equations

$$\delta_n(\varphi_{n-1}=\varphi' \mid \varphi_0=\varphi'')\equiv(\varphi_n=\varphi' \mid \varphi_0=\varphi'')-(\varphi_{n-1}=\varphi' \mid \varphi_0=\varphi'')$$
$$=i\Big(\varphi_{n-1}=\varphi' \;\Big|\; \frac{g}{3} T(n)\cdot U_n d\omega_n \;\Big|\; \varphi_0=\varphi''\Big).$$

Thus we get in case of $J(x)\equiv 0$

$$S_U=\sum_{n=0}^{\infty}{}' \frac{(i)^n}{n!\,3^n} g^n \int_{-\infty}^{\infty} \prod_{j=1}^{n} U_j \cdot T\{\prod_{j=1}^{n}\varphi_0(x_j)\varphi_0(x_j')\varphi_0(x_j'')\}d\omega_1\cdots d\omega_n, \tag{1·6}$$

and

$$S_U=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(i)^{n+m}g^n}{n!\,m!\,3^n}\int_{-\infty}^{\infty}\prod_{j=1}^{n}U_j\prod_{k=1}^{m}J(y_k)$$
$$\times T\{\prod_{j=1}^{n}\varphi_0(x_j)\varphi_0(x_j')\varphi_0(x_j'')\prod_{k=1}^{m}\varphi_0(y_k)\}d\omega_1\cdots d\omega_n\cdot dy_1\cdots dy_m \tag{1·6}'$$

in case of $J\neq 0$.

In the graphical representation we shall show the form factor U by a small triangle (being called a U-vertex). Since the conservation of energy-momentum of internal lines being incident on any U-vertex holds also as in case of local interactions, it is easily seen that in the momentum representation the only difference betwen S_U and that of the local interaction is the occurrence of the factor $U(k,l)$ at every U-vertex. Here $U(k,l)$ is the Fourier coefficient of $U(x,x',x'')$, and a symmetric function of k and l. (cf. Fig. 3). Therefore Dyson's line of reasoning concerning the primitive divergence is also applicable to our case. The necessary condition of the primitive divergent graph is

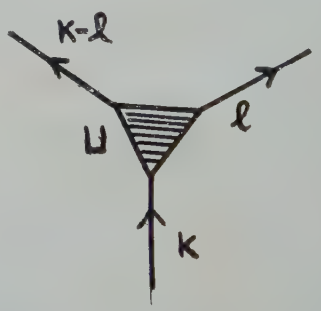

Fig. 3.

$$n(1+\lambda)+E+E'\leq 4, \tag{1·7}$$

where n is the number of U-vertices, E that of external lines, E' that of J's, and

finally λ is defined in such a way that $U(kl)$ has an asymptotic form $(const/k^{\lambda})$ for large k. From (1·7) the primitive divergence does not occur if we choose λ to be >2. For example,

$$U(k,\ l)=\{(1+r_0k^2)(1+r_0l^2)(1+r_0^2k^2)\}^{-1},\quad r_0=const.\ll\frac{1}{M}$$

has a sufficient ability to make the integral convergent.

From the reasons stated in the introduction, the propagation function $\Delta'_F(x, x')$ including the interaction effect is necessary for the investigation on the problem of convergence. According to the paper[5] of the present authors (in what follows this paper is cited as U. S. I.) Δ'_F is defined by

$$\Delta'_F(x,x')=\left(\Psi^*_{vac}T(S_U\varphi_0(x)\varphi_0(x'))\Psi_{vac}\right)\Big/\left(\Psi^*_{vac}S_U\Psi_{vac}\right)$$
$$-\left(\Psi^*_{vac}T(S_U\varphi_0(x))\Psi_{vac}\right)\left(\Psi^*_{vac}T(S_U\varphi_0(x))\Psi_{vac}\right)/(\Psi^*_{vac}S_U\Psi_{vac})^2,$$

where $T(S_U\cdot\varphi_0(x))$ means

$$\sum_{n,m=0}^{\infty}\frac{i^{n+m}}{n!\,m!\,3^n}\int_{-\infty}^{\infty}\prod_{j=1}^{n}U_j\prod_{k=1}^{m}J(y_k)\,T\left\{\varphi_0(x),\prod_{j=1}^{n}\varphi_0(x_j)\varphi_0(x'_j)\varphi_0(x''_j),\prod_{k=1}^{m}\varphi_0(y_k)\right\}$$
$$\times d\omega_1\cdots\cdots d\omega_n\cdot dy_1\cdots\cdots dy_m.$$

Following Schwinger's simple notation, we can rewrite the above expression as

$$i\Delta'_F(x_1\,x')=i\{\langle T(\varphi(x),\varphi(x'))\rangle-\langle\varphi(x)\rangle\langle\varphi(x')\rangle\}=G(x,x'). \qquad (1\cdot8)$$

This is just the Green-function introduced by Schwinger. Accordingly it is necessary to make a study of the behaviour of G for the discussion of the convergence of S-matrix.

We have been so far only concerned with the most simple case of the Lagrangian (1.2) (we call this Model I). In place of this simple case we shall investigate also a little more general model (in what follows this is called Model II) in which the Lagrangian density is

$$L(x)=-(\partial_\mu\bar\psi,\partial_\mu\psi+M^2\bar\psi\psi)-\frac{1}{2}((\partial_\mu A)^2+\mu^2A^2)$$
$$+g\int U(x,x',x'')\,T\{A(x).\,\bar\psi(x')\psi(x'')\}\,dx'dx''$$
$$+J(x)A(x)+\bar\rho(x)\psi(x)+\bar\psi(x)\rho(x), \qquad (1\cdot9)$$

A=neutral scalar field

ψ=complex scalar field: $J,\rho,\bar\rho$=c-number functions.

In this case there occurs also no primitive divergence if the form factor U satisfies the same conditions with those of model I.

It must be noted that the S-matrix S_l defined by $(1\cdot6)'$ is clearly not unitary. However we believe that this defect of non-unitarity has no influence on our conclusion which will be stated in the last section.

§ 2. Green function

Following Schwinger's new method of quantization, we get the q-number field equation

$$(\Box - M^2)\varphi(x) + g\int_{-\infty}^{\infty} U(x, x', x'')\, T\{\varphi(x')\varphi(x'')\}\, dx'dx'' = -J(x). \quad (2\cdot1)$$

From this the following system of equations is derived

$$(\Box - M^2)\langle\varphi(x)\rangle + g\int_{-\infty}^{\infty} U(x, x', x'')\langle\varphi(x')\rangle\langle\varphi(x'')\rangle dx'dx''$$

$$-ig\int U(x, x', x'')\, G(x', x'')dx'dx'' = -J(x), \quad (2\cdot2)$$

$$(\Box - M^2)G(x, x') + 2g\int U(x, y', y'')\langle\varphi(y')\rangle G(y'', x')dy'dy''$$

$$-\int \sum^*(xy)\, G(yx')dy = -\delta(x - x'), \quad (2\cdot3)$$

$$\sum^*(x, y) = \sum^*(y, x)$$

$$= 2ig^2\int U(x, x', x'')\, G(x', z')\, G(x'', z'')\, \Gamma(z'z''y)dx'dx''dz'dz'', \quad (2\cdot4)$$

$$\Gamma(x, y, z) = U(x, y, z) - \frac{1}{2g}\frac{\delta\sum^*(x, y)}{\delta\langle\varphi(z)\rangle} = -\frac{1}{2g}\frac{\delta G^{-1}(x, y)}{\delta\langle\varphi(z)\rangle}, \quad (2\cdot5)$$

$$G(x, y) = \frac{\delta\langle\varphi(x)\rangle}{\delta J(y)} = G(y, x). \quad (2\cdot6)$$

The function $\sum^*$ corresponds to the proper self-energy part in Dyson-Feynman graph, and Γ represents the vertex part. Γ is symmetric with regard to its three arguments. Here it must be noted that in deriving $(2.2)\sim(2\cdot6)$, the relation

$$\frac{\delta\langle\varphi(x)\rangle}{\delta J(x')} = i\{\langle T(\varphi(x), \varphi(x'))\rangle - \langle\varphi(x)\rangle\langle\varphi(x')\rangle\} \quad (2\cdot7)$$

has been used. Although this relation is a consequence of Schwinger's new formalism in case of local fields, it is not justified that this relation holds also in our case. We propose that (2.7) also holds in our case. This proposal is necessary for us to make "the Green-function" $(2\cdot6)$ identical with that of $(1\cdot8)$. In fact we can derive all these equations $(2\cdot2)\sim(2\cdot6)$ from the definition $(1\cdot8)$ by using the similar method of analysis with that used in U. S. I.

$(2\cdot3)$ can be transformed into a non-linear integral equation

$$G(x, y) = i\Delta_F(x - y) + 2ig\int \Delta_F(x - x')U(x', x'', x''')\langle\varphi(x'')\rangle G(x'''y)dx'dx''dx'''$$

$$-i\int \Delta_F(x - x')\sum^*(x'x'')\, G(x''y)dx'dx'', \quad (2\cdot8)$$

under the following conditions;

$\lim_{t=\infty} G(x,x') = G$ contains only positive frequencies

$$\text{i. e.} \qquad \int_0^\infty G(\nu) e^{-i\nu t} d\nu,$$

$\lim_{t\to\infty} G(x,x') = G$ contains only negative frequencies.

Here Δ_F is defined as follows

$$\Delta_F(x) = \frac{1}{2}(\Delta^{(1)}(x) + i\varepsilon(x)\Delta(x)).$$

Similarly (2·2) turns out to be

$$\begin{aligned}\langle\varphi(x)\rangle &= i\int \Delta_F(x-y) J(y) dy \\ &+ ig\int \Delta_F(x-x') U(x', x'', x''') \langle\varphi(x'')\rangle \langle\varphi(x''')\rangle \, dx' dx'' dx''' \\ &+ g\int \Delta_F(x-x') U(x', x'', x''') G(x'' x''') dx' dx'' dx'''. \end{aligned} \tag{2·9}$$

Taking account of the fact that Γ should depend only on G, we get from (2·5)

$$\begin{aligned}\Gamma(x,x',x'') &= U(x,x',x'') \\ &- \int \frac{\delta\sum^*(x,x')}{\delta G(y.\, y')} G(yz)\, \Gamma(z, z', x'')\, G(z'y') dy dy' dz dz'. \end{aligned} \tag{2·10}$$

The right-hand side of this equation can be simply written in the following symbolic form

$$\Gamma = U - 4ig^2 UG\Gamma G\Gamma G - 2ig^2 UGG\frac{\delta\Gamma}{\delta G} G\Gamma G. \tag{2·11}$$

where the second term of the right-hand side in (2·11) is shown graphically in Fig. 4, and the last term in (2·11) is shown in Fig. 5.

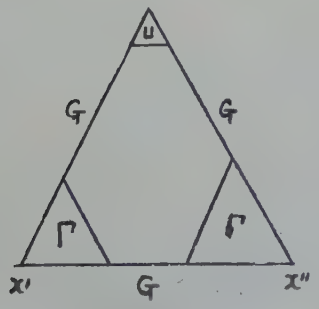

Fig. 4.

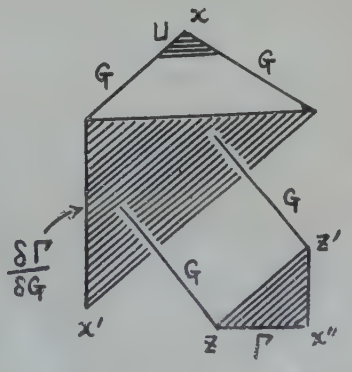

Fig. 5.

Suppose now that (2·11) is solved with regard to Γ by using the iteration method, then Γ will be obtained in a form of a power series in $g^2U^2G^3$. Each term of this series is graphically represented by a Feynman-Dyson graph which consists of $(2n+1)$ U-vertices, and $3n$ internal G-lines. The graphs here considered are in general different from the ordinary Feynman-Dyson graphs in the following points; the internal lines in the ordinary graphs correspond to the propagation function Δ_F not including the effect of interaction, whereas those in the graphs in our case correspond to G which includes the effect of interaction (such a internal line is called an internal G-line). Accordingly our graph does not contain in it any self-energy part or branch lines ending at points associated with external source J. That is to say, our graphs are irreducible with respect to the self-energy parts (in the following the term "irreducible" will be meant by this type of graphs).

Thus $\Gamma(x, x', x'')$ has the expression as

$$\Gamma(x, x', x'') = \sum\nolimits' c_n g^{2n} \times \sum_{\text{irr.graph}} \int \prod_{j=1}^{2n+1} U(x_j, x_j', x_j'') \prod_{k=1}^{3n} G(\xi_k, \eta_k) \prod_{j=1}^{2n+1} dx_j' dx'' \prod_{l=4}^{2n+1} dx_l, \tag{2·12}$$

where c_n's are yet undetermined numerical constants and $x=x_1$, $x'=x_2$, $x''=x_3$. Here $6n$ variables $(\xi_1 \cdots \xi_{3n}, \eta_1 \cdots \eta_{3n})$ must be chosen out by permuting $6n$ variables $(x_4 \cdots x_{2n+1}, x_1' \cdots x_{2n+1}' x_1'' \cdots x_{2n+1}'')$ in such a way that each integrand in (2·12) corresponds to a particular one of irreducible proper vertex parts in the sense stated above. The second summation $\sum'(irr.\ graph)$ means to sum up over all the contributions arising from the integrals corresponding to all the possible proper irreducible graphs. It may happen that there is a set of graphs any one of which can be made identical with any other one of this set by changing the labels of $6n$ points. In this case, however, the summation must be carried out independently over all the graphs of this set. Here it must be noted that every integrand with fixed n in (2·12) has the same sign because of the character of Boson fields. (c. f. (3.9))

Let us substitute with (2·12) into (2·4), then we get for the self-energy operator the expression

$$\Sigma^*(x,x') = ig^2 \sum_{n=0} c_n g^{2n} \sum_{irr.grap} \times \int \prod_{j=1}^{2n+2} U(x_j, x_j', x_j'') \prod_{k=1}^{3n+2} G(\xi_k, \eta_k) \prod_{j=3}^{2n+2} dx_j \prod_{k=1}^{2n+2} dx'_k dx''_k, \tag{2·13}$$

where $x_1=x$, $x_2=x'$. Here $6n+4$ variables $(\xi_1 \cdots \xi_{3n+2}, \eta_1 \cdots \eta_{3n+2})$ must be chosen in a similar way as that in (2·12), and $\sum'(irr.\ graph)$ has the same meaning with that in (2·12).

Now we have an alternative expression of the self-energy operator Σ^* which can be derived by following a similar line of reasoning used in U. S. I. When the external source is absent, Σ^* has the expression

$$\sum^*(x,x') = -ig^2\int U(x,y,z)U(x',y',z')$$
$$\times \langle T(\varphi_0(y)\varphi_0(z), \varphi_0(y')\varphi_0(z'), S_U)\rangle_{prop.conn.} dydz\cdot dy'dz'$$
$$= -ig^2\sum_{n=0}\frac{(i)^{2n}}{(2n)!\,3^{2n}}g^{2n}\int\prod_{j=1}^{2n+2}U(x_jx_j'x_j'')\prod_{j=3}^{j=2n+2}dx_j\prod_{k=1}^{2n+2}dx_k'dx_k''$$
$$\times \langle T\{\varphi_0(x_1')\varphi_0(x_1'')\varphi_0(x_2')\varphi_0(x_2'')\prod_{k=3}^{2n+2}\varphi_0(x_k)\varphi_0(x_k')\varphi_0(x_k'')\}\rangle_{prop.conn.}$$
$$\text{with}\quad x=x_1\cdot x'=x_2, \tag{2·14}$$

where $\langle\ \ \rangle_{prop.conn.}$ means that in decomposing this T-product into the sum of normal constituents we should retain only those terms which correspond to proper self-energy diagrams including reducible and irreducible ones. (2·14) can be rewritten also in the form

$$\sum^*(x,x') = -ig^2\sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!\,3^{2n}}g^{2n}\sum_{graph(prop.)}$$
$$\times\int\prod_{j=1}^{2n+2}U(x_jx_j'x_j'')\prod_{k=1}^{2n+2}\Delta(\xi_k\cdot\eta_k)\prod_{k=3}^{2n+2}dx_k\prod_{j=1}^{2n+2}dx_j'dx_j'', \tag{2·14}'$$

where $(6n+4)$ variables $(\xi_1\cdots\xi_{2n+2}:\eta_1\cdots\eta_{2n+2})$ must be chosen out of $(6n+4)$ variables $(x_3\cdots x_{2n+2}:x_1'\cdots x_{2n+2}':x_1''\cdots x_{2n+2}'')$ in such a way that each integrand in (2·14)′ corresponds to a particular one of self-energy diagrams, and $\sum(graph\ prop.)$ means to sum up over all the contributions arising from all these proper diagrams.
Now let us divide the series (2·14)′ into two parts

$$\sum^* = -ig^2\sum_{n=0}^{\infty}\frac{i^{2n}\cdot g^{2n}}{(2n)!\,3^{2n}}\{\sum_{graph(irr.)}+\sum_{graph(red.)}\}\int\Pi\cdot U\cdot\Pi\Delta_F dx,$$

where the first part $\sum_{(irr.)}$ consists of the contributions from all the possible irreducible graphs, whereas the second part those from the reducible diagrams. By putting

$$G_0(x,y) = i\Delta_F(x-y)$$

we can transform the first part into

$$\sum^*{}_{irr.}(x,x') = ig^2\sum_{n=0}^{\infty}\frac{(-i)^ng^{2n}}{(2n)!\,3^{2n}}$$
$$\times\sum_{graph\ irr.}\int\prod_{j=1}^{2n+2}U(x_jx_j'x_j'')\prod_{k=1}^{2n+2}G_0(\xi_k,\eta_k)\prod_{j=3}^{2n+2}dx_j\prod_{k=1}^{2n+2}dx_k'dx_k''. \tag{2·15}$$

From the comparison of (2·15) with (2·13), we get

$$C_n = -\frac{(-i)^n}{(2n)!\,3^{2n}}. \tag{2·16}$$

§ 3. Discussion on divergence of G

In what follows the external source J is assumed to be absent. In this case the Green-function G appears to be

$$G(x,y)=G(x-y)$$

and similarly it is obtained that

$$\sum^*(x,y)=\sum^*(x-y), \text{ and } \langle\varphi(x)\rangle=\text{const.}$$

Let us represent these quantities in the energy-momentum space in the following way;

$$G(x-y)=\frac{1}{(2\pi)^4}\int G(k)e^{ik(x-y)}d^4k,$$

$$\langle\varphi(x)\rangle=\frac{1}{(2\pi)^2}\varphi_0, \quad \varphi_0=\text{const.}$$

$$\sum^*(x-y)=(2\pi)^{-4}\int\sum^*(k)e^{ik(x-y)}d^4k, \tag{3·1}$$

$$U(x,y,z)=(2\pi)^{-8}\int U(k,l)e^{ik(x-y)+il(x-z)}\,d^4k\,d^4l,$$

$$i\Delta_F(x-y)=\frac{1}{(2\pi)^4}\int\frac{e^{ik(x-y)}}{k^2+M^2}d^4k=\frac{1}{(2\pi)^4}\int G_0(k)e^{ik(x-y)}d^4k.$$

Then (2·9) turns into

$$\varphi^0=\frac{g}{M^2}\left[U(0,0)\varphi_0^2-i\int U(k,-k)G(k)\,dk\right], \tag{3·2}$$

similarly (2·8) becomes

$$G(k)=\frac{1}{k^2+M^2}[1+2g\varphi_0U(0,k)G(k)-\sum^*(k)G(k)]. \tag{3·3}$$

From (3·3), $G(k)$ can be written as

$$G(k)=\frac{1}{k^2+M^2}\left\{1+\frac{\sum^*(k)-2g\varphi_0U(0.\,k)}{k^2+M^2}\right\}^{-1} \tag{3·4}$$

provided that $\sum^*$ and φ_0 are assumed to be finite quantities. The right-hand side of (3·4) can be expanded as

$$G(k)=\frac{1}{k^2+M^2}\sum_{n=0}^{\infty}\{(2g\varphi_0U(0,k)-\sum^*(k))/k^2+M^2\}^n \tag{3·4}'$$

if and only if

$$|\sum^*(k)-2g\varphi_0U(0,k)|<|k^2+M^2| \tag{3·5}$$

holds for all k's. If we further expand φ_0 and $\sum^*$ in (3·4)′ into power series in g

and rearrange the terms of this double series in an ascending power series in g, then we get just the expression of G which is usually derived by using the perturbation method. These various transformations of the expressions are allowable only if both $\sum'^*$ and φ_0 are finite (i. e. $\sum'^*$ and φ_0 are absolutely convergent series in g) and at the same time the condition $(3\cdot5)$ is satisfied. That is to say, the convergence of $\sum'^*$ is the necessary condition for the applicability of the perturbation method. It is seen, however, that this condition is also sufficient. In fact, we can show that if $\sum'^*$ is absolutely convergent then G turns to an absolutely convergent power series in g. (This proof will be published in this journal in the near future.) Thus our main task is to investigate the expandability of $\sum'^*$ in powers of g.

Let us substitute with $(3\cdot1)$ into $(1\cdot14)'$, then we get

$$\sum'^*(k)\,\delta(k-k') = ig^2 \sum_{n=0}^{\infty}{}' c_n g^{2n} \{A_n(k)+B_n(k)\}\,\delta(k-k') \qquad (3\cdot6)$$

with the abbreviation

$$\begin{aligned} A_n(k) &= \sum_{irr.\ graph}{}' \int \prod_{j=1}^{2n+2} U(k_j l_j) \prod_{k=1}^{2n+2} G_0(p_k) \prod_{j=1}^{n+1} d^4 q_j \\ B_n(k) &= \sum_{red.\ graph}{}' \int \Pi\ U\ \Pi\ G_0\ \Pi\ dq \end{aligned} \qquad (3\cdot7)$$

Fig. 6

and $k_1=-k$, $k_{2n+2}=k'$. Here $k_j l_j h_j$ are the energy-momenta carried by the internal lines which are incident at the j-th U-vertex, and have plus or minus sign according to whether these internal lines are going to or out of this vertex. Futther p_k is the energy-momentum carried by the k-th internal line, and $q_1 \cdots q_{n+1}$ are the linearly independent variables chosen out of p's.

Now suppose that $k^2+M^2>0$, in other words, the incoming energy is so small that the so-called displaced poles can not occur, then we can transform the fourth components of all the integration variable q's in $(3\cdot7)$ as

$$q_{0j} \to i q_{4j} \quad (j=1, 2\cdots n+1),$$

where q_{4j}'s$(j=1\cdots n+1)$ are real variables. Of course, in this case we must similarly transform k_0 and k_0' in $(3\cdot7)$ into pure imaginary variables. In these transformations of variables there occurs no displaced pole in $(3\cdot7)$ on account of the assumption stated above. It is sufficient for our purpose, as easily seen, to make a study of the convergence of $\sum'^*$ only in this case.[7]

After this transformation $(3\cdot7)$ runs

$$A_n(k)+B_n(k) = \{\sum_{graph(irr.)}{}' + \sum_{graph(red.)}{}'\}\, i^{n+1} \int_{Euclid} \cdot\, d^4q_1 \cdots d^4 q_{n+1} \prod_{j=1}^{2n+2} U(k_j l_j) \prod_{k=1}^{2n+2} G_0(p_k),$$

or

$$\delta(k-k')\{A_n(k)+B_n(k)\}$$
$$=\{\sum_{graph(irr.)}+\sum_{red.}\}i^{n+1}\int_{Euclid}dp_1\cdots dp_{3n+2}\prod_{j=1}^{2n+2}U(k_j\cdot l_j)\delta^4(k_j+l_j+h_i)\prod_{k=1}^{3n+2}G_0(p_k), \tag{3·8}$$

with

$$G_0(p)=\frac{1}{p^2+p_4^2+M^2}, \qquad p_4=\text{real},$$

where the integration is to be carried out in a $4(3n+2)$-dimensional Euclidian space E.

Let us assume that $U(k,l)$ is a positive definite function in E, and bound the domain of integration in (3·8) within a hyper sphere S_n with a large but finite radius Q_n, the center of which lies on the origin of the system of coordinates (p). Then we get (c. f. page 438)

$$\delta(k-k')|A_n(k)+B_n(k)|>\delta(k-k')|A_n(k)|$$
$$>\sum_{graph(irr.)}\int dp_1\cdots dp_{3n+2}\prod_{j=1}^{3n+2}G_0(p_j)\prod_{j=1}^{2n+2}\{U(k_j\cdot l_j)\delta^4(k_j+l_j+h_j)\}$$
$$>K_n\cdot L_n\cdot\sum_{graph\ irr.}\int dp_1\cdots dp_{3n+2}\prod_{j=1}^{2n+2}\delta^4(k_j+l_j+h_j). \tag{3·9}$$

Here we put

$$K_n=\underset{p\in S_n}{Min}\left(\prod_{k=1}^{3n+2}G_0(p_k)\right)=G_0^{3n+2}$$

$$L_n=\underset{p\in S_n}{Min}\left(\prod_{j=1}^{2n+2}U(k_jl_j)\right),$$

where $\boldsymbol{p}$ is a vector in E with components $(p_{1,1}p_{2,1}\cdots p_{4,1}, \cdots, p_{1,3n+2}\cdots p_{4,3n+2})$ and G_0 stands for

$$1\Big/\left(\frac{Q_n^2}{3n+2}+M^2\right).$$

If we choose Q_n as

$$Q_n=\sqrt{3n+2}\cdot Q, \qquad (Q \text{ being independent of } n),$$

then we have

$$K_n=\cdot(Q^2+M^2)^{-(3n+2)}.$$

In this case we can put L_n into the form

$$L_n=(U)^{2n+2},$$

where U is a function of Q but independent of n. For example, if we take the expression

$$U(k_j\cdot l_j)=U_0\exp\left[-r_0^2(k_j^2+l_j^2+h_j^2)\right] \qquad U_0>0,$$

then

$$\prod_{j=1}^{2n+2} U(k_j l_j) = U_0^{2n+2} e^{-2r_0^2(k^2+\sum_{j=1}^{3n+2} l_j^2)} > U_0^{2n+2} e^{-2r_0^2\{k^2+(3n+2)Q^2\}}$$

$$> \{U_0 e^{-r_0^2(k^2+3Q^2)}\}^{2n+2}$$

i. e.

$$U = U_0 e^{-r_0^2(k^2+3Q^2)}$$

In general if U has the expression

$$U(k_j l_j) = F(k_j^2) F(l_j^2) F(h_j^2),$$

then we have

$$U \approx F(Q^2)^3.$$

Now let us further bound the domain of integration in (3·9) within a particular sphere $\Omega_n(graph)$ with a radius $\sqrt{3n+2}\,R$ (R is independent of n and smaller than Q). This sphere Ω_n must be chosen in such a way as to be included within the sphere S_n. (c. f. Fig. 7.) The origin o' of Ω_n in general changes its position according to the structure of the irreducible graph considered in A_n. It will be explained in § 4 how to choose the position of the origin o' corresponding to a particular irreducible graph.

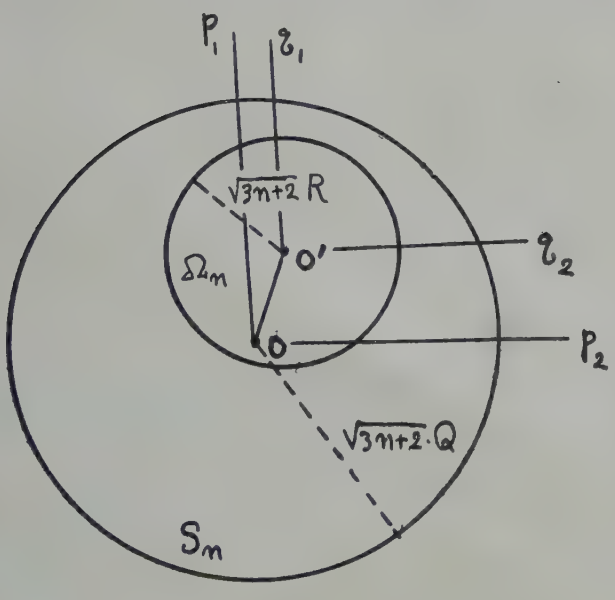

Fig. 7.

In any case, we have the following inequality:

$$\delta(k-k')\,|\,A_n + B_n\,| > G_0^{3n+2} U^{2n+2} \sum_{irr.graph} V_n(graph) \qquad (3\cdot10)$$

with the abbreviation:

$$V_n(graph) = \int_{\Omega_n(graph)} dp_1 \ldots dp_{3n+2} \prod_{j=1}^{2n+2} \delta^4(k_j + l_j + h_j). \qquad (3\cdot11)$$

As we shall show in § 4, the lower bound of V_n for all the irreducible graphs with fixed n is given by

$$\delta(k-k')\,\boldsymbol{V}_n=\delta(k-k')\frac{(3n+2)^{2n+2}(8n+3)!\,(2n+1)(\sqrt{\pi}\ R)^{4n+4}}{3^{4n}2^{8n+4}\cdot(6n+4)!\,(4n+2)!}<I_n(graph).$$

(for every irreducible graph.). (3·12)

Consequently, if we put

$$\sum_{graph(irr)}=(\text{the total number of the irreducible graphs of the proper self-energy type with fixed } n)=N'_n, \tag{3·13}$$

then we get finally

$$|A_n(k)+B_n(k)|>\boldsymbol{G}_0^{3n+2}\boldsymbol{U}^{2n+2}N'_n\cdot\boldsymbol{V}_n, \tag{3·14}$$

where the constant Q appearing in $\boldsymbol{G}_0$, $\boldsymbol{U}$ is chosen as

$$Q=\sqrt{k^2}+R.$$

The estimation of the lower bound of N_n' is given in § 4, the result of which is

$$|c_n|\,N_n'>2\cdot 6^n\cdot n!. \tag{3·15}$$

Accordingly we have

$$|c_n(A_n(k)+B_n(k))|>2\cdot 6^n\cdot n!\,\boldsymbol{V}_n\boldsymbol{G}_0^{3n+2}\cdot\boldsymbol{U}^{2n+2}. \tag{3·16}$$

Now the necessary condition for the convergence of the series (3·6) is

$$\lim_{n\to\infty}g^{2n}\,|c_n(A_n(k)+B_n(k))|=0. \tag{3·17}$$

Therefore, from (3·16)

$$\lim_{n\to\infty}g^{2n}\cdot 2\cdot 6^n\cdot n!\ \boldsymbol{V}_n\cdot\boldsymbol{G}_0^{3n+2}\cdot\boldsymbol{U}^{2n+2}=0 \tag{3·18}$$

is necessary for the convergence of $\sum^*$. But unfortunately the left-hand side of (3·18) behaves for sufficiently large n as $(const)^n\cdot n!$. Thus (3·18) does not hold except in the case $g=0$.

Consequently we get the conclusion that the expansion of G in powers of g is forbidden however small g may be.

In concluding this paragraph, some remarks should be given on the case of local interactions. Since the integrals in A_n with $n\geq 1$ have no primitive divergence in case of local interactions by virtue of (1·7) with $\lambda=0$, we can replace the form factor U's by δ-functions. In this case $U(k,l)$ turns out to be

$$U(k,l)=1.$$

Thus the same conclusion is reached also in case of local interactions, in other words, we arrive at the following two conclusions:

i) the non-unitarity of S_U has no influence on our results obtained above,

ii) the renormalization procedure can not make the series $G=\sum g^nG_n$ converge though each term G_n becomes finite on account of this procedure.

§ 4. Estimation of V_n and N_n'

i) Estimation of V_n

Let us consider any one of irreducible graphs of proper self-energy type with $2n+2$ U-vertices (c. f. Fig. 8) In this graph the j-th internal line is supposed to be associated with an energy-momentum p_j, and to have an arbitrary but fixed direction denoted by an arrow. Then the integrand $\Pi\delta^4(k_j+l_j+h_j)$ in V_n can be expressed as follows

Fig. 8

$$\prod_{j=1}^{2n+2}\delta^4(k_j+l_j+h_j)$$
$$=\prod_{\nu=1}^{2n+2}\delta^4\Big(\sum_{j=1}^{3n+2}a_{\nu j}p_j-\delta_{\nu 1}k+\delta_{\nu 2n+2}k'\Big). \qquad (4\cdot1)$$

In this section, we employ the following notations: Greek subscripts assume values ranging from 1 to $2n+2$, whereas Latin ones from 1 to $3n+2$, and a repeated index is to be summed up over its range. Further we omit indices denoting vector-components whenever no confusion occurs. The rectangular matrix $a_{\nu j}$ has the following property:

$$a_{\nu j}=\begin{cases}+1 & \text{when the } j\text{-th line goes to the } \nu\text{-th } U\text{-vertex,}\\ -1 & \text{when the } j\text{-th line goes out of the } \nu\text{-th } U\text{-vertex,}\\ 0 & \text{otherwise.}\end{cases}$$

From the graphical consideration it is easily seen that

$$\sum_{\nu=1}^{2n+2}a_{\nu j}=0 \quad (j=1\cdots3n+2) \qquad (4\cdot2)$$

and further each row has only three non-vanishing elements except the first and last rows both of which have only two non-vanishing elements. It is of importance that every graph here considered has these two characters.

Now $V_n(graph)$ can be transformed into

$$V_n(graph)$$
$$=\frac{1}{(2\pi)^{4(2n+2)}}\int_{-\infty}^{\infty}d^4\alpha_1\cdots d^4\alpha_{2n+2}\int_{\Omega_{3n}(graph)}\exp\big[i(a_\nu\alpha_{\nu j}p_j-\alpha_1k+\alpha_{2n+2}k')\big]\,d^4p_1\cdots d^4p_{3n+2}.$$

By putting

$$a_\nu=\beta_\nu+\alpha_{2n+2}\quad \nu=1\cdots2n+1,$$

$$\alpha=\alpha_{2n+2},$$

and taking account of (4·2), we get

$$V_n(graph)=\frac{1}{(2\pi)^{4(2n+2)}}\int_{-\infty}^{\infty}d^4a\ e^{ia(k'-k)}\int_{-\infty}^{\infty}d^4\beta_1\cdots d^4\beta_{2n+1}$$

$$\times\int_{\Omega_n(graph)}d^4p_1\cdots d^4p_{3n+2}\exp\left[i\left(\sum_{\nu=1}^{2n+1}\beta_\nu a_{\nu j}p_j-\beta_1 k\right)\right].$$

In what follows we assume that a repeated Greek index is to be summed up from 1 to $2n+1$. Let us consider the transformation

$$p_{(\rho)j}=q_{(\rho)j}+C_{(\rho)j}\quad(\rho)=1,2,3,4$$
$$(j=1\cdots 3n+2),\tag{4·3}$$

where $C_{(\rho)j}$'s are restricted by the condition

$$a_{\nu j}C_{(\rho)j}=\delta_{\nu 1}k_{(\rho)}\quad(\rho)=1,2,3,4$$
$$\nu=1.\ 2\cdots 2n+1.\tag{4·4}$$

Then V_n turns out to be

$$V_n(graph)=\delta(k-k')V_n'(graph)$$

$$=\delta(k-k')\frac{1}{(2\pi)^{4(2n+1)}}\int_{-\infty}^{\infty}d\beta_1\ldots d\beta_{2n+1}\int_{\Omega_n(graph)}e^{i\beta_\nu a_{\nu j}q_j}dq_1\ldots dq_{3n+2}.\tag{4·5}$$

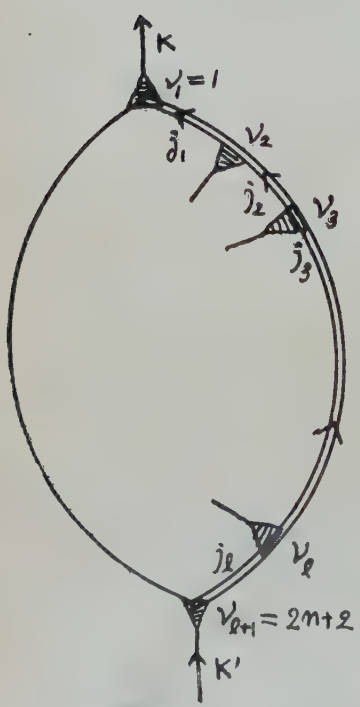

Fig. 9.

Suppose now that a directed doubled line is drawn in a way as in Fig. 9. The labels of the U-vertices and internal lines lying on this doubled line are respectively denoted by $\nu_1(=1), \nu_2, \ldots\nu_l, \nu_{l+1}(=2n+2)$, and $j_1\ldots j_l$.

Let C_j's with $j\neq(j_1\ldots j_l)$ be equal to zero. Then (4·4) can be written as

$$\nu=\nu_1=1\quad C_{(\rho)j_1}=k_{(\rho)}$$
$$\nu=\nu_k\quad C_{(\rho)j_k}=C_{(\rho)j_{k-1}}$$
$$\nu\neq(\nu_1\ldots\nu_l)\quad 0=0$$
$$(\rho)=1\ldots4,\quad k=2\ldots l,$$

i. e., we have the following solution of (4·4):

$$C_{(\rho)j}=\begin{cases}k_{(\rho)} & \text{for } j=j_1\ldots j_l\\ 0 & \text{for } j\neq j_1\ldots j_l.\end{cases}\tag{4·6}$$

Now we choose the position of the origin o' of the sphere $\Omega_n(graph)$ in such a way that the vector $\overrightarrow{oo'}$ in E has components $(C_1 C_2\ldots\ldots C_{3n+2})$ defined by (4·6). Since this vector has the length

$$|\overrightarrow{oo'}|=\sqrt{\sum_{j=1}^{3n+2} C_j^2}=\sqrt{l\cdot k^2}<\sqrt{(3n+2)k^2}.$$

$\mathcal{Q}_n(graph)$ can be included within the sphere S_n for every irreducible graph with fixed n by virtue of the relation

$$Q_u=\sqrt{3n+2}\,Q=\sqrt{3n+2}(R+\sqrt{k^2})>\sqrt{3n+2}R+|\overrightarrow{oo'}|,$$

where we have put

$$Q=R+\sqrt{k^2}.$$

The integral in (4·5) with respect to q's can be transformed into

$$\int_{\sum_j q_j^2\leq(3n+2)k^2} e^{i\sum_j \xi_j q_j}\,dq_1\cdots dq_{3n+2}=\frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}\int_{-R'}^{R'} e^{i\rho u}(R'^2-u^2)^{\frac{N-1}{2}}du, \tag{4·7}$$

with the abbreviation

$$\xi_{(\rho)j}=\beta_\nu^{(\rho)}a_{\nu j}\quad(j=1\cdots 3n+2\ \ (\rho)=1\cdots 4)$$

$$N=4(3n+2),\quad R'=\sqrt{3n+2}\,R,$$

and

$$\rho=\sqrt{\sum_{j=1}^{3n+2}\xi_j^2}\ .$$

Here the well known property of Γ-function has been utilized. Since the right-hand side of (4·7) can be rewritten in the following form by using Bessel-function :

$$\pi^{\frac{N}{2}}R'^N J_{\frac{N}{2}}(\rho R')\Big/\left(\frac{\rho R'}{2}\right)^{\frac{N}{2}},$$

we have

$$V_n'(graph)=\frac{\pi^{\frac{N}{2}}R'^{\frac{N}{2}}2^{\frac{N}{2}}}{(2\pi)^{4(2n+1)}}\int_{-\infty}^{\infty}J_{\frac{N}{2}}(\rho R')\cdot(\rho R')^{-\frac{N}{2}}d^4\beta_1\cdots d^4\beta_{2n+1}. \tag{4·8}$$

Now the positive definite quantity ρ^2 can be written as

$$\rho^2=\beta_\mu A_{\mu\nu}\beta_\nu\quad\text{with}\quad A_{\mu\nu}=A_{\nu\mu}=a_{\mu j}a_{\nu j}.$$

Accordingly we see that the $(2n+1)\times(2n+1)$ matrix $A_{\mu\nu}$ is non-singular and has positive eigenvalues $\lambda_1\cdots\lambda_{2n+1}$. Let it be supposed that the orthogonal transformation

$$\boldsymbol{A}=(A_{\mu\nu})\to\boldsymbol{SAS}^{-1}=\boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_{2n+1}\end{pmatrix}$$

is carried out, then β's are changed to

$$\beta_\mu^{(\rho)}\to x_\mu^{(\rho)}=S_{\mu\nu}\beta_\nu^{(\rho)}\quad(\rho)=1,\ 2,\ 3,\ 4,$$

and ρ^2 becomes

$$\rho^2=\lambda_\mu x_\mu{}^2=\sum_{\mu=1}^{2n+1} y_\mu{}^2 \quad \text{with } y_\mu^{(\rho)}=\sqrt{\lambda_\mu}\cdot x_\mu^{(\rho)}.$$

Then (4·8) turns into

$$V_n'(graph)=\frac{\pi^{\frac{N}{2}}R'^{\frac{N}{2}}2^{\frac{N}{2}}}{(2\pi)^{4(2n+1)}(\det \boldsymbol{A})^2}\int d\Omega\int_0^\infty \frac{J_{N/2}(\rho R')}{(\rho R')^{N/2}}\rho^{8n+3}d\rho,$$

or by putting

$$t=\rho R'$$

V_n' runs

$$V_n'(graph)=\frac{(R')^{4(n+1)}}{(2\pi)^{2n}(\det \boldsymbol{A})^2}\int d\Omega \lim_{\lambda\to 0}\int_0^\infty e^{-\lambda t}\cdot t^{2n-1}J_{6n+4}(t)\,dt$$

$$=\frac{(3n+2)^{2n+2}R^{4n+4}(8n+3)!}{(\det \boldsymbol{A})^2(2\pi)^{2n}\cdot 2^{6n+4}(6n+4)!}\cdot {}_2F_1\left(4n+2,\ \frac{4n+5}{2},\ 6n+5,\ 1\right)\int d\Omega. \tag{4·9}^{8)}$$

Here $d\Omega$ means the element of the solid angle in the $4(2n+1)$ dimensional Euclidian space, and ${}_2F_1$ is the hyper-geometric series which is convergent in our case because of

$$\gamma-(\alpha+\beta)>0,\ \alpha=4n+2,\ \beta=\frac{4n+5}{2},\ \gamma=6n+5.$$

Now from the definition of $a_{\mu j}$, it is easily seen that

$$A_{11}=\sum_j a_{1j}{}^2=2$$

$$A_{\mu\mu}=\sum_j a^2{}_{\mu j}=3 \quad (\mu=2,\ 3\cdots 2n+1)$$

$$A_{\substack{\mu\nu\\(\mu\neq\nu)}}=\sum_j a_{\mu j}a_{\nu j}=\begin{cases} -1 & \text{when the } \mu\text{- and } \nu\text{-th } U\text{-veritices are directly} \\ & \text{connected by an internal line,} \\ 0 & \text{otherwise.}\end{cases}$$

Therefore we obtain the upper bound of $\det \boldsymbol{A}$ owing to the very definition of $\boldsymbol{A}$:

$$\det \boldsymbol{A}<\prod_{\nu=1}^{2n+1} A_{\nu\nu}=2\cdot 3^{2n} \quad \text{(c. f. Hadamard's theorem)}.$$

Taking into account of the fact that

$${}_2F_1>1$$

and

$$\int d\Omega=\frac{4(2n+1)}{(4n+2)!}\pi^{4n+2},$$

we get finally the lower bound of $V_n'(graph)$:

$$V_n'(graph) >$$

$$\frac{(3n+2)^{2n+2}(8n+3)!\,(2n+1)\cdot(\sqrt{\pi}R)^{4n+4}}{3^{4n}2^{8n+4}(6n+4)!\,(4n+2)!} \equiv V_n \qquad (4\cdot10)$$

(for every graph with fixed n).

This is just the expression which has been used in § 3.

ii) Estimation of N_n'

Let N_n be the number of irreducible graphs of the proper vertex-type with fixed n in (2·12). Then from (2·4) it is easily seen that the relation

$$N_n' = 2N_n \qquad (4\cdot11)$$

holds. If we regard Γ, U, and G not as matrix-elements with suffices x, x', x'' etc., but as c-numbers, then the right-hand side in (2·12) turns out to be a simple power series

$$\Gamma = \sum_{n=0}^{\infty} c_n N_n g^{2n} U^{2n+1} G^{3n}. \qquad (4\cdot12)$$

This expression results from the fact that by looking upon G, U as c-numbers the N_n different terms under $\sum(irr.\ graph)$ in (2·12) have been reduced to a single term with an expression $U^{2n+1}G^{3n}$. From the same reason, (2·11) becomes to be an ordinary differential equation

$$\Gamma = U - 4ig^2UG^3\Gamma^2 - 3ig^2UG^4\Gamma\frac{d\Gamma}{dG}. \qquad (4\cdot13)$$

Since Γ with the expression (2·12) is a solution of (2·11), the expression (4·12) must be a solution of (4·13) under the condition

$$\Gamma(g, G)_{g=0} = U.$$

Therefore it is sufficient for our purpose to solve the eq. (4·13) in the form of a power series in g. By putting

$$y = \frac{\Gamma}{U} \qquad x = ig^{\frac{2}{3}}U^{\frac{2}{3}}G,$$

(4·13) becomes

$$y = 1 + 4x^3y^2 + 2x^4y\frac{dy}{dx} = 1 + \frac{d}{dx}(x^4y^2), \qquad (4\cdot13)'$$

with the condition

$$y)_{x=0} = 1,$$

and (4·12) turns to

$$y = \sum_{n=0}^{\infty} a_n x^{3n} \qquad \text{with } a_n = \frac{N_n}{(2n)!\,3^{2n}} = |c_n|\cdot N_n. \qquad (4\cdot12)'.$$

Substituting with (4·12)′ into (4·13)′, we get

$$a_0=1$$

$$a_{n+1}=(4+3n)\sum_{k=0}^{n} a_{n-k}a_k > 2(3n+4)a_n > 6(n+1)a_n,$$

where the last inequality is due to the fact that $a_n > 0 (n \geq 0)$. Thus it holds that

$$a_n \geq 6^n n\,!.$$

Hence

$$|c_n|\, N_n' = 2|c_n|\, N_n = 2a_n \geq 2\cdot 6^n\, n\,!$$

holds which has been already used in § 3.

It will be of some interest that if we omit in $(4\cdot 13)'$ the term with dy/dx, then $(3\cdot 18)$ holds. Speaking in more detail, we have the equation

$$y=1+4x^3y^2$$

in place of $(4\cdot 13)'$, the solution of which is

$$y=\frac{1-\sqrt{1-16x^3}}{x^3}.$$

This solution can be expanded in powers of x in the following way if the condition $|x| > 16^{-1/3}$ is satisfied:

$$y=\sum_{n=0}^{\infty} a_n' x^{3n} \qquad a_n'=\frac{16^n\cdot(2n)!}{2^{2n}\cdot n!\ (n+1)!}=(N_n\,|\,c_n\,|).$$

Substituting with this expression into $(3\cdot 18)$, we get

$$\lim_{n\to\infty} \frac{g^{2n+2}\cdot N'_{n+1}c_{n+1}\cdot \boldsymbol{V}_{n+1}\boldsymbol{G}_0^{3n+5}\cdot \boldsymbol{U}^{2n+1}}{g^{2n}N_n'c_n\boldsymbol{V}_n\boldsymbol{G}_0^{3n+2}\boldsymbol{U}^{2n+2}}=\frac{2^6}{3^8}e^2\cdot g^2\cdot \boldsymbol{G}_0^3\boldsymbol{U}^2(\sqrt{\pi}R)^4.$$

Thus $(3\cdot 18)$ holds if we choose

$$g^2 < \frac{3^8}{2^6e^2\boldsymbol{G}_0^3\boldsymbol{U}^2(\sqrt{\pi}R)^4}.$$

In fact, we can prove in this case that $\sum^*$ and G turns out to be convergent. The detailed explanation will be given elsewhere by the present author.

§ 5. Case of model II

We have been so far only concerned with the model I. The similar reasoning is also applicable to the case of model II. In this section we shall give an outline of the arguments, and omit the explanation of notations which have been used in the preceding sections. The fundamental eqs. corresponding to $(2\cdot 1)\sim(2\cdot 6)$ are

$$(\Box-\mu^2)\langle A(x)\rangle - ig\int U(y'y'':x)G(y''y')\,dy'dy''=-J(x),$$

$$(\Box-\mu^2)\,\mathfrak{G}(x\,x') - \int P(xy)\mathfrak{G}(yx')\,dy=-\delta(x-x'),$$

$$(\Box-M^2)G(xx')+g\int U(xy':y)\langle A(y)\rangle G(y'x')dydy'$$

$$-\int \sum{}^*(xy)G(yx')dy=-\delta(x-x'),$$

$$G(xx')=\frac{\delta\langle\varphi(x)\rangle}{\delta\rho(x')}\Big)_{\rho=0} \qquad \mathfrak{G}(xx')=\frac{\delta\langle A(x)\rangle}{\delta J(x')}\Big)_{\rho=0},$$

$$\Gamma(xx':y)=U(xx':y)-\frac{1}{g}\frac{\delta\sum{}^*(xx')}{\delta\langle A(y)\rangle},$$

$$\sum{}^*(xx')=ig^2\int U(xy:y')G(yz)\Gamma(zx':y'')\mathfrak{G}(y''y')dydy'dy''dz,$$

$$P(xx')=ig^2\int U(y'y'':x)G(y''z'')\Gamma(z''z':x')G(z'y')dy'dy''dz'dz''.$$

Let us further introduce a function θ in a way as

$$\theta(xx':y)=-\frac{1}{g^3}\frac{\delta\mathfrak{G}^{-1}(xx')}{\delta\langle A(y)\rangle}=-\frac{1}{g^3}\frac{\delta P(xx')}{\delta\langle A(y)\rangle}. \tag{5·1}$$

Since I' is a functional depending only on G and $\mathfrak{G}$, we have the relation

$$\frac{\delta\Gamma(xx':x'')}{\delta\langle A(y)\rangle}=\int\frac{\delta\Gamma(xx':x'')}{\delta G(y'y'')}\frac{\delta G(y'y'')}{\delta\langle A(y)\rangle}dy'dy''$$

$$+\int\frac{\delta I'(xx':x'')}{\delta\mathfrak{G}(y'y'')}\frac{\delta\mathfrak{G}(y'y'')}{\delta\langle A(y)\rangle}dy'dy'',$$

or writing down in a symbolic form, we have

$$\frac{\delta\Gamma}{\delta\langle A\rangle}=\frac{\delta I'}{\delta G}\cdot g\cdot G\Gamma G+g^3\frac{\delta I'}{\delta\mathfrak{G}}\mathfrak{G}\theta\mathfrak{G} \tag{5·2}$$

by virtue of (5·1). By taking account of (5·2), $\sum^*$ and θ can be expressed in the following symbolic forms:

$$\Gamma=U-ig^2UG\Gamma G\mathfrak{G}\Gamma'-ig^4UG\mathfrak{G}\theta\mathfrak{G}\Gamma'$$

$$-ig^2UG\mathfrak{G}\frac{\delta\Gamma}{\delta G}G\Gamma G-ig^4G\mathfrak{G}\frac{\delta\Gamma}{\delta\mathfrak{G}}\mathfrak{G}\theta\mathfrak{G} \tag{5·3}$$

$$\theta=-i\left[2UG\Gamma G\Gamma+UGG\frac{\delta I'}{\delta G}G\Gamma G+g^2UGG\frac{\delta I'}{\delta\mathfrak{G}}\mathfrak{G}\theta\mathfrak{G}\right]. \tag{5·4}$$

In order to investigate on the convergence of G and $\mathfrak{G}$, it is necessary to make a study on the behaviour of $\sum^*$ and P as in case of model I. In case of $J=0$, $\sum^*$ can be written in the momentum space as

$$\sum{}^*(k)=ig^2\sum_{n=0}^{\infty}c_ng^{2n}(A_n(k)+B_n(k)) \qquad c_n=\frac{(-i)^n}{(2n)!},$$

where

$$A_n(k)=\sum_{graph(irr)}\int\prod_{j=1}^{2n+2}U(k_jl_j)\delta^4(k_j+l_j+h_j)\prod_{k=1}^{2n+1}G_0(p_k)\prod_{j=1}^{n+1}\mathfrak{G}_0(q_j)\prod_{k=1}^{2n+1}dp_k\prod_{j=1}^{n+1}dq_j,$$

with the abbreviation

$$G_0(p)=\frac{1}{p^2+M^2},\qquad \mathfrak{G}_0(q)=\frac{1}{q^2+\mu^2}.$$

B_n has the similar meaning with that of (3·7). Now in this case the total number N_n of the irreducible graphs of the proper vertex type with $(2n+1)$ U-vertices is equal to that of the proper self-energy graphs with $(2n+2)$ U-vertices.

N_n can be estimated in the following way by using the similar method with that of § 4.

By putting

$$\Gamma=U\cdot y(x)\qquad x=-ig^2U^2G^2\mathfrak{G},$$

θ becomes

$$\theta=-2iU^3G^3\frac{y\left(y+x\dfrac{dy}{dx}\right)}{\left(1-x^2\dfrac{dy}{dx}\right)}.$$

Substituting with this expression into (5·3), we get

$$y=1-x^2\frac{dy}{dx}+\frac{3}{2}x^{\frac{4}{3}}\frac{d}{dx}(x^{\frac{2}{3}}y^2)+x\frac{d}{dx}(x^2y^3). \tag{5·5}$$

The formal solution of (5·5) can be obtained in a form of power series

$$y=\sum^{\infty}a_nx^n,\qquad a_n=|c_n|\,N_n.$$

The coefficient a_n's satisfy the recursion formula

$$a_0=1,$$

$$a_{n+1}=-na_n+\frac{3}{2}\left(n+\frac{2}{3}\right)\sum_{k=0}^{n}a_{n-k}a_k.$$

As it is easily seen that each a_n is positive, we get

$$a_{n+1}>-na_n+\frac{3}{2}\left(n+\frac{2}{3}\right)2a_na_0=2(n+1)a_n,$$

consequently, we obtain

$$a_n>2^n\cdot n\,!. \tag{5·6}$$

If we assume $M>\mu$, then the following inequality holds

$$|c_n|\cdot|A_n(k)+B_n(k)|>|c_nA_n(k)|$$

$$>U^{2n+2}\cdot G_0^{3n+2}\,|c_n|\cdot N_n\cdot V_n, \tag{5·7}$$

where the relation

$$G_0(p) < \mathfrak{G}_0(p)$$

has been taken into account.

The quantities $U\ G_0\ V_n$ in (5·7) agree with those given in § 3. Thus we get

$$\lim_{n\to\infty} | c_n(A_n(k) + B_n(k)) | = \infty$$

on account of (5·6). Accordingly the element of S-matrix of this model is also divergent.

§ 6. Discussion

In the preceding sections we have proved that the S-matrix $S_U = \sum g^n S_n$ of our model diverges on account of the rapid increase of the number N_n of graphs though each term S_n is finite by virtue of the form factor U. This proof is essentially based on the fact that all the terms with fixed n in (3·8) have the same sign because of the character of the scalar field φ(or φ and A). Accordingly in case of Fermion fields, the grade of divergence might be weakened owing to the anticommuting character of field operators. Further it is easily seen that when the form factor $U(k, l)$ is an oscillating function, our arguments do not hold. The convergence of the S-matrix S_U may be expected when oscillating function $U(k, l)$ are chosen out in such a way that the each integral in (3·8) is made to decrease rapidly with the increase in n. In this sense it may be seen that our result imposes a new restriction upon the form factor U at least in case of scalar fields.

Here it may be of some interest to conjecture in what case the S-matrix will be convergent. For simplicity, let us consider a harmonic oscillator with non-linear self-interaction, the equation of which is given by (in Heisenberg representation)

$$\frac{d^2x}{dt^2} = -\frac{dV(x)}{dx} \qquad V(x) = \frac{\nu^2}{2}x^2 + gx^n \qquad (6\cdot1)$$

(examples with $n=3$ and 4 were considered by Dr. T. Hayasi). The potential function V is concave in the upward direction when

i) n=even >2 and $g>0$

ii) $n=2$ and $|g| < \frac{\nu^2}{2}$

iii) $n=1$ and $g \leqq 0$.

In these cases we should have stationary solutions though we could not write down these solutions explicitly. On the other hand, if

i) n=even >2 and $g<0$

ii) $n=$ odd >1 and $g \leqq 0$,

(6·1) should have no stationary solutions. Thus the stationary solutions in case of $n=$even >2 and $g>0$ turn out to be singular when g tends to zero. Hence we can not express these solutions in a form of power series in g. Accordingly we arrive at the conclusion that the perturbation method is applicable only in the case where the interaction potential $gF(x)$ appearing in $l^r=\frac{\nu^2}{2}x^2+gF(x)$ does not change the asymptotic behaviour of l^r at $x=\pm\infty$ without any concern with the choice of the sign of g.

For example, we may adopt the following potential:

$$V(x)=\frac{\nu^2}{2}x^2+ax^2+bx+c \qquad \text{with} \frac{\nu^2}{2}>|a|,$$

or

$$l^r(x)=\frac{\nu^2}{2}x^2+ge^{-\lambda x^2}.$$

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7) Here the form factor U should be chosen in such a way as to give no displaced poles in this transformation. For example,
$$U(k,l)=(1+r_0^2k^2)^{-1}(1+r_0^2l^2)^{-1}(1+r_0^2k^2)^{-1} \qquad r_0\ll 1/M.$$
8) C. f. Watson, "*Theory of Bessel Functions*" p. 385.

Note added in proof

After completion of the present paper, Dr. S. F. Edwards informed us of that Hurst, Thirring and Petermann made investigations on the same problem with that of ours. We wish to thank Dr. Edwards for his kindness.

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On the Scattering Problem in Pseudo-scalar Meson in Pseudo-vector Coupling*

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The theoretical investigation of meson nucleon scattering given by Brueckner[1] which fits very well with the experimental data is based on the phenomenological considerations with the mixed use of weak and strong coupling theories. The weak coupling treatment was abandoned owing to its failure to explain the well known scattering data even qualitatively.[2] These weak coupling calculations are Born approximation. The well known threshold rise of the cross-section is given by pseudo scalar meson with pseudo-vector coupling, but there is no securence that the Born approximation is allowed. In the weak coupling case, the effective-scattering Hamiltonian can be easily obtained by a canonical transformation (for example, assuming the coupling to be weak, retain only second order term), but if one regards this as a "potential" which causes the scattering of meson, the singularity of this "potential" is so strong that the Born approximation do nothing, and just as in the theory of nuclear force one should solve the problem exactly with this "potential". (The word "potential" does not mean that the meson scattering can be represented by potential scattering; we call the quantity which appears in place of potential as "potential".) We now perform a calculation firstly deriving the "potential" under the weak coupling assumption and then solve the dynamical problem of the system correctly (not in Born approximation) for the problem of meson-nucleon scattering. This is just the way taken up to date in the meson theory of nuclear force, i.e., deriving the nuclear potential under the weak coupling assumption and solve the Schroedinger equation with this potential exactly; we want to perform the same procedure in the meson-nucleon scattering

The "potential" derived from pseudo-scalar meson with pseudo-vector coupling is very singular and changes its sign and magnitude for different total isotopic spin and angular- momentum of the system. It is seen that the virtual level resonance takes place for $I=3/2$ and $J=3/2$ state. The conclusion is roughly the same as the theory by Brueckner.[1]

The divergences appear many places but can be summed up to the effective coupling constant[3] aside from the position of virtual level.

§ 1. Elimination of the first order interaction

In this paper we treat the scattering problem only, so the quantity which depends only on the nucleon variable (self-energy, nuclear-potential, etc.) is omitted from the expression in the following. The pseudo-scalar meson in interaction with nucleon through pseudo-vector coupling has the following Hamiltonian in symmetrical theory (dummy indices mean sum over that variables)

$$\bar{H}=\bar{H}_0+\bar{H}_1, \qquad (1\cdot1)$$

* This is the English translation of article appeared in "Soryushiron-kenkyu", January, 1953 (in Japanese).

$$\bar{H}_0=\frac{1}{2}\int(\pi_\alpha^2+(\nabla\phi_\alpha)^2+\mu^2\phi_\alpha^2)d\boldsymbol{x}+\int\psi^*(\boldsymbol{\alpha p}+\beta m)\psi\,d\boldsymbol{x}\,, \tag{1·2}$$

$$\bar{H}_1=\frac{g}{\mu}\int\psi^*(\rho_1\pi_\alpha+(\boldsymbol{\sigma\nabla})\phi_\alpha)\tau_\alpha\psi\,d\boldsymbol{x}+O(g^2), \tag{1·3}$$

ϕ_α, π_α are the canonical variable of meson, $O(g^2)$ is so-called surface dependent term and depends only on nucleon variables.

Then under the assumption of weak coupling we perform the following canonical transformation to remove the first order interaction term;

$$\Psi=e^{-i\bar{F}}\chi\,;\ \bar{F}=\int\psi^*F\psi\,d\boldsymbol{x}\,, \tag{1·4}$$

where F satisfies the following equation;

$$i[F,\int\phi_\alpha^*(\boldsymbol{q})\phi_\alpha(\boldsymbol{q})\varepsilon_q d\boldsymbol{q}+\boldsymbol{\alpha p}+\beta m]=-\frac{g}{\mu}(\rho_1\pi_\alpha+(\boldsymbol{\sigma\nabla})\phi_\alpha)\tau_\alpha\,, \tag{1·5}$$

where

$$\phi_\alpha(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{q}}{(2\varepsilon_q)^{1/2}}(e^{i\boldsymbol{qx}}\phi_\alpha(\boldsymbol{q})+e^{-i\boldsymbol{qx}}\phi_\alpha^*(\boldsymbol{q}))\,,\ \text{etc.,} \tag{1·6}$$

$\varepsilon_q=\sqrt{\mu^2+\boldsymbol{q}^2}$ is the meson energy.

Hamiltonian for χ becomes;

$$e^{i\bar{F}}He^{-i\bar{F}}=\bar{H}_0+\frac{i}{2}[\bar{F},\bar{H}_1]_-+\cdots\cdots\,. \tag{1·7}$$

(first order terms are completely removed). We take as F,

$$F(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{q}}{(2\varepsilon_q)^{1/2}}(e^{i\boldsymbol{qx}}\phi_\alpha(\boldsymbol{q})G_\alpha(\boldsymbol{q})+G_\alpha^*(\boldsymbol{q})\phi_\alpha^*(\boldsymbol{q})e^{-i\boldsymbol{qx}}), \tag{1·8}$$

then G becomes the integro-differential operator. Expanding G by $\boldsymbol{\sigma}$ and ρ, and substituting into the left hand side of (1·5), and then putting the coefficient of ρ and $\boldsymbol{\sigma}$ to be equal, we have as the solution of (1·5);

$$\begin{aligned}G_\alpha(\boldsymbol{q})=g\Big[&\frac{1}{\mu}\rho_1+\frac{\varepsilon_q}{(1+p^2/m^2)\varepsilon^2-(\mu^2-2(\boldsymbol{pq})/2m)^2}\\&\times\Big\{-\frac{\varepsilon_q}{\mu}\rho_1-i\frac{\mu}{2m}\rho_2-\frac{(\boldsymbol{\sigma q})}{\mu}+\frac{\mu^2-(\boldsymbol{pq})}{2m\mu\varepsilon_q}\rho_3(\boldsymbol{\sigma q})\\&+\frac{\varepsilon_q}{m\mu}\rho_3(\boldsymbol{\sigma p})+\frac{(\boldsymbol{\sigma\cdot q\times p})}{m\mu}\rho_2\Big\}\Big]\tau_\alpha\,,\end{aligned} \tag{1·9}$$

where $p=-i\nabla$, and this operator in the denominator means the integral operation.[4]

Up to now, the formulas are exact, and here we simplify the calculation by taking the momentum of the nucleon to be neglected. In (1·9) this means to take $\boldsymbol{p}=0$,[5] and correspondingly $\psi^*\psi\to\delta(\boldsymbol{x})$, which means one nucleon at the origin.

Then, $\bar{F}$ becomes ;

$$\bar{F}=-\left(\frac{\mu}{2m}\right)^2 g\mu\rho_1\phi_\alpha^{(-2)}\tau_\alpha+\left(\frac{\mu}{2m}\right)g\rho_2\pi_\alpha^{(-2)}\tau_\alpha$$
$$-\left(\frac{g}{\mu}\right)(\sigma\nabla)\pi_\alpha^{(-2)}\tau_\alpha+\left(\frac{\mu}{2m}\right)g\rho_3(\sigma\nabla)\pi_\alpha^{(-2)}\tau_\alpha\,, \tag{1·10}$$

where

$$\phi_\alpha^{(-2)}=\frac{1}{(2\pi)^{3/2}}\int\frac{dq}{(2\varepsilon_q)^{1/2}}\frac{1}{\varepsilon_q^2}\left(e^{iqx}\phi_\alpha(q)+e^{-iqx}\phi_\alpha^*(q)\right)|_{x=0}\,, \tag{1·11}$$

$$\pi_\alpha^{(-2)}=\frac{1}{(2\pi)^{3/2}}\int\frac{dq}{(2\varepsilon_q)^{1/2}}\frac{(-i\varepsilon_q)}{\varepsilon_q^2}\left(e^{iqx}\phi_\alpha(q)-e^{-iqx}\phi_\alpha^*(q)\right)|_{x=0}\,. \tag{1·12}$$

Thus, we obtain the Hamiltonian in order g^2 (from (1·7))

$$\bar{H}^{g^2}=\frac{i}{2}\left[\bar{F},\frac{g}{\mu}(\rho_1\pi_\alpha(0)+(\sigma\nabla)\phi_\alpha(0))\tau_\alpha\right]_-$$
$$=\frac{i}{2}\left\{-\left(\frac{\mu}{2m}\right)^2 g^2[\phi_\alpha^{(-2)}\tau_\alpha,\ \pi_\beta(0)\tau_\beta]_- -i\rho_3\frac{g^2}{2m}\{\pi_\alpha^{(-2)}\tau_\alpha,\ \pi_\beta(0)\tau_\beta\}_+\right.$$
$$\left.-\left(\frac{g}{\mu}\right)^2[(\sigma\nabla)\pi_\alpha^{(-2)}\tau_\alpha,\ (\sigma\nabla)\phi_\beta(0)\tau_\beta]_- +\frac{g^2}{2m}\rho^3[(\sigma\nabla)\pi_\alpha^{(-3)}\tau_\alpha,\ (\sigma\nabla)\phi_\beta(0)\tau_\beta]_-\right\}, \tag{1·13}$$

where terms containing ρ_1 and ρ_2 was rejected. In this expression, the first term may be rejected compared with the second and the last may be omitted compared with the 3rd. Then, putting $\rho_3=1$;

$$\bar{H}^{g^2}=\frac{g^2}{2m}\pi_\alpha^{(-2)}\pi_\alpha^{(0)}-\frac{i}{2}\left(\frac{g}{\mu}\right)^2[(\sigma\nabla)\pi_\alpha^{(-2)}\tau_\alpha,\ (\sigma\nabla)\phi_\beta(0)\tau_\beta]_-\,. \tag{1·14}$$

The total Hamiltonian in second order in coupling constant is in ϕ, ϕ^*, π, π^*, ϕ_0 π_0 representation ;

$$\bar{H}=\int(\pi^*\pi+\phi^*(\mu^2-\Delta)\phi)\,dx+\frac{1}{2}\int(\pi_0^2+\phi_0(\mu^2-\Delta)\phi_0)\,dx$$
$$+\frac{g^2}{2m}(\pi^{*(-2)}\pi(0)+\pi^{(-2)}\pi^*(0)+\pi_0^{(-2)}\pi_0(0))$$
$$-i\left(\frac{g}{\mu}\right)^2\left[(\nabla\pi^{*(-2)}\cdot\nabla\phi^*(0)-\nabla\pi^{(-2)}\cdot\nabla\phi(0))\tau_3\right.$$
$$\left.+(\nabla\pi_0^{(-2)}\cdot\nabla\phi(0)-\nabla\pi^{*(-2)}\cdot\nabla\phi_0(0))\sqrt{2}\,\tau_{PN}+(\nabla\pi^{(-2)}\cdot\nabla\phi_0-\nabla\pi_0^{(-2)}\cdot\nabla\phi^*(0))\sqrt{2}\,\tau_{NP}\right]$$
$$+\left(\frac{g}{\mu}\right)^2\left[(\sigma\cdot\nabla\pi^{*(-2)}\times\nabla\phi^*(0))+(\sigma\cdot\nabla\pi^{(-2)}\times\nabla\phi(0))+(\sigma\cdot\nabla\pi_0^{(-2)}\times\nabla\phi_0(0))\right]. \tag{1·15}$$

In momentum space ;

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{dq}{(2\varepsilon_q)^{1/2}}\left(e^{iqx}\phi^{(+)}(q)+e^{-iqx}\phi^{(-)*}(q)\right),$$

$$\pi^*(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{dq}{(2\varepsilon_q)^{1/2}}(-i\varepsilon_q)\left(e^{iqx}\phi^{(+)}(q)-e^{-iqx}\phi^{(-)*}(q)\right), \quad (1\cdot16)$$

$$\begin{aligned}\bar{H}=&\int(\phi^{(+)*}(q)\phi^{(+)}(q)+\phi^{(-)*}(q)\phi^{(-)}(q)+\phi^{(0)*}(q)\phi^{(0)}(q))\varepsilon_q dq\\
&+\frac{1}{(2\pi)^3}\iint\frac{dq\,dp}{(2\varepsilon_q)^{1/2}(2\varepsilon_p)^{1/2}}\cdot\left[\frac{g^2}{2m}\left(\frac{\varepsilon_p}{\varepsilon_q}+\frac{\varepsilon_q}{\varepsilon_p}\right)(\phi^{(+)*}(q)\phi^{(+)}(p)\right.\\
&+\phi^{(-)*}(q)\phi^{(-)}(p)+\phi^{(0)*}(q)\phi^{(0)}(p)-\phi^{(-)}(q)\phi^{(+)}(p)-\phi^{(-)*}(q)\phi^{(+)*}(p)\\
&-\frac{1}{2}\phi^{(0)}(q)\phi^{(0)}(p)-\frac{1}{2}\phi^{(0)*}(q)\phi^{(0)*}(p))-\left(\frac{g}{\mu}\right)^2(pq)\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_p}\right)\\
&\times\{\tau_3(\phi^{(+)*}(q)\phi^{(+)}(p)-\phi^{(-)*}(q)\phi^{(-)}(p))+\sqrt{2}\,\tau_{PN}(\phi^{(0)}(q)\phi^{(-)*}(p)\\
&-\phi^{(0)*}(q)\phi^{(+)}(p))+\sqrt{2}\,\tau_{NP}(\phi^{(0)*}(q)\phi^{(-)}(p)-\phi^{(0)}(q)\phi^{(+)*}(p))\}\\
&+\left(\frac{g}{\mu}\right)^2(pq)\left(\frac{1}{\varepsilon_q}-\frac{1}{\varepsilon_p}\right)\{\tau_3(\phi^{(+)}(q)\phi^{(-)}(p)-\phi^{(-)*}(q)\phi^{(+)*}(p))\\
&+\sqrt{2}\,\tau_{PN}(\phi^{(0)}(q)\phi^{(+)}(p)-\phi^{(0)*}(q)\phi^{(-)*}(p))+\sqrt{2}\,\tau_{NP}(\phi^{(0)*}(q)\phi^{(+)*}(p)\\
&-\phi^{(0)}(q)\phi^{(-)}(p))\}+\left(\frac{g}{\mu}\right)^2 i(\sigma\cdot q\times p)\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_p}\right)(\phi^{(+)*}(q)\phi^{(+)}(p)\\
&+\phi^{(-)*}(q)\phi^{(-)}(p)+\phi^{(0)*}(q)\phi^{(0)}(p))+\left(\frac{g}{\mu}\right)^2 i(\sigma\cdot q\times p)\left(\frac{1}{\varepsilon_q}-\frac{1}{\varepsilon_p}\right)\\
&\times(\phi^{(+)}(q)\phi^{(-)}(p)-\phi^{(+)*}(q)\phi^{(-)*}(p)-\frac{1}{2}\phi^{(0)*}(q)\phi^{(0)*}(p)\\
&\left.+\frac{1}{2}\phi^{(0)}(q)\phi^{(0)}(p))\right].\end{aligned} \quad (1\cdot17)$$

This is our working Hamiltonian derived under the weak coupling assumption. It is clear that the τ_3 dependent term is of opposite sign for π^+ and π^- mesons but its absolute value is the same. Thus, the solution not in Born approximation gives the charge dependence of scattering cross-section.

§ 2. Solution of the Schroedinger equation

The Schroedinger equation we must now solve is

$$\bar{H}\chi=E\chi, \quad (2\cdot1)$$

where $\bar{H}$ is (1·17). Then we divide

$$\bar{H}=\bar{H}_0+\bar{H}_s+\bar{H}', \quad (2\cdot2)$$

where $\bar{H}_0$ is the free meson, $\bar{H}_s$ is the scattering part of (1·17) (the "potential" energy acting on meson) and $\bar{H}'$ is the remainder of (1·17) which contains the pair-formation

and annihilation terms only. We neglect in the following the term $\bar{H}'$. (The effect of which is to be smaller than $\bar{H}_s$ since from (1·17) $\bar{H}'$ contains the energy difference).[6] This does not mean to neglect the two meson contribution completely, since $\bar{H}_s$ contains the scattering "potential" which comes from emitting firstly and then absorbing.

Our problem now becomes to solve

$$(\bar{H}_0+\bar{H}_s)\psi_\lambda=E_\lambda\psi_\lambda\,, \tag{2·3}$$

from the solution of

$$\bar{H}_0\phi_\lambda=E_\lambda\phi_\lambda\,. \tag{2·4}$$

We treat only the scattering problem, we can use Møller's wave matrix, the property of which is

$$\psi_\lambda=W\phi_\lambda=\sum_\mu\phi_\mu(\mu|W|\lambda)$$

$$(\mu|W_\pm|\lambda)=\delta(\mu-\lambda)+\lim_{\delta\to 0}\frac{1}{E_\lambda-E_\mu\pm i\delta}(\mu|H_sW|\lambda),$$

Taking the state with one meson with momentum $\boldsymbol{q}$, charge b (+, 0 or −) and one nucleon (at rest) with charge β (+ or 0), z-component of spin ρ, (this is correct since $\bar{H}_s$ does not change the meson number), we have

$$\begin{aligned}(\boldsymbol{q},\,b\,;\,\beta,\,\rho\,|\,W_\pm\,|\,\boldsymbol{q}_0,\,a\,;\,\alpha,\,\sigma)&=\delta(\boldsymbol{q}-\boldsymbol{q}_0)\,\delta_{b\alpha}\delta_{\beta\alpha}\delta_{\rho\sigma}\\&+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\sum_\nu\sum_c\sum_\gamma\int d\boldsymbol{l}\,(\boldsymbol{q},\,b\,;\,\beta,\,\rho\,|\,\bar{H}_s\,|\,\boldsymbol{l},\,c\,;\,\gamma,\,\nu)\\&\times(\boldsymbol{l},\,c\,;\,\gamma,\,\nu\,|\,W_\pm\,|\,\boldsymbol{q}_0,\,a\,;\,\alpha,\,\sigma).\end{aligned} \tag{2·5}$$

Here writing the nucleon spin function as χ_ρ, charge spin function t_β, and the meson states with charge +, 0, and − as

$$T^{(+)}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad T^{(0)}=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad T^{(-)}=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{2·6}$$

We have as $(\boldsymbol{q},\,b\,;\,\beta,\,\rho\,|\,W_\pm\,|\,\boldsymbol{q}_0,\,a\,;\,\alpha,\,\sigma)$;

$$(\boldsymbol{q},\,b\,;\,\beta,\,\rho\,|\,W_\pm\,|\,\boldsymbol{q}_0,\,a\,;\,\alpha,\,\sigma)=((T^{(b)}t_\beta\chi_\rho)^*(\boldsymbol{q}\,|\,W_\pm\,|\,\boldsymbol{q}_0)\,(T^{(a)}t_\alpha\chi_\sigma)), \tag{2·7}$$

$$(\boldsymbol{q}\,|\,W_\pm\,|\,\boldsymbol{q}_0)=\delta(\boldsymbol{q}-\boldsymbol{q}_0)+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\int d\boldsymbol{l}\,(\boldsymbol{q}\,|\,\bar{H}_s\,|\,\boldsymbol{l})\,(\boldsymbol{l}\,|\,W_\pm\,|\,\boldsymbol{q}_0),$$

and from (1·7)

$$\begin{aligned}(\boldsymbol{q}\,|\,H_s\,|\,\boldsymbol{l})=&\frac{1}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}\Big[\frac{g^2}{2m}\Big(\frac{\varepsilon_q}{\varepsilon_l}+\frac{\varepsilon_l}{\varepsilon_q}\Big)\mathbf{1}\\&+\Big(\frac{g}{\mu}\Big)^2\Big(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_l}\Big)(i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{l})\cdot\mathbf{1}-(\boldsymbol{q}\boldsymbol{l})\,(\tau\omega))\Big],\\&\omega=(\omega_1,\,\omega_2,\,\omega_3)\,;\;\tau=(\tau_1,\,\tau_2,\,\tau_3)\end{aligned}$$

$$\omega_1=\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},\quad \omega_2=\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix},\quad \omega_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2·8}$$

$$(\tau\boldsymbol{\omega})(\tau\boldsymbol{\omega})=(\boldsymbol{\omega})^2-(\tau\boldsymbol{\omega})=2-(\tau\boldsymbol{\omega}), \tag{2·9}$$

where $\mathbf{1}$ is the unit matrix in meson charge space. Then to perform the partial wave analysis of $(\boldsymbol{q}\,|\,W_\pm\,|\,\boldsymbol{q}_0)$, we transform

$$(\boldsymbol{q}\,|\,W_\pm\,|\,\boldsymbol{q}_0)=(Y_0^0(\theta,\varphi);\ Y_1^1(\theta,\varphi),\ Y_1^0(\theta,\varphi),\ Y_1^{-1}(\theta,\varphi);\cdots)$$
$$\times(q\,|\,W_\pm\,|\,q_0)\begin{pmatrix} Y_0^0\ (\theta_0\varphi_0) \\ Y_1^1\ (\theta_0\varphi_0) \\ Y_1^0\ (\theta_0\varphi_0) \\ Y_1^{-1}(\theta_0\varphi_0) \\ \vdots \end{pmatrix}^*, \tag{2·10}$$

$$\sum_{l,m} Y_l^m(\theta,\varphi)\,Y_l^{m*}(\theta_0,\varphi_0)=\delta(\cos\theta-\cos\theta_0)\,\delta(\varphi-\varphi_0), \tag{2·11}$$

(then $(q\,|\,W_\pm\,|\,q_0)$ is matrix). In this notation,

$$i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{l})=(Y_0^0(\theta,\varphi);\ Y_1^1(\theta,\varphi),\ Y_1^0(\theta,\varphi),\ Y_1^{-1}(\theta,\varphi);\cdots)$$
$$\times\left(-\frac{4\pi}{3}\right)ql(\boldsymbol{\sigma}\boldsymbol{L})\begin{pmatrix} Y_0^0(\theta_l\varphi_l) \\ Y_1^1(\theta_l\varphi_l) \\ \vdots \end{pmatrix}^*, \tag{2·12}$$

where $\boldsymbol{L}=(L_1, L_2, L_3)$ is the same form as $\omega_1\sim\omega_3$ in (2·9) and has only values in shaded part of the following matrix;

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & & & & 0 & 0 & \vdots \\ 0 & & & & 0 & 0 & \vdots \\ 0 & & & & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Also

$$(\boldsymbol{ql})=(\theta,\varphi)\cdot\frac{4\pi}{3}ql\,\boldsymbol{p}\begin{pmatrix}\theta_l \\ \varphi_l\end{pmatrix}^*, \tag{2·13}$$

where $\boldsymbol{p}$ is the unit matrix in the shaded area of the above matrix.

By using these notations we have

$$(q\,|\,W_\pm\,|\,q_0)=\frac{1}{q^2}\delta(q-q_0)+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i0}\int_0^\infty l^2dl\,(q\,|\,\bar{H}_s\,|\,l)\,(l\,|\,W_\pm\,|\,q_0), \tag{2·14}$$

$$(q|\bar{H}_s|l)=\frac{1}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}\left[4\pi\frac{g^2}{2m}\left(\frac{\varepsilon_l}{\varepsilon_q}+\frac{\varepsilon_q}{\varepsilon_l}\right)\cdot\boldsymbol{S}\cdot\mathbf{1}\right.$$
$$\left.-\frac{4\pi}{3}\left(\frac{g}{\mu}\right)^2 ql\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_l}\right)(\mathbf{1}\cdot(\boldsymbol{\sigma L})+(\tau\omega)\boldsymbol{p})\right], \qquad (2\cdot15)$$

where $\boldsymbol{S}$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}.$$

The problem thus reduces to solve (2·15). $(\tau\omega)$, $(\boldsymbol{\sigma L})$ has the eigen-value 1 and -2, and corresponds to total charge- and angular- momentum $3/2$ and $1/2$ states. And thus from (2·15) it is clear that the state with $I=J=3/2$ only suffers the attractive "potential" and the other states are in repulsive "potential". (This is the same conclusion from the strong coupling theory since when the coupling is very large, real binding may occur only for states $I=J=3/2$).[8]

Putting

$$(q|W_\pm|q_0)=(q|S|q_0)\boldsymbol{S}$$
$$+(q|L_{1/2}|q_0)\frac{1}{3}(\boldsymbol{p}-(\boldsymbol{\sigma L}))$$
$$+(q|L_{3/2}|q_0)\frac{1}{3}(2\boldsymbol{p}+(\boldsymbol{\sigma L}))$$
$$+(q|R|q_0)\boldsymbol{r}, \qquad (2\cdot16)$$

where $\boldsymbol{r}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix},$$

we have from (2·15);

$$(q|S|q_0)=\frac{1}{q^2}\delta(q-q_0)+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\int\frac{dl}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}$$
$$\times\left(\frac{g^2}{2m}\right)\left(\frac{\varepsilon_q}{\varepsilon_l}+\frac{\varepsilon_l}{\varepsilon_q}\right)(l|S|q_0), \qquad (2\cdot17)$$

$$(q|L_{1/2}|q_0)=\frac{1}{q^2}\delta(q-q_0)+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\int\frac{dl}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}$$
$$\times\left(\frac{g}{\mu}\right)^2\left(-\frac{ql}{3}\right)\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_l}\right)(-2\mathbf{1}+(\tau\omega))(l|L_{1/2}|q_0), \qquad (2\cdot18)$$

$$(q|L_{3/2}|q_0)=\frac{1}{q^2}\delta(q-q_0)+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\int\frac{dl}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}$$
$$\times\left(\frac{g}{\mu}\right)^2\left(-\frac{ql}{3}\right)\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_l}\right)(1+(\tau\omega))\,(e|L_{3/2}|q_0). \qquad (2\cdot19)$$

Moreover, putting

$$(q|L_{1/2}|q_0)=(q|L_{1/2}T_{1/2}|q_0)\frac{1}{3}(1-(\tau\omega))$$
$$+(q|L_{1/2}T_{3/2}|q_0)\frac{1}{3}(2\mathbf{1}+(\tau\omega)), \qquad (2\cdot20)$$

$$(q|L_{3/2}|q_0)=(q|L_{3/2}T_{1/2}|q_0)\frac{1}{3}(1-(\tau\omega))$$
$$+(q|L_{3/2}T_{3/2}|q_0)\frac{1}{3}(2\mathbf{1}+(\tau\omega)). \qquad (2\cdot21)$$

Then, (2·18) and (2·19) become;

$$(q|\begin{pmatrix}L_{1/2}T_{1/2}\\L_{1/2}T_{3/2}\\L_{3/2}T_{1/2}\\L_{3/2}T_{3/2}\end{pmatrix}|q_0)=\frac{1}{q^2}\delta(q-q_0)$$
$$+\frac{1}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\int\frac{dl}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_l)^{1/2}}\frac{ql}{3}\left(\frac{g}{\mu}\right)^2\left(\frac{1}{\varepsilon_q}+\frac{1}{\varepsilon_l}\right)$$
$$\times\begin{pmatrix}4\\1\\1\\-2\end{pmatrix}(l|\begin{pmatrix}L_{1/2}T_{1/2}\\L_{1/2}T_{3/2}\\L_{3/2}T_{1/2}\\L_{3/2}T_{3/2}\end{pmatrix}|q_0), \qquad (2\cdot22)$$

(2·17) and (2·22) can be solved immediately (since the kernel is the product of two variables), and we have;

$$(q|S|q_0)=\frac{1}{q^2}\delta(q-q_0)+\frac{4\pi}{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}\frac{g^2/2m}{(2\pi)^3(2\varepsilon_q)^{1/2}(2\varepsilon_{q_0})^{1/2}}$$
$$\times\frac{\frac{\varepsilon_{q_0}}{\varepsilon_q}\left(1-V^{(1)}(q_0)+\frac{V^{(-1)}(q_0)}{\varepsilon_{q_0}^2}\right)+\frac{\varepsilon_q}{\varepsilon_{q_0}}(1-l^{r(1)}(q_0)+\varepsilon_{q_0}^2 l^{r(3)}(q_0))}{(1-V^{(1)}(q_0))^2-V^{(-1)}(q_0)V^{(3)}(q_0)}, \qquad (2\cdot23)$$

$$l^{r(\lambda)}(q_0)=\frac{(g^2/4\pi)}{2\pi m}\int_0^\infty\frac{l^2dl}{\varepsilon_l^{(\lambda)}(\varepsilon_{q_0}-\varepsilon_l\pm i\delta)}, \qquad (2\cdot24)$$

$$(q|\lambda|q_0)=(q|\begin{pmatrix}L_{1/2}T_{1/2}:\lambda=4\\L_{1/2}T_{3/2}:\lambda=1\\L_{3/2}T_{1/2}:\lambda=1\\L_{3/2}T_{3/2}:\lambda=-2\end{pmatrix}|q_0)$$

$$= \frac{1}{q^2}\delta(q-q_0) + \frac{4\pi/3}{\varepsilon_{q_0}-\varepsilon_q \pm i\delta} \frac{\lambda(g/\mu)^2 q q_0}{(2\pi)^3 (2\varepsilon_q)^{1/2}(2\varepsilon_{q_0})^{1/2}}$$

$$\times \frac{\frac{1}{\varepsilon_{q_0}}(1-\lambda\Omega^{(2)}(q_0)+\lambda\varepsilon_{q_0}\Omega^{(3)}(q_0)) + \frac{1}{\varepsilon_q}\left(1-\lambda\Omega^{(2)}(q_0)+\lambda\frac{\Omega^{(1)}(q_0)}{\varepsilon_{q_0}}\right)}{(1-\lambda\Omega^{(2)}(q_0))^2-\lambda^2\Omega^{(1)}(q_0)\Omega^{(3)}(q_0)}, \quad (2\cdot25)$$

$$\Omega^{(n)}(q_0) = \frac{(g^2/4\pi)}{3\pi\mu^2}\int_0^\infty \frac{l^4 dl}{\varepsilon_l^{(n)}(\varepsilon_{q_0}-\varepsilon_l \pm i\delta)}. \quad (2\cdot26)$$

Then denoting the scattered wave by $(q|\bar{S}|q_0)$, $(q|\bar{\lambda}|q_0)$, we have

$$\begin{aligned}(q|W_\pm|q_0) = {} & \frac{1}{q^2}\delta(q-q_0) + (q|\bar{S}|q_0)\boldsymbol{S}\cdot\mathbf{1} \\ & + (q|\overline{L_{1/2}T_{1/2}}|q_0)\frac{1}{3}(\mathbf{1}-\tau\omega)\frac{1}{3}(\boldsymbol{p}-\sigma\boldsymbol{L}) \\ & + (q|\overline{L_{1/2}T_{3/2}}|q_0)\frac{1}{3}(2\mathbf{1}+\tau\omega)\frac{1}{3}(\boldsymbol{p}-\sigma\boldsymbol{L}) \\ & + (q|\overline{L_{3/2}T_{1/2}}|q_0)\frac{1}{3}(\mathbf{1}-\tau\omega)\frac{1}{3}(2\boldsymbol{p}+\sigma\boldsymbol{L}) \\ & + (q|\overline{L_{3/2}T_{3/2}}|q_0)\frac{1}{3}(2\mathbf{1}+\tau\omega)\frac{1}{3}(2\boldsymbol{p}+\sigma\boldsymbol{L}). \end{aligned} \quad (2\cdot27)$$

We have the state vector from (2·7) by constructing the matrix $(\boldsymbol{q}|W|\boldsymbol{q}_0)$ by using (2·10);

$$\psi_{\boldsymbol{q},b;\beta,\rho} = \sum_\nu \sum_\sigma \sum_\tau \int d\boldsymbol{l}\,\phi_{l,c;\tau,\nu}(\boldsymbol{l},c;\gamma,\nu|W_\pm|\boldsymbol{q},b;\beta,\rho).$$

§ 3. Cross-section of scattering

$(\boldsymbol{q}|W_\pm|\boldsymbol{q}_0)$ is given by using (2·27) from (2·10), (2·12) and (2·13)

$$\begin{aligned}(\boldsymbol{q}|W_\pm|\boldsymbol{q}_0) = {} & \delta(\boldsymbol{q}-\boldsymbol{q}_0) + \frac{1}{4\pi}(q|\bar{S}|q_0)\cdot\mathbf{1} \\ & + \frac{3}{4\pi q q_0}\Big[(q|\overline{L_{1/2}T_{1/2}}|q_0)\frac{1}{3}(\mathbf{1}-\tau\omega)\frac{1}{3}(\boldsymbol{q}\boldsymbol{q}_0+i(\sigma\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\ & + (q|\overline{L_{1/2}T_{3/2}}|q_0)\frac{1}{3}(2\mathbf{1}+\tau\omega)\frac{1}{3}(\boldsymbol{q}\boldsymbol{q}_0+i(\sigma\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\ & + (q|\overline{L_{3/2}T_{1/2}}|q_0)\frac{1}{3}(\mathbf{1}-\tau\omega)\frac{1}{3}(2\boldsymbol{q}\boldsymbol{q}_0-i(\sigma\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\ & + (q|\overline{L_{3/2}T_{3/2}}|q_0)\frac{1}{3}(2\mathbf{1}+\tau\omega)\frac{1}{3}(2\boldsymbol{q}\boldsymbol{q}_0-i(\sigma\cdot\boldsymbol{q}\times\boldsymbol{q}_0))\Big]. \end{aligned} \quad (3\cdot1)$$

Since the state functional is given by (2·28), the scattering matrix element is given by dropping $1/(2\pi)^3(\varepsilon_{q_0}-\varepsilon_q\pm i\delta)$ from $(q|\bar{S}|q_0)$, $(q|\overline{LT}|q_0)$,

$$
\begin{aligned}
H^{\text{scatt}}_{qq_0} &= (q|H_s|q_0)\mathbf{1} \\
&+ (q|H_{1/21/2}|q_0)\cdot\frac{1}{3}(\mathbf{1}-\tau\boldsymbol{\omega})\frac{1}{3}(\boldsymbol{qq}_0+i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\
&+ (q|H_{1/23/2}|q_0)\cdot\frac{1}{3}(2\mathbf{1}+\tau\boldsymbol{\omega})\frac{1}{3}(\boldsymbol{qq}_0+i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\
&+ (q|H_{3/21/2}|q_0)\cdot\frac{1}{3}(\mathbf{1}-\tau\boldsymbol{\omega})\frac{1}{3}(2\boldsymbol{qq}_0-i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{q}_0)) \\
&+ (q|H_{3/23/2}|q_0)\cdot\frac{1}{3}(2\mathbf{1}+\tau\boldsymbol{\omega})\frac{1}{3}(2\boldsymbol{qq}_0-i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{q}_0)),
\end{aligned}
\tag{3·2}
$$

where $(q|H|q_0)$ is in (2·23) and (2·25) the scattered amplitude multiplied by $\frac{\varepsilon_{q_0}-\varepsilon_q\pm i\delta}{4\pi}(2\pi)^3$ and $\frac{3(\varepsilon_{q_0}-\varepsilon_q\pm i\delta)}{4\pi}(2\pi)^3$ respectively. We can simplify $(q|H|q_0)$ by using the following relations (We put $\varepsilon_{q_0}=\varepsilon_q=\varepsilon$) ;

$$
\begin{aligned}
&V^{(1)}(q)=\varepsilon^2V^{(3)}(q)-\varepsilon v^{(3)}-v^{(2)}, \\
&V^{(-1)}(q)=\varepsilon^4V^{(3)}(q)-\varepsilon^3v^{(3)}-\varepsilon^2v^{(2)}-\varepsilon v^{(1)}-v^{(0)}, \\
&v^{(n)}=\frac{g^2/4\pi}{2\pi m}\int_0^\infty\frac{l^2\,dl}{\varepsilon_l^{(n)}},
\end{aligned}
\tag{3·3}
$$

$$
\begin{aligned}
&\Omega^{(2)}(q)=\varepsilon\Omega^{(3)}(q)-\omega^{(3)}, \\
&\Omega^{(1)}(q)=\varepsilon^2\Omega^{(3)}(q)-\varepsilon\omega^{(3)}-\omega^{(2)}, \\
&\omega^{(n)}=\frac{g^2/4\pi}{3\pi\mu^2}\int_0^\infty\frac{l^4\,dl}{\varepsilon_l^{(n)}}
\end{aligned}
\tag{3·4}
$$

(where v, ω do not depend on the meson energy).

$$
(q|H_s|q_0)=\frac{g_s^{\,2}}{2m\varepsilon}\cdot\frac{1}{1-2\varepsilon^2V^{(3)}(g_s^{\,2})}, \tag{3·5}
$$

$$
g_s^{\,2}=g^2\frac{1+\frac{\varepsilon}{2}v'^{(3)}+\frac{1}{2}v^{(2)}-\frac{1}{2\varepsilon}v^{(1)}-\frac{1}{2\varepsilon^2}v^{(0)}}{(1+v^{(2)})^2+\varepsilon v^{(3)}(2v^{(2)}+\varepsilon v^{(3)})}, \tag{3·6}
$$

$$
(q|\begin{pmatrix}H_{1/21/2}:\lambda=4\\ H_{1/23/2}:\lambda=1\\ H_{3/21/2}:\lambda=1\\ H_{3/23/2}:\lambda=-2\end{pmatrix}|q_0)=(q|H_\lambda|q_0)=\frac{\lambda g_\lambda^{\,2}}{\mu^2\varepsilon^2}\cdot\frac{1}{1-2\lambda\varepsilon\Omega^{(3)}(g_\lambda^{\,2})}, \tag{3·7}
$$

$$
g_\lambda^{\,2}=g^2\frac{1+\frac{\lambda}{2}\left(\omega^{(3)}-\frac{1}{\varepsilon}\omega^{(2)}\right)}{(1+\lambda\omega^{(3)})^2}, \tag{3·8}
$$

where in the denominator of (3·5) and (3·7), $I^{(3)}(g_s^2)$, $\Omega^{(3)}(g_\lambda^2)$ means to put g_s^2, g_λ^2 instead of g^2 in (2·24) and (2·26) respectively. Thus, the divegences appeared are summed to the effective coupling constant and the effective coupling constant is different for each states with $I=3/2$, $1/2$ and $J=3/2$ and $1/2$, and also depends on the meson energy.

From (3·2) we have

$$\begin{aligned}H_{qq_0}^{\text{scatt}}=&(q\,|\,H_s\,|\,q_0)\mathbf{1}\\&+1/9[2(2\mathbf{1}+\tau\boldsymbol{\omega})(q\,|\,H_2\,|\,q_0)\\&+(4\mathbf{1}-\tau\boldsymbol{\omega})(q\,|\,H_1\,|\,q_0)+(\mathbf{1}-\tau\boldsymbol{\omega})(q\,|\,H_4\,|\,q_0)](\boldsymbol{q}\boldsymbol{q}_0)\\&+1/9[-(2\mathbf{1}+\tau\boldsymbol{\omega})(q\,|\,H_2\,|\,q_0)\\&+(\mathbf{1}+2\tau\boldsymbol{\omega})(q\,|\,H_1\,|\,q_0)+(\mathbf{1}-\tau\boldsymbol{\omega})(q\,|\,H_4\,|\,q_0)]i(\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{q}_0),\end{aligned}\tag{3·9}$$

The scattering cross-section becomes

$$d\sigma=2\pi\,|\,H^{\text{Scatt}}\,|^2\frac{\varepsilon^2}{(2\pi)^3}d\Omega=\left|\frac{2\varepsilon}{4\pi}H^{\text{Scatt}}\right|^2 d\Omega\,.\tag{3·10}$$

Writing $\Omega^{(3)}$ and $I^{(3)}$ explicitly and taking $(2\varepsilon/4\pi)H=K$, we obtain the following expressions;

$$K_s=\frac{2\varepsilon}{4\pi}H_s=\frac{f_s^2}{m}\frac{1}{1-\varepsilon^2\dfrac{f_s^2}{\pi m}\displaystyle\int_0^\infty\frac{l^2\,dl}{\varepsilon_l^3(\varepsilon-\varepsilon_l\pm i\delta)}}\approx\frac{f_s^2}{m}\,,\tag{3·11}$$

$$K_\lambda=\frac{2\varepsilon}{4\pi}H_\lambda=\frac{2\lambda f_\lambda^2}{\mu^2\varepsilon}\cdot\frac{1}{+1\dfrac{2\lambda f_\lambda^2}{3\pi\mu^2}\left(\varepsilon l_{\max}+\varepsilon\log\left(\dfrac{l_{\max}+\varepsilon_{\max}}{\mu}\right)-\dfrac{\mu^3}{\varepsilon}\tan^{-1}\left(\dfrac{l_{\max}}{\mu}\right)-\mu^2-\dfrac{q^3}{\varepsilon}\log\left(\dfrac{q+\mu}{\mu}\right)+\pi i\dfrac{q^3}{\varepsilon}\right)},$$

where $l_{\max}$ is the cut-off momentum and $f^2=g^2/4\pi$, and since the denominator of the (3·11) converges and contains small factor μ/m, we may put it as $\approx f_s^2/m$. By using these K's we have the cross-sections from (3·10);

(i) $\tau\omega=1$ $\quad(\pi^+ + P\to\pi^+ + P,\ \pi^- + N\to\pi^- + N)$

$$\begin{aligned}d\sigma/d\Omega=&\,|\,K_s\,|^2+1/9\,\{|\,K_2\,|^2-(K_{-2}^*K_1+\text{c.c.})+|\,K_1\,|^2\}\,q^4\\&+1/3\,\{K_s^*(2K_{-22}+K_1)+\text{c.c.}\}\,q^2\cos\theta\\&+1/9\,\{3\,|\,K_{-2}\,|^2+3(K_{-2}^*K_1+\text{c.c.})\}\,q^4\cos^2\theta\,,\end{aligned}\tag{3·12}$$

(ii) $\tau\omega=-1$ $\quad(\pi^- + P\to\pi^- + P,\ \pi^+ + N\to\pi^+ + N)$

$$\begin{aligned}d\sigma/d\Omega=&\,|\,K_s\,|^2+1/81\,\{|\,K_{-2}\,|^2+(K_{-2}^*K_1+\text{c.c.})+|\,K_1\,|^2\\&-2(K_1^*K_4+\text{c.c.})+4\,|\,K_4\,|^2-2(K_4^*K_{-2}+\text{c.c.})\}\,q^4\\&+1/9\,\{K_s^*(2K_{-2}+5K_1+2K_4)+\text{c.c.}\}\,q^2\cos\theta\end{aligned}$$

$$+1/81\{3|K_{-2}|^2+9(K_{-2}^*K_1+\text{c.c.})+24|K_1|^2$$
$$+12(K_1K_4^*+\text{c.c.})+6(K_4^*K_{-2}+\text{c.c.})\}q^4\cos^2\theta\,, \qquad (3\cdot13)$$

(iii) $\tau\omega=\sqrt{2}\quad(\pi^-+P\rightleftarrows\pi^0+N),\ -\sqrt{2}\ (\pi^++N\rightleftarrows\pi^0+P)$

$$d\sigma/d\Omega=2/81\{|K_{-2}|^2-2(K_{-2}^*K_1+\text{c.c.})+4|K_1|^2$$
$$-2(K_1^*K_1+\text{c.c.})+|K_1|^2+(K_4^*K_{-2}+\text{c.c.})\}q^4$$
$$+2/81\{3|K_{-2}|^2-3|K_1|^2+3(K_1^*K_4+\text{c.c.})$$
$$-3(K_4^*K_{-2}+\text{c.c.})\}q^4\cos^2\theta\,, \qquad (3\cdot14)$$

(iv) $\tau\omega=0\quad(\pi^0+(P,N)\rightarrow\pi^0+(P,N)$

$$d\sigma/d\Omega=|K_s|^2+1/81\{4|K_{-2}|^2-2(K_{-2}^*K_1+\text{c.c.})$$
$$+|K_1|^2+(K_1^*K_4+\text{c.c.})+|K_4|^2-2(K_4^*K_{-2}+\text{c.c.})\}q^4$$
$$+1/9\{K_s^*(3K_{-2}+4K_1+K_4)+\text{c.c.}\}q^2\cos\theta$$
$$+1/81\{12|K_{-2}|^2+18(K_{-2}^*K_1+\text{c.c.})+15|K_1|^2+3(K_1^*K_4+\text{c.c.})$$
$$+6(K_4^*K_{-2}+\text{c.c.})\}q^4\cos^2\theta\,. \qquad (3\cdot15)$$

In these expressions only K_{-2} contains the resonance denominator[1] and hence if this only contributes the ratio 9 : 2 1 and the angular dependence $1+3\cos^2\theta$ follows immediately.

Total cross-section is

(a) $\pi^++P\rightarrow\pi^++P,\ \pi^-+N\rightarrow\pi^-+N.$

$$\sigma=4\pi[|K_s|^2+1/9\{2|K_{-2}|^2+|K_1|^2\}q^4]\,, \qquad (3\cdot16)$$

(b) $\pi^-+P\rightarrow\pi^-+P,\ \pi^++N\rightarrow\pi^++N$

$$\sigma=4\pi[|K_s|^2+1/81\{2|K_{-2}|^2+4(K_{-2}^*K_1+\text{c.c.})+9|K_1|^2$$
$$+2(K_1K_4^*+\text{c.c.})+4|K_4|^2\}q^4]\,, \qquad (3\cdot17)$$

(c) $\pi^-+P\rightleftarrows\pi^0+N,\ \pi^++N\rightleftarrows\pi^0+P.$

$$\sigma=4\pi[2/81\{2|K_{-2}|^2-2(K_{-2}^*K_1+\text{c.c.})+3|K_1|^2$$
$$-(K_1K_4^*+\text{c.c.})+|K_4|^2\}q^4]\,, \qquad (3\cdot18)$$

(d) $\pi^0+(P,N)\rightarrow\pi^0+(P,N).$

$$\sigma=4\pi[|K_s|^2+1/81\{8|K_{-2}|^2+4(K_{-2}^*K_1+\text{c.c.})+6|K_1|^2$$
$$+2(K_1^*K_4+\text{c.c.})|K_4|^2\}q^4]\,, \qquad (3\cdot19)$$

Total cross-section for π^-+P becomes ; (or π^++N)

$$\sigma=4\pi\left[|K_s|^2+\frac{1}{9}\left\{\frac{2}{3}|K_{-2}|^2+\frac{5}{3}|K_1|^2+\frac{2}{3}|K_4|^2\right\}q^4\right]. \qquad (3\cdot20)$$

Expressions (3·16) and (3·20) coincides with Brueckner's expression (21)[1] except for S-wave scattering and small changes. The changes are that in our formula the "effective coupling constant" (3·6) and (3·8) appears and the non-resonating term contains the

different denominator compared to the Brueckner's. Apart from the resonance, the $I=J=3/2$ state is favored from the coupling constant, because (3·8) shows larger coupling constant in this state. But owing to the ambiguous treatment of the diverging quantities, we cannot put this conclusion conclusively.

§ 4. Conclusions

We showed that the effective Hamiltonian derived under the assumption of weak coupling gives the good result for meson-scattering when treated exactly (not in Born approximation). Since the treatment of this paper is essentially non-relativistic, we cannot derive any reliable numerical value to be compared with meson scattering experiments, our only conclusion is that the qualitative features of meson scattering can be explained in standard meson theory. Other ambiguous point is that we have not renormalized the self-energy hence the effective coupling constant contains this effect.

But if it is allowed to use our approximations made in this paper, pseudo-scalar meson with pseudo-vector coupling is qualitatively good for meson scattering. That we performed in this paper is so to speak the treatment of nucleon in second order nuclear force. The higher order terms are to be expected to be the correction to this or will play definite role in the higher energy region. Non the less, the effective Hamiltonian derived under weak coupling assumption gives the right qualitative results for both nuclear force and meson scattering will serve to support the treatment in this way.

In conclusion, I express my heartful thanks to Professor S. Tomonaga for his kind discussions and to members of Tokyo University and Tokyo Bunrika Daigaku for their critical discussions.

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3) The charge renormalization of this type was performed by Goldberger Phys. Rev. **84** (1951), 429, in the Blatt's example.
4) The nuclear force can be derived immediately from (1.9), the result is the same as those given by G. Araki.
5) The result is not identical with putting ρ_1, ρ_2 matrices zero from the beginning. The S-wave contribution can only be obtained in this way.
6) The omission corresponds in some feature to "adiabatic" treatment according to Dr. N. Fukuda.
7) $[\omega_1, \omega_2]=i\omega_3$ (cyclic)
8) In the strong coupling theory the isobar-energy is given by
$$E_{J=I}=\frac{3}{2}\cdot\frac{\mu a_0}{g^2}\left[J(J+1)-\frac{3}{4}\right], \quad (J=I), \quad a_0=\frac{\pi}{2}l_{\max}.$$
9) If one retains only the most severe divergence, the denominator of K_{-2} is $1-\frac{4f_{-2}^2}{3\pi\mu^2}\varepsilon l_{\max}-\frac{4f_{-2}^2}{3\mu^2}\times\frac{q^3}{\varepsilon}i$ and is the same for photo-meson production treated by Brueckner and Case, Phys. Rev. 83 (1951), 1141.

Letters to the Editor

On the Proton-neutron Interaction

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and

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March 26, 1953

In the phenomenological treatments of nuclear force, non-singular or Yukawa potentials have generally been used for the sake of simplicity. These potentials will be sufficient so long as the energy of a system is low. However, to secure the better fit with experiments up to high energies, it seems necessary that investigations are extended to the various types of potentials, singular as well as non singular. As the preliminary work we have computed the magnetic moment and quadrupole moment using the potential with r^{-2} singularity. It was shown that the results did not contradict with the known facts concerning the deuteron ground state.

We shall assume the nuclear potential:

$$V(\vec{r}) = \frac{1}{3}(\vec{\tau}_1 \cdot \vec{\tau}_2) V_0 \left(I + \gamma' S_{12} \frac{I}{\varkappa r}\right) \frac{e^{-\varkappa r}}{\varkappa r} .$$

This is the modification of the pseudo-scalar meson potential with a parameter γ', the r^{-2} singularity being for the practical simplicity. If we take the eigenfunction of the deuteron ground state as

$$\psi(x) = x^{-1}[u(x)\,{}^3Z_1{}^{(0)}1 + W(x)\,{}^3Z_1{}^{(2)}1]^1(\tau)_0,$$

then the radial functions $u(x)$ and $w(x)$ satisfy the well known coupled differential equations:

$$\frac{d^2u}{dx^2} - \varepsilon^2 u + a\frac{e^{-x}}{x}u = -2\sqrt{2}\,\gamma\frac{e^{-x}}{x^2}w$$

$$\frac{d^2w}{dx^2} - \varepsilon^2 w + a\frac{e^{-x}}{x}w - 2\gamma\frac{e^{-x}}{x^2}w - \frac{6}{x^2}w = -2\sqrt{2}\,\gamma\frac{e^{-x}}{x^2}u$$

where we set the abbreviation

$$x = \varkappa r, \qquad a = \frac{V_0 M}{\mu^2 c^2}, \qquad \gamma = a\gamma', \qquad \varepsilon^2 = \frac{|E|M}{\mu^2 c^2}.$$

The equations were numerically integrated. By taking the binding energy of the deuteron and the meson mass as 2.237 Mev and 273 m respectively, we have computed the magnetic moment and quadrupole moment according to the zero cut off and hard core approximation. The results are given in the Table.

			Zero cut off			Hard core		
a	V_0 in Mev	γ/a	x_0	Percent D	Q in 10^{-27} cm^2	x_0	Percent D	Q in 10^{-27} cm^2
2.0	41.382	1	0.489,2	2.855	1.871			
2.2	45.521	1	0.543,6	2.954	2.009	0.239,7	3.678	
2.4	49.659	1	0.605,6	3.046	2.163	0.274,1	3.952	2.296
2.6	53.797	1	0.655,7	3.114	2.300_5	0.299,8	4.079	2.425
3.0	62.074	1	0.755,6	3.229	2.556	0.359,5	4.399	2.732
3.5	72.419	1	0.860,1	3.455	2.855	0.428,8	4.737	

From the above figures we see that in the zero cut off approx. the cut off radius becomes too large and D-state probability and quadrupole moment are small, while in the hard core approx. the cut off radius is satisfactory and the obtained values of the magnetic moment and quadrupole moment are well fitted with the experimental values.

A more detailed discussion on the assumed interaction will be investigated from the proton-neutron scattering and photo-disintegration of the deuteron in the high energies.

On the Pion-nucleon Scattering

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March 29, 1953

Field theories with non-localized interactions have been investigated by many authors[1]. Their main purpose has been the construction of a convergent field theory, while applications to practical problems have been of secondary importance. On the other hand, recent experimental findings on pion-nucleon scattering[2),3)] differ widely from the perturbation theoretical predictions and many works have been done on this subject. In this note we consider the non-localized interaction as a possible mechanism of providing a consistent description of a pion-nucleon interaction and investigate to what extent we can approach to the satisfactory interpretation of the pion-nucleon scattering phenomena.

As mentioned above we consider the interaction between pion and nucleon to be non-local with the following interaction Lagrangian:

$$L'(x)\equiv\int\int\int L'(x',x'',x''';x)\,(dx')\,(dx'')\,(dx''')$$

$$=-\frac{ig\alpha}{\varkappa_0}\int\int\int\bar{\psi}(x')\xi(x'-x)\gamma_5\gamma_\mu\tau_\alpha\xi(x-x'')\times$$

$$\times\psi(x'')\frac{\partial\zeta(x-x''')}{\partial x_\mu}\phi_\alpha(x''')\,(dx')\,(dx'')\,(dx'''). \tag{1}$$

Here $\xi(x)$ and $\zeta(x)$ are the Lorentz invariant extensions of the $\delta^4(x)$'s and subject to the following conditions:

$$\xi(x)=\frac{1}{2}\,(\xi_0(x)+\bar{\xi}_0(x)),\ \bar{\xi}_0(x)=\gamma_4\xi_0^{\dagger}(-x)\gamma_4, \tag{2}$$

$$(\gamma_\mu\partial_\mu+\mu)\xi_0(x)=(\mu-\mu_0)\delta^4(x), \tag{3}$$

$$\zeta(x)=\frac{1}{2}(\zeta_0(x)+\zeta_0^{\dagger}(x)), \tag{4}$$

$$(\Box-\varkappa^2)\zeta_0(x)=(\varkappa_0^2-\varkappa^2)\delta^4(x), \tag{5}$$

where μ_0 and $\varkappa_0$ denote the nucleon mass and meson mass respectively and μ and $\varkappa$ are newly introduced parameters which we regard as complex quantities.

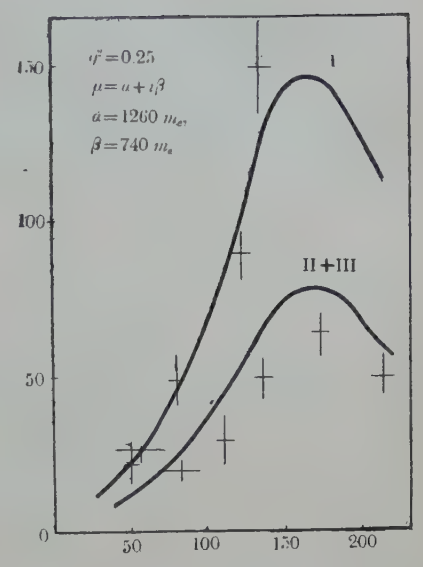

Fig. 1. Total cross sections.
I. $\pi^+ + P \to \pi^+ + P$.
II. $\pi^- + P \to \pi^- + P$.
III. $\pi^- + P \to \pi^0 + N$.

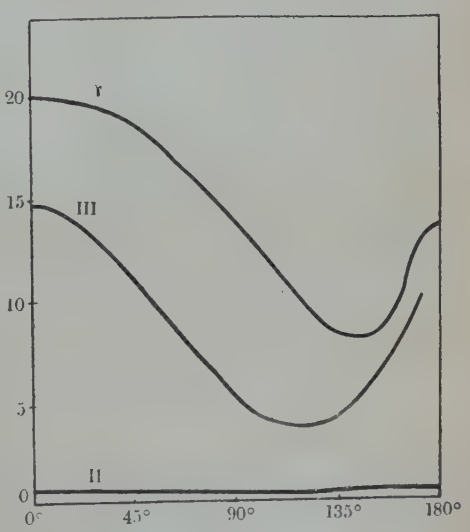

Fig. 2.
Angular distributions in the centre of mass system at meson kinetic energy 135 Mev.

For the derivation of the S-matrix we have used the method developed by Yang and Feldman[4] and by Källen[5]. We have calculated the matrix elements for the pion-proton scattering to the second order in the coupling constant and to this approximation the form factor $\zeta(x)$ plays no rôle and only $\xi(x)$ is essential. From (2) and (3) the Fourier transform of $\xi(x)$ is of the form:

$$\frac{1}{2}\{(\mu-\mu_0)(i\gamma k+\mu)^{-1}+(\mu^*-\mu_0)(i\gamma k+\mu^*)^{-1}\}, \tag{6}$$

and by rationalizing (6) we find this factor may contribute as if there were a resonance. We have determined μ (complex) and γ so that the calculated values of the total cross sections for the process I $\pi^+ + P \to \pi^+ + P$ may best fit the experimental values by Fermi et al. For the values of μ thus determined, (6) contributes merely as a cut off factor for the process II $\pi^- + P \to \pi^- + P$. This is because the momentum of the virtual nucleon is $k_\mu = f_\mu - q_\mu'$ in I while $k_\mu = f_\mu + q_\mu$ in II, where f_μ is the momentum of the initial nucleon and q_μ and q_μ' are those of the initial and final meson respectively. Using the symmetrical theory, the results thus obtained are given in Fig. 1 and Fig. 2. The total cross section for II+III is somewhat larger than the experimental one. This is due to the large cross section for III. The calculated ratio of σ_{I}, σ_{III} and σ_{II} at meson energy 135 Mev is 14:7:1, while the experiments by Fermi et al. give the ratio 9:2:1.

In the lowest order approximation here considered, the form factor $\zeta(x)$ contributes nothing. This will play an important part when we consider higher order terms or any other phenomena, e.g. the anomalous magnetic moment of the nucleon. It is also of great interest whether we can explain the $\gamma-\pi$ process consistently by using the values of parameters here adopted. But before entering these problems we must investigate the problem of gauge invariance.

The author is grateful to Mr. N. Shōno for his kind suggestions and instructive discussions and also to Mr. T. Ouchi for his valuable discussion.

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The Statistical Mechanics of Time-dependent Phenomena

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By using the laws of classical mechanics and the ensemble averaging procedure, a general statistical mechanical theory of time-dependent phenomena will be presented. The phenomenological behaviors of a system of molecules are described by a far smaller number of variables than the degrees of freedom of the system f. We shall first attempt to describe the motion of a dynamical system (a phase point is denoted by $X=(p,q)$) by an arbitrary set of a small number of dynamical variables $a(X)=\{a_1, \cdots, a_h(X)\}$ $(h<2f)$. The change with time of these dynamical variables is given by

$$a_l(X_s)=e^{-isL(X)}a_l(X) \tag{1·1}$$

and

$$L(X)=i\sum_{r=1}^{f}\left\{\frac{\partial H}{\partial p_r}\frac{\partial}{\partial q_r}-\frac{\partial H}{\partial q_r}\frac{\partial}{\partial p_r}\right\}, \tag{1·2}$$

where H is the Hamiltonian and X_s is the image of phase X after time $s(\geqq 0)$. By the probability distribution function $F(X,t)$, let us define

$$f(a,t)=\int dx\, F(X,t)\delta(a-a(X)), \tag{1·3}$$

$$W(a(X)/X;t)=F(X,t)/f(a(X),t) \tag{1·4}$$

and

$$T(a,t;a',t')=\langle\delta(a'-e^{-i(t'-t)L(X)}a(X))\rangle_a, \tag{1·5}$$

where $\delta(a-a(X))=\Pi\,\delta(a_l-a_l(X))$, δ is Dirac's δ-function and $\langle\cdots\rangle_a$ denotes $\int dx\,\delta(a-a(X))\times W(a|X;t)\cdots$. $f(a,t)$ is the probability that $a(X)$ has a set of values $a=(a_1,\cdots,a_h)$ at time t $W(a|X;t)$ is the "conditional probability" that the phase is X when $a(X)$ has a set of values a, and $T(a,t;a',t')$ $(t<t')$ expresses the "*transition probability*" that $a(X)$ having a at t attains to a' after time $(t'-t)$. Applying Liouville's theorem $F(X_{t'-t},t')=F(X,t)$, we obtain

$$f(a',t')=\int da\,f(a,t)\,T(a,t;a',t') \qquad (1\cdot6)$$

and

$$f(a,t)\,T(a,t;a',t')=f(a',t')\,T(a',t';a,t). \qquad (1\cdot7)$$

Because of $\exp[-i(t-t_0)L(X_0)]=\exp[-i(t-t')\times L(X')]\exp[-i(t'-t_0)L(X_0)]$, we obtain

$$T(a_0,t_0;a,t)=\int da'\,T(a_0,t_0;a',t')\times \times T(a',t';a,t), \qquad (1\cdot8)$$

which is nothing other than *Smoluchowski*'s equation for the continuous stochastic process if $t_0<t'<t$. If (1·4) does not explicitly depend on time, i.e., if the ensemble is in statistical equilibrium or one may admit the *adiabatic approximation*

$$F(X,t)=f(a(X),t)\,{}^0W(a(X)|X), \qquad (1\cdot9)$$

this process becomes *Markoffian* and Green's principle[1] is justified.

If we confine ourselves to the case of ${}^0F(X)=F\{H(X)\}$, we have

$${}^0F(X)={}^0F(-X), \quad -X\equiv(-p,q) \qquad (2\cdot1)$$

by the principle of dynamical reversibility $H(X)=H(-X)$. Then, we obtain

$${}^0f(a)={}^0f(-a), \quad {}^0W(a|X)={}^0W(-a|-X),$$
$${}^0T(a,a';s)={}^0T(-a,-a';-s), \qquad (2\cdot2)$$

where $a(X)$ is assumed to be composed of even and odd variables, $b(X)=b(-X)$ and $c(X)=-c(-X)$, $-a$ means $(b,-c)$ and $s=t'-t$. From (2·2) and (1·7) we have a generalization of the *principle of detailed balance*

$${}^0f(a)\,{}^0T(a,a';s)={}^0f(-a')\,{}^0T(-a',-a;s) \qquad (2\cdot3)$$

which, if we adopt $H(X)$ as $a(X)$, gives

$${}^0f(E)\,{}^0T(E,E';s)={}^0f(E')\,{}^0T(E',E;s),$$

E and E' being the values of energy of system. Then, Kolmogoroff's ergodic theorem gives

$$\lim_{s\to\infty}{}^0T(a_0,a;s)={}^0f(a). \qquad (2\cdot4)$$

The *macroscopic* rate of change with time of dynamical variable $a_i(t)$ is given by

$$v_i(a,t)=\Delta^{-1}\{\int da'\,T(a,t;a',t+\Delta)a'_i-a_i\}$$
$$=\Delta^{-1}\langle\int_0^\Delta V_i(X_s)\,ds\rangle_a, \qquad (3\cdot2)$$

$\Delta(\geqq\tau)$ being a short time interval in *macro-scale*, $a=a(t)$ and $V_i(X_s)=da_i(X_s)/ds$. (3·2) justifies the *double averaging procedure* which Kirkwood[2] used in analytical derivation of Langevin's equation for a Brownian particle. If we admit (1·9), (3·2) may be brought into

$$(3.2)={}^0\langle V_i\rangle_a+\sum_{l=1}^{h}{}^0f^{-1}(a)\frac{\partial}{\partial a_l}[{}^0f(a)\,{}^0\xi_{il}(a)] \qquad (3\cdot3)$$

under certain restrictions for the set $a(X)$. ${}^0\xi_{il}(a)$'s satisfy symmetry relations which are a generalization of Onsager's[1]. Thus we obtain generalized Langevin's equation

$$da_i/dt={}^0\langle V_i\rangle_a+ \sum_{l=1}^{h}{}^0f^{-1}(a)\frac{\partial}{\partial a_l}[{}^0f(a)\,{}^0\xi_{il}(a)]+K_i(a,t), \qquad (3\cdot4)$$

where

$${}^0\langle\Delta^{-1}\int_0^\Delta K_i(a,t+s)\,ds\rangle_a=0,$$
$${}^0\langle K_i(a,t)\,K_l(a,t+\Delta t)\rangle_a\simeq 2\,{}^0\bar{\xi}_{il}(a)\,\delta(\Delta t). \qquad (3\cdot5)$$

K_i is the generalized fluctuating force, and ${}^0\bar{\xi}_{il}$ is the symmetric part of ${}^0\xi_{il}$. These equations should be compared with Hashitsume's[3]. Langevin's equation obtained by Kirkwood[2] is, as a special case, derived from (3·4).

If we admit (1·9), we obtain from (1·8)

$$\partial\,{}^0T(a;t)/\partial t\equiv\Delta^{-1}\{{}^0T(a;t+\Delta)-{}^0T(a;t)\} \qquad (4\cdot1)$$

$$=\sum_i\frac{\partial}{\partial a_i}\Big[-\Big\{{}^0\langle V_i\rangle_a+ \sum_l{}^0f^{-1}(a)\frac{\partial}{\partial a_l}({}^0f(a)\,{}^0\xi_{il}(a))\Big\}\,{}^0T(a;t) +\sum_l\frac{\partial}{\partial a_l}\{{}^0\bar{\xi}_{il}(a)\,{}^0T(a;t)\}\Big], \qquad (4\cdot2)$$

where ${}^0T(a;t)$ denotes ${}^0T(a_0,a;t-t_0)$. For an

isolated system, Eq. (4·2) coincides with Fokker-Planck's equation obtained by Green[1]. From (4·2) and (1·6) we obtain the differential equation for the distribution function $f(a, t)$, coarse-grained in time, from which we may, as a special case, derive Kirkwood's[2] equation and Maxwell-Boltzmann's equation[4] *without* use of time averaging procedure of Kirkwood. The detailed investigations of the above-mentioned shall appear in this journal shortly.

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The Quantum-statistical Theory of Transport Phenomena, II*

—On the Theory of Metallic Conductivity—

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The main purpose of the present paper is to derive the Bloch integro-differential equation, upon which the modern electron theories of metallic conductivity are based, analytically from the principles of quantum-statistical mechanics. In the first place, a generalization of the phase-space distribution function of Wigner is obtained by means of the integral transformations which may be, in general, different from the Fourier transformation. And the starting point of the present theory is the equation of motion for the singlet generalized phase-space distribution function, which is derived from the Schrödinger equation for the quantized wave function. The Irving-Zwanzig equation for the Wigner distribution function is immediately obtained from the above-mentioned equation of motion. The generalized phase-space distribution functions of electrons are defined so as to be suitable for Bloch's band scheme, and the equation of motion for the singlet electronic distribution function is obtained. The macroscopic electric current density is defined from the quantized density field with the help of the time-averaging procedure over a microscopic interval. If one uses the customary approximation method together with the postulate of random *a priori* phases, the Bloch integro-differential equation is obtained from the equation of motion for the time-averaged electronic distribution function.

§ 1. Introduction

The consequences of the Bloch integro-differential equation[1) 2)], on which the electron theory of metallic conductivity is based, have been developed by many investigators since Bloch[3)] derived this equation about twenty years ago. It seems, however, that the relation between the Bloch equation and the quantum-mechanical Liouville equation for the density matrix is still obscure, and that there has never been an attempt to derive the basic equation of the electron theory of metallic conduction analytically from the principles of quantum-statistical mechanics. In the present paper, we shall undertake analytical derivation of the Bloch integro-differential equation by the method closely related to the one for analytical derivation of the Uehling-Uhlenbeck equation[4)] in the gas kinetics at very low temperature from the quantum-mechanical Liouville equation, which has been shown in the first paper of this series.[5)]** The quantum-statistical theory of transport developed in this paper is based on the time-averaging procedure over an interval, macroscopically short, but microscopically long in the sense of Kirkwood.[6)]

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* The main contents of the present paper was already published in Japanese (Busseiron Kenkyu, No. 51 (1952), 52) and reported at the meeting of the Kyushu Branch of the Physical Society of Japan held on June 21, 1952.

** We will refer this paper as I.

Although the method developed in I may be applicable to a system of Fermi particles as well as to one of Bose particles, the distribution function (abbreviated d.f.) of Wigner[7] is not suitable for treating electrons moving through the crystal lattice. To obtain an adequate description of electrons in metals, a generalization of the Wigner distribution function is obtained from the quantized density matrix by means of the integral transformation which has the other kernel than the Fourier kernel. The equation of motion for the irreducible generalized phase-space d.f., which serves our theory as a starting point, is derived from the Schrödinger equation for the quantized wave function in the second quantization. This equation of motion can be easily brought into the Irving-Zwanzig equation[8] for the irreducible phase-space d.f. of Wigner (§ 2). Applying the formulation in § 2 to the system of electrons in metals, one defines the generalized phase-space d.f. of electrons so as to be adapted for Bloch's band scheme,[9] and derives the equation of motion for the above-mentioned electronic d.f. In the present formalism, however, the customary one-electron approximation is not used, but the role of the Bloch functions is only to furnish the transformation in the same manner that the plane wave functions give rise to the Wigner d.f. (§ 3).

We define the electric current density from the quantized density field, and derive ordinary expression for the electric current, observable by actual macroscopic experiment, with the help of the time-averaging procedure (§ 4 and 5). The expression of the current density, which has been given by Irving and Zwanzig,[8] is equivalent to the one obtained from this definition, but is unsuitable for our present purpose because their derivation has been based on the ordinary phase-space d.f. of Wigner. The equation of motion for the time-averaged electronic d.f. is derived, and the Kikuchi-Nordheim term due to the external field, the term due to the interaction between electrons and phonons, and the term corresponding to the collisions between electrons are obtained.

For the present, the discussions are restricted to the electrical conductivity. But the present method applies to the thermal conductivity, on which we shall discuss in the later paper of this series.

§ 2. The generalized phase-space distribution function

We shall now consider a system composed of N identical Bose or Fermi particles. And we will denote by q_i the coordinates of the i-th particle and by q^N the set of the coordinates of the system; q will include the spin coordinates as well as the space coordinates. Let $\{\psi_l(q)\}$ be a complete orthonormal set of functions of the coordinates of single particle, and then a complete orthonormal set of symmetrized or antisymmetrized functions of the set of the coordinates of the N particles is given by

$$\Psi^{(\pm)}_{n_1, n_2, \cdots}(q^N) = \det{}^{(\pm)}[\psi_{l_1}(q_1)\cdots\psi_{l_N}(q_N)] \qquad (2\cdot1)$$

with

$$\sum_l n_l = N,$$

where $\det^{(+)}$ and $\det^{(-)}$ indicate a permanent and a determinant, respectively, and n_l is

the number of particles in the state represented by $\psi_\iota(q)$. If annihilation operators a_ι and creation operators a_ι^*, which are Hermitian conjugate to each other and both of which act upon $\Psi^{(\pm)}_{\{n_\iota\}}(q^N)$, are used, the system of identical particles may be described by the quantized wave function[9]:

$$\boldsymbol{\phi}(q)=\sum_\iota a_\iota \boldsymbol{\psi}_\iota(q), \qquad (2\cdot2)$$

which fulfills the canonical commutation rules, and which is so normalized that

$$\boldsymbol{N}=\int \boldsymbol{\phi}^*(q)\boldsymbol{\phi}(q)dq=\sum \boldsymbol{n}_\iota, \quad \boldsymbol{n}_\iota\equiv a_\iota^* a_\iota. \qquad (2\cdot3)$$

The Gibbs ensemble of the quantum-mechanical systems is described by the density matrix,[10] of which the representative in the q-representation is given by

$$\rho^{(N)}(q^N, q'^N; t)=\sum_j w_j \Phi_j^*(q'^N, t)\Phi_j(q^N, t), \qquad (2\cdot4)$$

where $\{\Phi_j(q^N, t)\}$ is a complete orthonormal set of wave functions, and w_j is the probability of the system being in the j-th state ($\sum_j w_j=1$). If the transformation to the N-representation is utilized, the representative of the density matrix becomes

$$\rho^{(N)}(\{n_\iota\}, \{n_\iota'\}; t)=\sum_j w_j C_j^*(\{n_\iota'\}, t)C_j(\{n_\iota\}, t), \qquad (2\cdot5)^*$$

where $C_j(\{n_\iota\}, t)$ is the transformed wave function in the N-representation and is defined by

$$\Phi_j(q^N, t)=\sum_{\substack{\{n_\iota\}\\ [\sum n_\iota=N]}} C_j(\{n_\iota\}, t)\,\Psi^{(\pm)}_{\{n_\iota\}}(q^N), \qquad (2\cdot6)$$

whence the above-mentioned two representatives of the density matrix are related by

$$\rho^{(N)}(q^N, q'^N; t)=\sum_{\substack{\{n_\iota\}}}\sum_{\{n_\iota'\}} \Psi^{(\pm)}_{\{n_\iota\}}(q^N)\rho^{(N)}(\{n_\iota\}, \{n_\iota'\}; t)\,\Psi^{(\pm)*}_{\{n_\iota'\}}(q'^N). \qquad (2\cdot7)$$
$$[\sum n_\iota=\sum n_\iota'=N]$$

The reduced density matrices are defined by

$$\rho^{(n)}(q_1, \cdots, q_n; q_1', \cdots, q_n'; t)$$
$$=\boldsymbol{\phi}^*(q_1')\cdots\boldsymbol{\phi}^*(q_n')\boldsymbol{\phi}(q_n)\cdots\boldsymbol{\phi}(q_1), \quad (N\geq n\geq 1), \qquad (2\cdot8)$$

which is the simplified form, i.e., the expression in the right-hand side should operate on the density matrix $\rho^{(N)}(\overline{q}^N, \overline{q}^N; t)$ or $\rho^{(N)}(\{n_\iota\}, \{n_\iota\}; t)$ according to such a rule that the quantized wave functions $\boldsymbol{\psi}(q)$'s operate only on $\Psi^{(\pm)}_{\{n_\iota\}}(\overline{q}^N)$ of $\rho^{(N)}(\overline{q}^N, \overline{q}^N; t)$ given by (2·7) or only on $C_j(\{n_\iota\}, t)$ of $\rho^{(N)}(\{n_\iota\}, \{n_\iota\}; t)$ given by (2·5) and should be integrated with respect to $\overline{q}^N$ or summed over the sets $\{n_\iota\}$. The expression (2·8) may

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* In the case of the pure state described by the wave function $\Phi(q^N, t)$, we may, in the following, use the density matrices

$$\rho^{(N)}(q^N, q'^N; t)=\Phi^*(q'^N, t)\,\Phi(q^N, t)$$

and

$$\rho^{(N)}(\{n_\iota\}, \{n_\iota'\}; t)=C^*(\{n_\iota'\}, t)\;C(\{n_\iota\}, t)$$

instead of (2·4) and (2·5).

be obtained from the algebraic theory of the density matrix,[11] and alternatively from the tedious but elementary calculation shown in Appendix I. The expression (2·8) with $n=N$ differs from $\rho^{(N)}(q^N, q'^N; t)$ by a constant factor $N!$ and is excluded to avoid confusion. Denoting the space coordinates by $\boldsymbol{x}$ and the spin coordinate by σ, and using the space function $\varphi_k(\boldsymbol{x})$ and the spin function $\chi_\mu(\sigma)$, we have

$$\psi_t(q)=\varphi_k(\boldsymbol{x})\chi_\mu(\sigma), \quad t=(\mu, k) \tag{2·9}$$

$$\left.\begin{aligned}\psi(q)&=\sum_\mu \psi^\mu(\boldsymbol{x})\,\chi_\mu(\sigma),\\ \psi^\mu(\boldsymbol{x})&\equiv\sum_k a_k^\mu \varphi_k(\boldsymbol{x}).\end{aligned}\right\} \tag{2·10}$$

The reduced density matrices in the configurational space are defined by

$$\begin{aligned}&\rho^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; \boldsymbol{x}_1', \cdots, \boldsymbol{x}_n'; t)\\ &\quad=\sum_{\sigma_1}\cdots\sum_{\sigma_n}\delta_{\sigma_1'\sigma_1}\cdots\delta_{\sigma_n'\sigma_n}\rho^{(n)}(q_1, \cdots, q_n; q_1', \cdots, q_n'; t)\\ &\quad=\sum_{\mu_1}\cdots\sum_{\mu_n}\psi^{\mu_1*}(\boldsymbol{x}_1')\cdots\psi^{\mu_n*}(\boldsymbol{x}_n')\,\psi^{\mu_n}(\boldsymbol{x}_n)\cdots\psi^{\mu_1}(\boldsymbol{x}_1),\end{aligned} \tag{2·11}$$

where $\delta_{\sigma'\sigma}$ is Kronecker's delta. The diagonal element of the irreducible density matrix $\rho^{(1)}(\boldsymbol{x}, \boldsymbol{x}'; t)$ gives the particle number density in the configurational space. Here we shall introduce the generalized phase-space d.f. which is defined by

$$\begin{aligned}&g^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; k_1, \cdots, k_n; t)\\ &=\int\cdots\int\varphi_{k_1}^*\Big(\boldsymbol{x}_1-\frac{1}{2}\boldsymbol{y}_1\Big)\cdots\varphi_{k_n}^*\Big(\boldsymbol{x}_n-\frac{1}{2}\boldsymbol{y}_n\Big)\rho^{(n)}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}; \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\Big)\\ &\qquad\times\varphi_{k_1}\Big(\boldsymbol{x}_1+\frac{1}{2}\boldsymbol{y}_1\Big)\cdots\varphi_{k_n}\Big(\boldsymbol{x}_n+\frac{1}{2}\boldsymbol{y}_n\Big)d\boldsymbol{y},\end{aligned} \tag{2·12}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ stand for the sets of the space coordinates $(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ and $(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n)$, respectively. And then we obtain immediately from (2·12)

$$\left.\begin{aligned}&\sum_{k_1}\cdots\sum_{k_n}g^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; k_1, \cdots, k_n; t)=\rho^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; t)\\ &\text{and}\\ &\int\cdots\int d\boldsymbol{x}_1\cdots d\boldsymbol{x}_n\, g^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; k_1, \cdots, k_n; t)=\boldsymbol{n}_{k_1}\cdots\boldsymbol{n}_{k_n},\end{aligned}\right\} \tag{2·13}$$

where $n_k=\sum_\mu n_k^\mu$, and no two of k_1 $\cdots$ and k_n are equal to each other. The first and second expressions of (2·13) give the multiplet probability densities of order n in the configurational space and in the k-space, respectively. If we choose the plane waves, $\varphi_k(\boldsymbol{x})=h^{-3/2}\exp(-\boldsymbol{p}\cdot\boldsymbol{x}/i\hbar)$, as the complete orthonormal set, we have

$$\begin{aligned}&g^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; \boldsymbol{p}_1, \cdots, \boldsymbol{p}_n; t)\\ &\quad=h^{-3n}\int^{(n)}\cdots\int e^{-\boldsymbol{p}\cdot\boldsymbol{y}/i\hbar}\rho^{(n)}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}; \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\Big)d\boldsymbol{y}\end{aligned}$$

$$=[N!/(N-n)]f^{(n)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n; \boldsymbol{p}_1, \cdots, \boldsymbol{p}_n; t), \qquad (2\cdot14)$$

where $\boldsymbol{p}\cdot\boldsymbol{y}=\sum_{i=1}^{n}\boldsymbol{p}_i\cdot\boldsymbol{x}_i$. $f^{(n)}$ is nothing other than the phase-space d.f. first introduced by Wigner[7] and extensively investigated by Moyal[12] for a pure state and applied to quantum hydrodynamics by Irving and Zwanzig.[5] From the above-mentioned, the function defined as (2·12) may be called the generalized phase-space d.f.

The equation of motion for $\rho^{(n)}$ is derived from the quantum-mechanical Liouville equation for the density matrix[13] or from the Schrödinger equation for the quantized wave function. Then, we may obtain the equation of motion for $g^{(n)}$ by means of the transformation (2·12). The singlet generalized phase-space d.f., which gives the particle number density in the $\boldsymbol{x}-k$ space, is

$$g(\boldsymbol{x}, k, t)=\int d\boldsymbol{y}\,\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right), \qquad (2\cdot15)^*$$

in which we are mainly interested for the present. Now we shall search for the equation of motion for $g(\boldsymbol{x}, k, t)$. If the Hamiltonian of the system does not contain the spin variables, the Schrödinger equation for the quantized wave function may be written as[9]

$$i\hbar\,\dot{\boldsymbol{\phi}}^{\mu}=\{-(\hbar^2/2m)\Delta+\phi(\boldsymbol{x})+G(\boldsymbol{x})\}\boldsymbol{\phi}^{\mu} \qquad (2\cdot16)$$

with

$$G(\boldsymbol{x})=\sum_{\nu}\int d\bar{\boldsymbol{x}}\,\boldsymbol{\phi}^{\nu*}(\bar{\boldsymbol{x}})\,u(\boldsymbol{x}, \bar{\boldsymbol{x}})\,\boldsymbol{\phi}^{\nu}(\bar{\boldsymbol{x}}),$$

where ϕ and u are the potential energy of the external field and the potential between particles, respectively. Thus, the equation of motion for the irreducible density matrix, $\rho(\boldsymbol{x}, \boldsymbol{x}'; t)=\sum_{\mu}\boldsymbol{\phi}^{\mu*}(\boldsymbol{x}')\,\boldsymbol{\phi}^{\mu}(\boldsymbol{x})$, is obtained as

$$\begin{aligned}
&\partial\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)/\partial t\\
&\quad-(\hbar/mi)\nabla_x\cdot\nabla_y\,\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)\\
&\quad+(i/\hbar)\left\{\phi\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)-\phi\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\right\}\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)\\
&\quad=(1/i\hbar)\int d\bar{\boldsymbol{y}}\left\{u\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)-u\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)\right\}\\
&\qquad\times\rho^{(2)}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{y}; \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}; t\right). \qquad (2\cdot17)
\end{aligned}$$

According to the transformation law (2·15), eq. (2·17) is brought into

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* Henceforth, the index(1) is omitted for simplicity's sake.

$$\partial g(\boldsymbol{x}, \boldsymbol{k}, t)/\partial t$$

$$-(\hbar/mi)\int d\boldsymbol{y}\,\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\nabla_y\cdot\nabla_x\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)$$

$$+(i/\hbar)\int d\boldsymbol{y}\,\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\left\{\phi\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)-\phi\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\right\}$$

$$\times\rho\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)$$

$$=(1/i\hbar)\int d\boldsymbol{y}\int d\bar{\boldsymbol{y}}\left\{u\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)-u\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)\right\}$$

$$\times\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\rho^{(2)}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}; \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}; t\right), \quad (2\cdot18)$$

which serves the present theory as the fundamental equation. If one adopts $\varphi_k=h^{-3/2}\exp(-\boldsymbol{p}\cdot\boldsymbol{x}/i\hbar)$ and expands ϕ in the power series of $(\boldsymbol{y}/2)$, the equation of motion (2·18) reduces to the one for the Wigner d.f.:

$$\partial f^{(1)}/\partial t+(\boldsymbol{p}/m)\cdot\nabla_x f^{(1)}-(2/\hbar)\sin\left[(\hbar/2)\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_p\right]\phi(\boldsymbol{x})f^{(1)}$$

$$=(1/i\hbar N h^3)\int d\boldsymbol{y}\int d\bar{\boldsymbol{y}}\left\{u\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)-u\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}\right)\right\}$$

$$\times e^{-\boldsymbol{p}\cdot\boldsymbol{y}/i\hbar}\rho^{(2)}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}; \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, \bar{\boldsymbol{y}}; t\right), \quad (2\cdot19)$$

where ∇_x operates only on the potential energy. This equation, which is equivalent to the one obtained by Irving and Zwanzig[8] from the Liouville equation for the density matrix in the q-representation and is an alternative form of eq. (3·7) of I, is the fundamental equation for the quantum-statistics of transport in gases.

Eqs. (2·11) (2·19) have the same forms irrespective of statistics of particles. It is, however, more convenient for the purpose of deriving the Bloch integral equation from the quantum-mechanical Liouville equation to adopt, instead of $\rho(\boldsymbol{x}, \boldsymbol{x}'; t)$ and $g(\boldsymbol{x}, \boldsymbol{k}, t)$, the following functions:

$$\rho^{\mu}(\boldsymbol{x}, \boldsymbol{x}'; t)=\phi^{\mu}(\boldsymbol{x}')\,\phi^{\mu}(\boldsymbol{x}) \quad (2\cdot20)$$

and

$$g^{\mu}(\boldsymbol{x}, \boldsymbol{k}, t)=\int d\boldsymbol{y}\,\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\rho^{\mu}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right). \quad (2\cdot21)$$

Then, the change of g^{μ} with time is determined by

$$\partial g^{\mu}(\boldsymbol{x}, \boldsymbol{k}, t)/\partial t$$

$$-(\hbar/mi)\int d\boldsymbol{y}\,\varphi_k^*\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)\varphi_k\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\nabla_y\cdot\nabla_x\rho^{\mu}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\right)$$

$$+(i/\hbar)\int d\boldsymbol{y}\,\varphi_k^*\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\Big)\varphi_k\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\Big)\Big\{\phi\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\Big)-\phi\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\Big)\Big\}$$
$$\times\rho^{\mu}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y},\,\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\,;\,t\Big)$$
$$=(1/i\hbar)\iint d\boldsymbol{y}\,d\bar{\boldsymbol{y}}\Big\{u\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y},\,\bar{\boldsymbol{y}}\Big)-u\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y},\,\bar{\boldsymbol{y}}\Big)\Big\}$$
$$\times\varphi_k^*\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\Big)\varphi_k\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\Big)\sum_{\nu}\phi^{\mu*}\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\Big)\varphi^{\nu*}(\bar{\boldsymbol{y}})\,\phi^{\nu}(\bar{\boldsymbol{y}})\,\phi^{\mu}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\Big), \tag{2·22}$$

which will be used in the following sections instead of (2·18).

§ 3. The states of system of electrons in metal

The existing theories on metallic conductivity and the cohesive energy of metals are usually based upon the one-electron approximation for the states of electrons. According to Bloch's band scheme,[2) 3)] the one-electron eigenfunctions, under the Born-von Kármán boundary conditions,* are determined by the Schrödinger equation:

$$\{-(\hbar^2/2m)\Delta+V(\boldsymbol{x})\}\varphi_k(\boldsymbol{x})=E_k\varphi_k(\boldsymbol{x}), \tag{3·1}$$

where the potential $V(\boldsymbol{x})$ consists of the electrostatic potential due to all remaining electrons as well as the total ion-core potential, having the periodicity of the lattice. The function $\varphi_k(\boldsymbol{x})$ has the form

$$\varphi_{nk}(\boldsymbol{x})=e^{i\boldsymbol{k}\cdot\boldsymbol{x}}u_{nk}(\boldsymbol{x}),\quad k\equiv(n,\,\boldsymbol{k}), \tag{3·2}$$

to which we refer as the Bloch function[3)]: $u_{nk}(\boldsymbol{x})$ has the translational periodicity of the lattice, and $\boldsymbol{k}$ and n are, respectively, the reduced wave vector and the number of the zone. As was reasonably reformulated by Slater[14)] recently, eq. (3·1) may be regarded as the Hartree-Fock equation for the system of myriad electrons in metal. And, according to Koopmann's theorem,[2)] we have

$$E\{n_k=0,\,n_l=1\}-E\{n_k=1,\,n_l=0\}=E_l-E_k \tag{3·3}$$

with

$$E\{n_k^{\mu}\}=\langle\{n_k^{\mu}\}\,|\,\boldsymbol{H}_e\,|\,\{n_k^{\mu}\}\rangle=\int\Psi^{(-)*}_{\{n_k^{\mu}\}}(q^N)\,\boldsymbol{H}_e\,\Psi^{(-)}_{\{n_k^{\mu}\}}(q^N)\,dq^N,$$

where $\boldsymbol{H}_e$ is the Hamiltonian of the electrons, and $E\{n_k^{\mu}\}$ is the energy of the state represented by $\Psi^{(-)}_{\{n_k^{\mu}\}}(q^N)$ which is given by (2·1) and $\psi_l(q)=\varphi_k(\boldsymbol{x})\chi_{\mu}(\sigma)$ ($\varphi_k(\boldsymbol{x})$ is the Bloch function!). Eq. (3·3) gives the energy required for the transition $k\to l$ in the one-electron energy diagram of the band scheme.

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* Usually, the infinite crystal is divided into cubes of the same dimension to which we refer as elementary regions. The Born-von Kármán condition is imposed on each of the cubes.

In order to define the electronic d.f. in accordance with Bloch's band scheme, we choose the Bloch function (3·2) as $\varphi_k(x)$ in (2·12), obtaining the singlet electronic d.f. :

$$g^{\mu}(\boldsymbol{x}, n\boldsymbol{k}, t)=\int d\boldsymbol{y}\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\, u^{*}_{n\boldsymbol{k}}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\Big)u_{n\boldsymbol{k}}\Big(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\Big)\rho^{\mu}\Big(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}; t\Big)\cdot \tag{3·4}$$

In the subsequent sections, we shall investigate the equation of motion for $g^{\mu}(\boldsymbol{x}, n\boldsymbol{k}, t)$, which is obtained from (2·22), from quantum-statistical point of view, and shall show how the Bloch integro-differential equation is derived from this equation of motion. It should, however, be kept in mind that the one-electron approximation is not used in the present formalism, and that we choose the Bloch functions as the complete orthonormal set of functions, in terms of which the wave functions of the system of electrons are expanded, instead of the plane waves in the case of gas kinetics.

The electronic system is perturbed by the lattice vibration. According to Bloch's theory, the perturbing potential can be represented as[1)]

$$\boldsymbol{H}_1=\sum_{i=1}^{N} v(\boldsymbol{x}_i), \quad v(\boldsymbol{x})=-\boldsymbol{u}(\boldsymbol{x})\cdot\nabla V(\boldsymbol{x}) \tag{3·5}$$

with

$$\boldsymbol{u}(\boldsymbol{x})=G^{-1/2}\sum_{\boldsymbol{s}}\sum_{j=1}^{3}(\hbar/2M\omega_{\boldsymbol{s}j})^{1/2}\,\boldsymbol{e}_{\boldsymbol{s}j}(a_{\boldsymbol{s}j}\,e^{i\boldsymbol{s}\cdot\boldsymbol{x}}+a^{*}_{\boldsymbol{s}j}\,e^{-i\boldsymbol{s}\cdot\boldsymbol{x}}), \tag{3·6}$$

where $a^{*}_{\boldsymbol{s}j}$ and $a_{\boldsymbol{s}j}$, which are Hermitian conjugate to each other and both of which act upon the wave functional[15)] $\Psi^{(+)}_{\{N_s\}}(\alpha)$ of system of phonons, are creation and annihilation operators of the phonons with the wave vector $\boldsymbol{s}$ and the direction of polarization $\boldsymbol{e}_{\boldsymbol{s}j}$, respectively. G^3 is the number of the unit cell in the elementary region,* and M and $\omega_{\boldsymbol{s}j}$ are the mass of ion-cores and the frequency of the phonon $\boldsymbol{s}_j$, respectively. Even if Bardeen's[16)] perturbing potential obtained from a self-consistent field method is used, no alteration is necessary in the following formalism.

§ 4. The time average over a microscopic interval of time

In deriving the macroscopic law from the microscopic law, we may use such a averaging procedure as time averaging over a finite interval of time and space averaging over a certain region of ordinary space. We shall, here, attempt to average the equation (2·22) over a microscopic interval of time in order to obtain the secular variation of the macroscopically observable properties of the system, in the same manner as in I and as in Kirkwood's theory on classical statistical mechanics of transport processes.[6)] The space averaging procedure is unnecessary here, because we confine ourselves to the case of uniform electric current.

* cf. footnote, p. 479.

Since the interaction between electrons and lattice vibration is expressed by (3·5) and (3·6), it is more convenient for taking into consideration the system of phonons to choose, as the base of functions of N-representation,

$$\Psi^{(-)}_{\{n^{\mu}_{k}\}}(q^{N})\,\Psi^{(+)}_{\{N_s\}}(a) \tag{4·1}$$

instead of $\Psi^{(-)}_{\{n^{\mu}_{k}\}}(q^{N})$. Then, the Hamiltonian of the whole crystal may be written as

$$\boldsymbol{H}=\boldsymbol{H}_{e}+\boldsymbol{H}_{ph}+\boldsymbol{H}_{I} \tag{4·2}$$

with

$$\left.\begin{aligned}
\boldsymbol{H}_{e}&=\sum_{\mu}\sum_{k,\bar{k}}\langle k|A|\bar{k}\rangle a^{\mu *}_{k}a^{\mu}_{\bar{k}}+\frac{1}{2}\sum_{\mu,\nu}\sum_{k,l,\bar{k},\bar{l}}\langle kl|u|\bar{k}\,\bar{l}\rangle a^{\mu *}_{k}a^{\nu *}_{l}a^{\nu}_{\bar{l}}a^{\mu}_{\bar{k}}\\
\boldsymbol{H}_{ph}&=\sum_{s}\left(N_s+\frac{1}{2}\right)\hbar\omega_s\\
\boldsymbol{H}_{I}&=\sum_{\mu}\sum_{k,\bar{k}}\langle k|v|\bar{k}\rangle a^{\mu *}_{k}a^{\mu}_{\bar{k}}\,,
\end{aligned}\right\} \tag{4·3}$$

where

$$A=-(\hbar^2/2m)\Delta+(\text{a potential energy due to external field})$$

and

$$\langle kl|u|\bar{k}\,\bar{l}\rangle=\iint\varphi_{k}^{*}(\boldsymbol{x}_1)\varphi_{l}^{*}(\boldsymbol{x}_2)u(\boldsymbol{x}_1,\boldsymbol{x}_2)\varphi_{\bar{k}}(\boldsymbol{x}_1)\varphi_{\bar{l}}(\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,, \tag{4·4}$$

and $\boldsymbol{H}_{ph}$ is the Hamiltonian of the phonons, the interaction between phonons being neglected. If $\langle k|v|\bar{k}\rangle$'s are actually calculated (c.f. (7·3)), the Hamiltonian $\boldsymbol{H}$ is brought into the complete quantized form. The set of integers $(\{n^{\mu}_{k}\},\ \{N_s\})$ shall be denoted by $(\varkappa)$ for simplicity's sake, in the following.

The time dependence of the d.f., (2·12), and the transport quantities, such as electric and heat current, can be represented in terms of the density matrix $\rho(\varkappa,\varkappa';t)$, which is determined by the Liouville equation. The change, coarse graining in time, of $\rho(\varkappa,\varkappa';t)$ with time may be given by

$$\bar{\rho}(\varkappa,\varkappa';t)=(1/\tau)\int_{0}^{\tau}\rho(\varkappa,\varkappa';t+\nu)\,d\nu\,, \tag{4·5}$$

where τ is the interval of the order of the magnitude of the mean life of the states corresponding to

$$\Psi^{(-)}_{\{n^{\mu}_{k}\}}(q_N)\,\Psi^{(+)}_{\{N_s\}}(a)\exp(E_{\varkappa}\cdot t/i\hbar), \tag{4·6}$$

in which

$$E_{\varkappa}=\langle\varkappa|\boldsymbol{H}|\varkappa\rangle=E\{n^{\mu}_{k}\}+\sum_{s}\left(N_s+\frac{1}{2}\right)\hbar\omega_s C\,, \tag{4·7}$$

and which changes with time by the perturbation due to the interactions between electrons and phonons and between electrons, i.e., the non-diagonal part of the Hamiltonian (4·2).

By means of the similar method to the customary theory of the time-dependent perturbation theory, we obtain, for the transformed wave function $C_j(x, t)$ of the crystal system,

$$C_j(x, t+\nu) = C_j(x, t)\exp(E_x\cdot\nu/i\hbar) - \Big[\sum_{x'(\neq x)} \langle \{n_i\}|\boldsymbol{H}_e|\{n_i'\}\rangle \delta(\{N_s'\}-\{N_s\}) + \sum_{x'}\langle x|H_I|x'\rangle\Big] \times \frac{\exp(E_{x'}\cdot\nu/i\hbar)-\exp(E_x\cdot\nu/i\hbar)}{E_x-E_{x'}} C_j(x', t) \qquad (4\cdot8)^*$$

with $\tau > \nu > 0$. Thus, in the same manner as in I, the time averaged values of "non-diagonal" elements of the density matrix in our N-representation are given by

$$\bar{\rho}(x, x'; t) = i\hbar\Big[\langle \{n_i\}|\boldsymbol{H}_e|\{n_i'\}\rangle \delta(\{N_s'\}-\{N_s\}) + \langle x|\boldsymbol{H}_I|x'\rangle\Big] \times \{\omega(E_x-E_{x'})/\tau\}\{\rho(x, x; t)-\rho(x', x'; t)\}, \qquad (4\cdot9)$$

where

$$\omega(x) = (1-e^{x\tau/i\hbar})/x^2+\tau/i\hbar x = \omega^*(-x) = \frac{1}{2}\{\Omega(x)+i\Theta(x)\} \qquad (4\cdot10)$$

with

$$\Omega(x) = \omega(x)+\omega^*(x) = 4\sin^2(x\tau/2\hbar)/x^2 = (2\pi\tau/\hbar)\,\delta(x) \qquad (4\cdot11)^{**}$$

$$\Theta(x) = i^{-1}\{\omega(x)-\omega^*(x)\} = (2\tau/\hbar x)\{\sin(x\tau/\hbar)/(x\tau/\hbar)-1\}, \qquad (4\cdot12)$$

where the postulate of random *a priori* phases,[17) 18)] which may correspond to the approximate character of our measurement or the disturbance of our object in the macroscopic observation, is used at the initial time $\nu=0$ in the integrand of the expression (4·5). From (4·12), it is easily seen that Θ/τ is the second order infinitesimal quantity with respect to τ and, therefore, may be neglected compared with Ω/τ. Thus we obtain

$$(1/\tau)\omega(x) = (1/2\tau)\Omega(x) = (\pi/\hbar)\,\delta(x). \qquad (4\cdot13)$$

The relation (4·13) will be used throughout the present paper, while the imaginary part of ω will play an important role in the quantum-statistical formalism of the optical property of solids, as will be seen in the next paper of this series. Since the present consideration on the change of state with time due to the interaction between particles is based on the similar method to the ordinary time-dependent perturbation method, the results of the present section are valid only in the case of gases.

§ 5. The macroscopic electric current density

Let us consider the case where the uniform electrostatic field is applied to a metal. For the system of electrons, we obtain the equation of continuity, from eq. (2·17),

* $\delta(\{N_s'\}-\{N_s\}) = \begin{cases}1, & \text{if the set } \{N_s'\} \text{ is identical with the set } \{N_s\}\\ 0, & \text{otherwise}\end{cases}$

** $\delta(x)$ is Dirac's δ function.

$$\partial\rho(\boldsymbol{x}, t)/\partial t+\nabla\cdot\boldsymbol{j}(x, t)=0\,, \tag{5·1}$$

where

$$\boldsymbol{j}(\boldsymbol{x}, t)=(e\hbar/2mi)\sum_{\mu}(\phi^{\mu *}\nabla\phi^{\mu}-\nabla\phi^{\mu *}\phi^{\mu}), \tag{5·2}$$

$\rho(\boldsymbol{x}, t)=e\rho(\boldsymbol{x}, \boldsymbol{x}, t)$ (e is electronic charge) being the charge density of electrons and $\boldsymbol{j}(\boldsymbol{x}, t)$ the corresponding current density. Using the Wigner phase-space d.f., (5·2) may be brought into the alternative form

$$\boldsymbol{j}(\boldsymbol{x}, t)=\iint\left[\frac{e}{m}\sum_{i=1}^{N}\bar{\boldsymbol{p}}_i\delta(\bar{\boldsymbol{x}}_i-\boldsymbol{x})\right]f^{(N)}(\bar{X}, \bar{P}, t)d\bar{X}\,d\bar{P},$$

which is the expression for the current density used by Irving and Zwanzig in quantum hydrodynamics. Then, the macroscopic electric current density in the metal may be given by

$$\boldsymbol{J}=\left[(1/\tau)\int_{0}^{\tau}d\nu\right]\left[(1/D)\int dx\right]\boldsymbol{j}(\boldsymbol{x}, t+\nu), \tag{5·3}$$

where τ is same as in the expression (4·5) and D is the volume of the elementary region.* Thus, using (5·2), (2·10), (4·9), and (4·13), the ordinary expression for the electric current density is obtained (cf. Appendix II) :

$$\boldsymbol{J}=(1/D)\sum_{\mu}\sum_{n,\,\boldsymbol{k}}\bar{g}^{\mu}(n\boldsymbol{k}, t)\,e\,\langle\boldsymbol{v}\rangle_{n\boldsymbol{k}}\,, \tag{5·4}$$

where

$$\langle\boldsymbol{v}\rangle_{n\boldsymbol{k}}\equiv\int d\boldsymbol{x}\,\varphi_{n\boldsymbol{k}}^{*}(\hbar/mi)\nabla_{x}\varphi_{n\boldsymbol{k}}=(1/\hbar)\nabla_{\boldsymbol{k}}E_{n}(\boldsymbol{k}) \tag{5·5}$$

and

$$g^{\mu}(k, t)\equiv\sum_{\boldsymbol{\varkappa}}n_{k}^{\mu}\,\bar{\rho}(\boldsymbol{\varkappa}, \boldsymbol{\varkappa}\,; t)=\int\bar{g}^{\mu}(\boldsymbol{x}, k, t)d\boldsymbol{x}\,. \tag{5·6}$$

Because of $E_{n}(\boldsymbol{k})=E_{n}(-\boldsymbol{k})$, we have $\boldsymbol{J}=0$, if

$$\bar{g}^{\mu}(n, \boldsymbol{k}, t)=\bar{g}^{\mu}(n, -\boldsymbol{k}, t), \tag{5·7}$$

which is correct for metals in thermal equilibrium except for superconducting metals.[19]

Carrying out time averaging procedure over the interval τ and integrating the electronic d.f., (3·4), and eq. (2·22) over the elementary region, we obtain the equation of motion for $\bar{g}^{\mu}(n\boldsymbol{k}, t)$:

$$\partial\bar{g}^{\mu}(n\boldsymbol{k}, t)/\partial t+S_{kin}^{\mu}+S_{e}^{\mu}+S_{ph}^{\mu}=S_{e}^{\mu} \tag{5·8}$$

with

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* cf. footnote, p. 479.

$$S^{\mu}_{kin}(n\boldsymbol{k}, t) = (\hbar/mi)\iint d\boldsymbol{x}d\boldsymbol{y}\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\nabla_{\boldsymbol{x}}\left[u^{*}_{n\boldsymbol{k}}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)u_{n\boldsymbol{k}}\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)\right]$$
$$\cdot\nabla_{\boldsymbol{y}}\bar{\rho}^{\mu}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, t\right), \tag{5·9}$$

$$S_{e}{}^{\mu}(n\boldsymbol{k}, t) = (\boldsymbol{F}/\hbar)\iint d\boldsymbol{x}d\boldsymbol{y}(\nabla_{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{y}})u^{*}_{n\boldsymbol{k}}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\right)u_{n\boldsymbol{k}}\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right)$$
$$\times\bar{\rho}^{\mu}\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}, \boldsymbol{x}+\frac{1}{2}\boldsymbol{y}, t\right), \tag{5·10}$$

$$S^{\mu}_{ph}(n\boldsymbol{k}, t) = (1/\hbar)\iint d\boldsymbol{x}d\boldsymbol{y}\,\varphi^{*}_{n\boldsymbol{k}}(\boldsymbol{y})\varphi_{n\boldsymbol{k}}(\boldsymbol{x})\{v(\boldsymbol{y})-v(\boldsymbol{x})\}\bar{\rho}(\boldsymbol{y}, \boldsymbol{x}, t) \tag{5·11}$$

and

$$S_{c}{}^{\mu}(n\boldsymbol{k}, t) = (1/i\hbar)\iint d\boldsymbol{x}\,d\boldsymbol{y}\int d\bar{\boldsymbol{y}}\,\varphi^{*}_{n\boldsymbol{k}}(\boldsymbol{y})\varphi_{n\boldsymbol{k}}(\boldsymbol{x})\{u(\boldsymbol{y}, \bar{\boldsymbol{y}})-u(\boldsymbol{x}, \bar{\boldsymbol{y}})\}$$
$$\times\sum_{\nu}\overline{\boldsymbol{\phi}^{\mu *}(\boldsymbol{x})\boldsymbol{\phi}^{\nu *}(\bar{\boldsymbol{y}})\boldsymbol{\phi}^{\nu}(\bar{\boldsymbol{y}})\boldsymbol{\phi}^{\mu}(\boldsymbol{y})}, \tag{5·12}$$

where the potential of the external field for the electrons is $\phi(\boldsymbol{x}) = -\boldsymbol{F}\cdot\boldsymbol{x}+v(\boldsymbol{x})$, and bars indicate time averaging. The calculation of each term of (5·8) will be performed in the following sections.

§ 6. The Kikuchi-Nordheim term

In the present section, it will be shown that S^{μ}_{kin} vanishes and $S_{e}{}^{\mu}$ gives the Kikuchi-Nordheim term. Using (2·20) and (2·10), we obtain

$$S^{\mu}_{kin}(n\boldsymbol{k}, t)$$
$$= (i\hbar/2m)\sum_{l,\boldsymbol{p}}\{\overline{a^{\mu}_{n\boldsymbol{k}}\, a^{\mu}_{l\boldsymbol{p}}}\int e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{x}}\nabla u^{*}_{n\boldsymbol{k}}\cdot\nabla u_{l\boldsymbol{p}}\,d\boldsymbol{x}-\text{c.c.}\}$$
$$-(\hbar/2m)\sum_{l,\boldsymbol{p}}(\boldsymbol{k}+\boldsymbol{p})\cdot\{\overline{a^{\mu}_{n\boldsymbol{k}}\, a^{\mu}_{l\boldsymbol{p}}}\int e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{x}}(\nabla u^{*}_{n\boldsymbol{k}})u_{l\boldsymbol{p}}\,d\boldsymbol{x}+\text{c.c.}\}.$$

Since $u_{n\boldsymbol{k}}(\boldsymbol{x})$'s have the periodicity of the lattice, the integral over an elementary region may be replaced by a sum of the integrals over the cells:

$$S^{\mu}_{kin}(n\boldsymbol{k}, t) = -(G^{3}/i\hbar)$$
$$\times\sum_{l(\neq n)}\overline{a^{\mu}_{n\boldsymbol{k}}\, a^{\mu}_{l\boldsymbol{k}}}\int_{D_0}u^{*}_{n\boldsymbol{k}}\{-(\hbar^{2}/2m)\nabla^{2}+(\hbar^{2}\boldsymbol{k}/mi)\cdot\nabla\}u_{l\boldsymbol{k}}\,d\boldsymbol{x}+\text{c.c.}$$

because of

$$\sum_{\boldsymbol{a}=(0)}^{(G-1)}e^{i(\boldsymbol{q}\cdot\boldsymbol{a})}=\begin{cases}G^{3}, & \text{if } \boldsymbol{q}=2\pi\boldsymbol{g}\\ 0, & \text{otherwise},\end{cases} \tag{6·1}$$

where D_0 represents integration over a unit cell, and $\boldsymbol{a}$ and $\boldsymbol{g}$ are lattice and reciprocal-lattice vectors, respectively. Since $\overline{a^{\mu *}_{n\boldsymbol{k}}\, a^{\mu}_{l\boldsymbol{k}}}\,(l\neq n)$ vanishes as shown in Appendix II, we obtain

$$S^{\mu}_{kin}(n\boldsymbol{k}, t)=0\,. \tag{6·2}$$

Using (5·6) and (3·4), we may express S_e^{μ} given by (5·10) in the form

$$\begin{aligned} S_e^{\mu}(n\boldsymbol{k}, t)&=(\boldsymbol{F}/\hbar)\cdot\nabla_k \bar{g}^{\mu}(n\boldsymbol{k}, t)\\ &\quad-(\boldsymbol{F}/\hbar)\cdot\iint e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}\bar{\rho}^{\mu}(\boldsymbol{x}, \boldsymbol{y}, t)\nabla_k[u^*_{nk}(\boldsymbol{x})u_{nk}(\boldsymbol{x})]\,d\boldsymbol{x}\,d\boldsymbol{y}\\ &=(\boldsymbol{F}/\hbar)\cdot\nabla_k \bar{g}^{\mu}(n\boldsymbol{k}, t), \end{aligned} \tag{6·3}$$

where the last line is obtained by considering that $\overline{a^{\mu *}_{nk}\, a^{\mu}_{lk}}$ vanishes if $l\neq n$. The above derivation of the Kikuchi-Nordheim term[20] is quite rigorous. Kikuchi and Nordheim have shown first that this term always obeys Pauli's exclusion principle if it obeys initially this principle, but here we need not prove particularly this conclusion, because (6·3) has been derived directly from the Schrödinger equation for the system of Fermi particles.

§ 7. The term due to the interaction with phonons

The S^{μ}_{ph} given by (5·11) may rewritten as

$$S^{\mu}_{ph}(n\boldsymbol{k}, t)=S^{\mu}(n\boldsymbol{k}, t)+S^{\mu\cdot k}(n\boldsymbol{k}, t), \tag{7·1}$$

where

$$S^{\mu}(n\boldsymbol{k}, t)=(i/\hbar)\sum_{l,\boldsymbol{p}}\sum_{\varkappa}\overline{a^{\mu}_{nk}a^{\mu}_{l\boldsymbol{p}}\langle n\boldsymbol{k}|v|l\boldsymbol{p}\rangle\,\rho(\varkappa, \varkappa, t)} \tag{7·2}$$

with

$$\begin{aligned} \langle n\boldsymbol{k}|v|l\boldsymbol{p}\rangle&=-G^{-3/2}\sum_{s}\sum_{j}\sum_{\boldsymbol{g}}(\hbar/2M\omega_{sj})^{1/2}K^{sj}_{nk,\,l\boldsymbol{p}}\\ &\quad\times[a_{sj}\delta(\boldsymbol{k}, \boldsymbol{p}+\boldsymbol{s}-2\pi\boldsymbol{g})+a^*_{sj}\delta(\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{s}-2\pi\boldsymbol{g})] \end{aligned} \tag{7·3}$$

$$K^{sj}_{nk,\,l\boldsymbol{p}}\equiv\boldsymbol{e}_{sj}\cdot\int_{P_0}u^*_{nk}(\boldsymbol{y})[\nabla V(\boldsymbol{y})]u_{l\boldsymbol{p}}(\boldsymbol{y})\,d\boldsymbol{y}\,, \tag{7·4}$$

(3·5) and (6·1) being used. In (7·3), the reciprocal-lattice vectors $\boldsymbol{g}\neq 0$ correspond to Peierls' umklapp process. Substituting (7·3) into (7·2), we obtain

$$\begin{aligned} S^{\mu}(n\boldsymbol{k}, t)&=(1/i\hbar)\sum_{l,\boldsymbol{p}}\sum_{\varkappa}\theta^{\mu}_{nk}\theta^{\mu}_{l\boldsymbol{p}}(1-n^{\mu}_{nk})n^{\mu}_{l\boldsymbol{p}}\\ &\quad\times\sum_{s}\sum_{j}\sum_{\boldsymbol{g}}(\hbar/2M\omega_{sj}G^3)^{1/2}K^{sj}_{nk,\,l\boldsymbol{p}}\\ &\quad\times[\delta(\boldsymbol{k}, \boldsymbol{p}+\boldsymbol{s}-2\pi\boldsymbol{g})\sqrt{N_{sj}}\,\rho(\varkappa;\cdots 1-n^{\mu}_{nk}, 1-n^{\mu}_{l\boldsymbol{p}}, N_{sj}-1\cdots;t)\\ &\quad+\delta(\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{s}-2\pi\boldsymbol{g})\sqrt{(N_{sj}+1)}\,\rho(\varkappa;\cdots 1-n^{\mu}_{nk}, 1-n^{\mu}_{l\boldsymbol{p}}, N_{sj}+1\cdots;t)], \end{aligned} \tag{7·5}$$

where θ represents the sign function which appears in the second quantization according to Fermi-Dirac statistics.[15] From (4·9), (4·13), (4·7), (3·3) and (4·3), we obtain

$$\begin{aligned}
&\bar{\rho}(\boldsymbol{x};\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}\mp 1\cdots;t)\\
&\quad=i\hbar\langle\boldsymbol{x}|\boldsymbol{H}_I|\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}\mp 1\cdots\rangle\\
&\quad\times(1/2\tau)\,\Omega(\pm\hbar\omega_{sj}+E_l(\boldsymbol{p})-E_n(\boldsymbol{k}))\\
&\quad\times[\rho(\boldsymbol{x},\boldsymbol{x};t)-\rho(\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}\mp 1\cdots;\ \text{〃}\ ;t)]\,, \qquad (7\cdot 6)\\
&\langle\boldsymbol{x}|\boldsymbol{H}_I|\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}\mp 1\cdots\rangle\\
&\quad=\theta^{\mu}_{lp}\theta^{\mu}_{nk}n^{\mu}_{lp}(1-n^{\mu}_{nk})\langle N_{sj}|\langle l\boldsymbol{p}|v|n\boldsymbol{k}\rangle|N_{sj}\mp 1\rangle\,, \qquad (7\cdot 7)
\end{aligned}$$

if $\bar{n}^{\mu}_{nk}=0$ and $n^{\mu}_{lp}=1$, and, from (7·3),

$$\begin{aligned}
&\langle N_{sj}|\langle l\boldsymbol{p}|v|n\boldsymbol{k}\rangle|N_{sj}\mp 1\rangle\\
&\quad=-(\hbar/2M\omega_{sj}G^3)^{1/2}K^{sj*}_{nk,lp}\begin{cases}\sqrt{N_{sj}}\,\delta(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{s}-2\pi\boldsymbol{g})\\ \sqrt{N_{sj}+1}\,\delta(\boldsymbol{k},\boldsymbol{p}-\boldsymbol{s}-2\pi\boldsymbol{g}).\end{cases} \qquad (7\cdot 8)
\end{aligned}$$

Thus, substituting (7·6), (7·7) and (7·8) into (7·5), we have

$$\begin{aligned}
S^{\mu}(n\boldsymbol{k},t)=&-\frac{1}{2}\sum_{l,p}{}'\sum_{\boldsymbol{x}}{}'(1-\bar{n}^{\mu}_{nk})n^{\mu}_{lp}\sum_{s}{}'\sum_{j}{}'\sum_{g}{}'(\hbar/2M\omega_{sj}G^3)|K^{sj}_{nk,lp}|^2\\
&\times[N_{sj}\{\Omega(\hbar\omega_{sj}+E_{lp}-E_{nk})/\tau\}\,\delta(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{s}+2\pi\boldsymbol{g})\\
&\times\{\rho(\boldsymbol{x};\boldsymbol{x};t)-\rho(\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}-1\cdots;\ \text{〃}\ ;t)\}\\
&+(N_{sj}+1)\{\Omega(-\hbar\omega_{sj}+E_{lp}-E_{nk})/\tau\}\,\delta(\boldsymbol{p},\boldsymbol{k}+\boldsymbol{s}+2\pi\boldsymbol{g})\\
&\times\{\rho(\boldsymbol{x};\boldsymbol{x};t)-\rho(\cdots 1-n^{\mu}_{nk},\ 1-n^{\mu}_{lp},\ N_{sj}+1\cdots;\ \text{〃}\ ;t)\}]\,. \qquad (7\cdot 9)
\end{aligned}$$

The time averaged functions for the distributions departing only slightly from thermal equilibrium may, as was pointed out in I, fulfill the relation

$$\begin{aligned}
&\sum_{\boldsymbol{x}}{}'\overline{(1-n^{\mu}_{nk})n^{\mu}_{lp}N_{sj}\rho(\boldsymbol{x};\boldsymbol{x};t)}\\
&\qquad=\{1-\bar{g}^{\mu}(n\boldsymbol{k},t)\}\,\bar{g}^{\mu}(l\boldsymbol{p},t)\langle\!\langle N_{sj}\rangle\!\rangle\,, \qquad (7\cdot 10)
\end{aligned}$$

where

$$\langle\!\langle N_{sj}\rangle\!\rangle=\sum_{\boldsymbol{x}}{}'N_{sj}\bar{\rho}(\boldsymbol{x};\boldsymbol{x};t)\,. \qquad (7\cdot 11)$$

This relation corresponds to the principle of molecular chaos in quantum-statistics. Substitution of the expression for $S^{\mu}(n\boldsymbol{k},t)$ into (7·1) and time averaging of thus obtained expression over such an interval that the relation (7·10) may be justified give

$$\begin{aligned}
\bar{S}^{\mu}_{\mathrm{coll}}(n\boldsymbol{k},t)=&-(\hbar/2MG^3)\sum_{l}{}'\sum_{s}{}'\sum_{j}{}'\sum_{g}{}'(1/\omega_{sj})\\
&\times[|K^{sj}_{nk,lp}|^2\{\Omega(E_{nk}-E_{lp}+\hbar\omega_{sj})/\tau\}\\
&\times\{\bar{g}^{\mu}_{lp}(1-\bar{g}^{\mu}_{nk})(\langle\!\langle N_{sj}\rangle\!\rangle+1)-\bar{g}^{\mu}_{nk}(1-\bar{g}^{\mu}_{lp})\langle\!\langle N_{sj}\rangle\!\rangle\}\\
&+|K^{sj}_{nk,lq}|^2\{\Omega(E_{nk}-E_{lq}-\hbar\omega_{sj})/\tau\}\\
&\times\{\bar{g}^{\mu}_{lq}(1-\bar{g}^{\mu}_{nk})\langle\!\langle N_{sj}\rangle\!\rangle-\bar{g}^{\mu}_{nk}(1-\bar{g}^{\mu}_{lq})(\langle\!\langle N_{sj}\rangle\!\rangle+1)\}]\,, \qquad (7\cdot 12)^*
\end{aligned}$$

* g^{μ}_{nk}'s indicate $g^{\mu}(n\boldsymbol{k},t)$'s given by (5·6).

where

$$\boldsymbol{p}=\boldsymbol{k}+\boldsymbol{s}+2\pi\boldsymbol{g},\ \boldsymbol{q}=\boldsymbol{k}-\boldsymbol{s}+2\pi\boldsymbol{g}\ .$$

As with the customary theory of metallic conduction, we may neglect the transversal lattice vibrations and the umklapp processes. Then (7·12) agrees with Bloch's expression, as far as we consider only the first zone.

§ 8. The term due to the collision between electrons

The expression (5·12) for $S_c^{\mu}(n\boldsymbol{k}, t)$ can be brought into the form

$$S_c^{\mu}(n\boldsymbol{k}, t)=S^{\mu}(n\boldsymbol{k}, t)+S^{\mu *}(n\boldsymbol{k}, t),. \tag{8·1}$$

where

$$S^{\mu}(k, t)=(1/i\hbar)\sum_{\nu}\sum_{q, r, s}\overline{a_k^{\mu *}a_q^{\nu *}a_s^{\nu}a_r^{\mu}}\langle kq|u|rs\rangle\ , \tag{8·2}$$

in which the set $(n, \boldsymbol{k})$, for the sake of simplicity, is denoted by k and also q etc. similarily represent the set of the reduced wave number and the zone number. From (4·9), (4·3) and (3·3) we obtain

$$\begin{aligned}&\overline{a_k^{\mu *}a_q^{\nu *}a_s^{\nu}a_r^{\mu}}\\&\quad=i\hbar\{\langle rs|u|kq\rangle-\langle rs|u|qk\rangle\delta_{\mu\nu}\}\\&\quad\times(1/\tau)\omega(E_r+E_s-E_q-E_k)\\&\quad\times\sum_{\varkappa}\{n_r^{\mu}n_s^{\nu}(1-n_q^{\nu})(1-n_k^{\mu})-(1-n_r^{\mu})(1-n_s^{\nu})n_q^{\nu}n_k^{\mu}\}\rho(\varkappa, \varkappa, t).\end{aligned} \tag{8·3}$$

From the property of the Bloch function $\varphi_k(\boldsymbol{x})=e^{i\boldsymbol{k}_k\cdot\boldsymbol{x}}u_{\boldsymbol{k}_k}(\boldsymbol{x})$, we have the conservation law of momentum:

$$\langle rs|u|kq\rangle=\delta(\boldsymbol{k}_r+\boldsymbol{k}_s-\boldsymbol{k}_k-\boldsymbol{k}_q)\langle rs|u|kq\rangle\ . \tag{8·4}$$

If we substitute (8·3) into (8·2) and if we average the obtained expression over such an interval that the relation

$$\begin{aligned}&\sum_{\varkappa}\overline{n_r^{\mu}n_s^{\nu}(1-n_q^{\nu})(1-n_k^{\mu})\rho(\varkappa, \varkappa, t)}\\&\qquad=\bar{g}^{\mu}(r, t)\bar{g}^{\nu}(s, t)\{1-\bar{g}^{\nu}(q, t)\}\{1-\bar{g}^{\mu}(k, t)\}\end{aligned} \tag{8·5}$$

may be fulfilled, we have, making use of (8·4),

$$\begin{aligned}&\bar{S}_c^{\mu}(k, t)\\&\quad=\sum_{\nu}\sum_{q}\sum_{\substack{(r, s)\\[\boldsymbol{k}_r+\boldsymbol{k}_s=\boldsymbol{k}_k+\boldsymbol{k}_q]}}\{\bar{g}_r^{\mu}\bar{g}_s^{\nu}(1-\bar{g}_q^{\nu})(1-\bar{g}_k^{\mu})-(1-\bar{g}_r^{\mu})(1-\bar{g}_s^{\nu})\bar{g}_q^{\nu}\bar{g}_s^{\mu}\}\\&\quad\times|\langle rs|u|kq\rangle-\langle rs|u|qk\rangle\delta_{\mu\nu}|^2\\&\quad\times(2\pi/\hbar)\delta(E_r+E_s-E_q-E_k),\end{aligned} \tag{8·6}$$

where $\sum_{(r,s)}'$ means the summation with respect to pairs (r, s). It may easily be seen that this expression has intimate relation with the probabilities for transition by molecular collisions.[17]

§ 9. Conclusion

Substituting (6·2) and (6·3) into eq. (5·8), and averaging thus obtained equation over such an interval of time that eqs. (7·10) and (8·5) may be fulfilled, we obtain the equation for the electronic d.f. $\bar{g}^{\mu}(n\boldsymbol{k}, t)$:

$$\begin{aligned}&\partial\bar{g}^{\mu}(n\boldsymbol{k}, t)/\partial t+(\boldsymbol{F}/\hbar)\cdot\nabla_{k}\bar{g}^{\mu}(n\boldsymbol{k}, t)\\&\qquad=-\bar{S}_{p\hbar}^{\mu}(n\boldsymbol{k}, t)+\bar{S}_{e}^{\mu}(n\boldsymbol{k}, t),\end{aligned}\tag{9·1}$$

where the explicit expression for $\bar{S}_{p\hbar}^{\mu}$ and $\bar{S}_{e}^{\mu}$ are given by (7·12) and (8·6), respectively. We have derived eq. (9·1) analytically from the quantum-mechanical Liouville equation for the density matrix by means of [I] *the time averaging procedure over a certain interval of time* (§ 4, 7, 8), [II] *the postulate of random a priori phases* (§ 4), [III] *Bloch's band scheme* (§ 3), *and* [IV] *the assumption that* $E_n(\boldsymbol{k})\neq E_l(\boldsymbol{k})$ *if* $n\neq l$ (§ 6).

Here, let us confine ourselves to the case of steady state, where $\partial\bar{g}^{\mu}/\partial t=0$, and omit the term corresponding to the collisions between electrons, $\bar{S}_{e}^{\mu}$. Then, if we neglect the interaction between electrons and transversal vibrations and umklapp processes as in the customary theory of metallic conduction, we obtain, as far as we concern only the first zone,

$$\begin{aligned}(\boldsymbol{F}/\hbar)\cdot\nabla_{k}\bar{g}^{\mu}(\boldsymbol{k})&=(\hbar C^{2}/2MG^{3})\sum_{s}{}'(S^{2}/\omega_{s})\\&\times[\{\Omega(E_{k}-E_{p}+\hbar\omega_{s})/\tau\}\\&\times\{\bar{g}(\boldsymbol{p})(1-\bar{g}(\boldsymbol{k}))(\langle\!\langle N_{s}\rangle\!\rangle+1)-\bar{g}(\boldsymbol{k})(1-\bar{g}(\boldsymbol{p}))\langle\!\langle N_{s}\rangle\!\rangle\}\\&+\{\Omega(E_{k}-E_{q}-\hbar\omega_{s})/\tau\}\\&\times\{\bar{g}(\boldsymbol{q})(1-\bar{g}(\boldsymbol{k}))\langle\!\langle N_{s}\rangle\!\rangle-\bar{g}(\boldsymbol{k})(1-\bar{g}(\boldsymbol{q}))(\langle\!\langle N_{s}\rangle\!\rangle+1)\}],\end{aligned}\tag{9·2}$$

where

$$\boldsymbol{p}=\boldsymbol{k}+\boldsymbol{s}\,,\quad \boldsymbol{q}=\boldsymbol{k}-\boldsymbol{s}$$

and

$$C=(2/3)(\hbar^{2}/2m)\int_{D_{0}}|\nabla u|^{2}\,d\boldsymbol{x}$$

is 2/3 times Sommerfeld and Bethe's coupling constant,[1] and $\Omega(x)$ is defined by (4·11). Eq. (9·2) is nothing other than the Bloch integro-differential equation, on which the customary theory[1] of metallic conduction is based.

Here, it should be noted that the term corresponding to the collisions between electrons is omitted and one concerns only the first zone in the derivation of eq. (9·2).

The latter approximation may be admitted only in the case of mono-valent metal. Eqs. (9·1) and (9·2) might become invalid at the very low temperature, where the *postulate of random a priori phases* might not be acceptable.

In conclusion, the author wishes to express his sincere thanks to Assist. Prof. S. Ono for his continual guidance and advice, and to Mr. T. Takemura and K. Kamiya for their helpfull discussions.

Appendix I. The quantized density matrix

Husimi and Nishiyama[11] have obtained the expression (2·8) from the algebraic theory of the density matrix. Since this method is, however, not familiar to most physicists, we shall here show an elementary but tedious method of obtaining the expression (2·8). We have, from (2·7),

$$\rho^{(n)}(q_1, \cdots, q_n; q_1', \cdots, q_n'; t)$$
$$\equiv [N!/(N-n)!]\int^{N-n}\cdots\int \rho^{(N)}(q^n, \bar{q}_{n+1}, \cdots, \bar{q}_N; q'^n, \bar{q}_{n+1}, \cdots, \bar{q}_N; t)\, d\bar{q}_{n+1}\cdots d\bar{q}_N \tag{AI·1}$$

$$= [N!/(N-n)!]\sum_{\{n_t\}}\sum_{\{n_t'\}} \rho^{(N)}(\{n_t\}; \{n_t'\}; t)$$
$$\times \int^{(N-n)}\cdots\int \Psi^{(\pm)\nu}_{\{n_t'\}}(q'^n, \bar{q}_{n+1}, \cdots, \bar{q}_N)\, \Psi^{(\pm)}_{\{n_t\}}(q^n, \bar{q}_{n+1}, \cdots, \bar{q}_N)\, d\bar{q}_{n+1}\cdots d\bar{q}_N. \tag{AI·2}$$

Substituting the expression (2·1) into (AI·2) and performing the integration, we obtain (2·8). For the sake of simplicity, let us consider $\rho^{(2)}(q_1, q_2; q_1', q_2'; t)$ for the case of the Bose-Einstein statistics. The expression (2·1) may alternatively be written as

$$\Psi^{(+)}_{\{n_t\}}(q^N) = [1/N!\cdot\prod_t n_t!]^{1/2}\sum P\psi_{t_1}(q_1)\cdots\psi_{t_N}(q_N)$$
$$= [\prod_t n_t!/N]^{1/2}\sum{}' P\psi_{t_1}(q_1)\cdots\psi_{t_N}(q_N),$$

where P represents permutation of the coordinates $q^N = (q_1, \cdots, q_N)$ and $\sum$ and $\sum'$ indicate the summation extended over all the permutations and the one extended over only the permutations giving the terms different to each other, respectively. Substituting the first and second expressions into $\Psi^{(+)}_{\{n_t\}}$ and $\Psi^{(+)\nu}_{\{n_t'\}}$, respectively, we obtain the following expressions, in the same manner as in the second quantization,[9]

$$\rho^{(2)}(q_1, q_2; q_1', q_2'; t) = \sum_{\{n_t\}}\sum_{\{n_t'\}} \rho^{(N)}(\{n_t\}; \{n_t'\}; t)$$
$$\times [\prod n_t'!/\prod n_t!]^{1/2}\sum_r\sum_s n_r(n_s-\delta_{r,s})\sum_p\sum_q \psi_p^*(q_1')\psi_q^*(q_2')\psi_r(q_1)\psi_s(q_2)$$
$$\times \delta_{n_r', n_{r-1}}\cdot\delta_{n_p', n_{p+1}}\cdot\delta_{n_s', n_{s-1}}\cdot\delta_{n_q', n_{q+1}}$$
$$= \sum_{\{n_t\}}\sum_{p,q,r,s}[n_r(n_s-\delta_{s,r})(n_q+1-\delta_{q,r}-\delta_{q,s})(n_p+1-\delta_{p,r}-\delta_{p,s}+\delta_{p,q})]^{1/2}$$
$$\times \psi_p^*(q_1')\psi_q^*(q_2')\psi_r(q_1)\psi_s(q_2)\rho^{(N)}(\{n_t\}; \cdots n_r-1, n_s-1, n_p+1, n_q+1\cdots; t)$$

$$= \sum_{\{n_t\}} \sum_{p,q,r,s} \psi_p^*(q_1')\psi_q^*(q_2')\psi_r(q_1)\psi_s(q_2)\, a_p^* a_q^* a_s a_r \rho^{(N)}(\{n_t\};\{n_t\};t), \tag{AI·3}$$

where a_t^* and a_t satisfy the relations

$$\left.\begin{aligned} a_t^*\,\rho^{(N)}(\{n_t\};\{n_t\};t) &= \sqrt{n_t+1}\;\rho^{(N)}(\{n_t\};\cdots n_t+1\cdots;t) \\ a_t\,\rho^{(N)}(\{n_t\};\{n_t\};t) &= \sqrt{n_t}\;\rho^{(N)}(\{n_t\};\cdots n_t-1\cdots;t). \end{aligned}\right\} \tag{AI·4}$$

Using the expression (2·2), we may rewrite (AI·3) in the form

$$\rho^{(2)}(q_1, q_2; q_1', q_2'; t) = \sum_{\{n_t\}} \boldsymbol{\phi}^*(q_1')\boldsymbol{\phi}^*(q_2')\boldsymbol{\phi}(q_2)\boldsymbol{\phi}(q_1)\rho^{(N)}(\{n_t\};\{n_t\};t). \tag{AI·5}$$

Alternatively, this may be written as

$$\rho^{(2)}(q_1, q_2; q_1', q_2'; t) = \sum_{\{n_t\}} \sum_{\{n_t'\}} \rho^{(N)}(\{n_t\};\{n_t'\};t)$$
$$\times \int\cdots\int \Psi^{(+)*}_{\{n_t'\}}(\overline{q}^N)\boldsymbol{\phi}^*(q_1')\boldsymbol{\phi}^*(q_2')\boldsymbol{\phi}(q_2)\boldsymbol{\phi}(q_1)\Psi^{(+)}_{\{n_t\}}(\overline{q}^N)\,d\overline{q}^N, \tag{AI·6}$$

where

$$\left.\begin{aligned} a_t^*\,\Psi^{(+)}_{\cdots,n_t,\cdots}(q^N) &= \sqrt{n_t+1}\;\Psi^{(+)}_{\cdots,n_t+1,\cdots}(q^N) \\ a_t^*\,\Psi^{(+)}_{\cdots,n_t,\cdots}(q^N) &= \sqrt{n_t}\qquad\Psi^{(+)}_{\cdots,n_t-1,\cdots}(q^N), \end{aligned}\right\} \tag{AI·7}$$

which are identical with the definitions of the operators a_t^* and a_t given in § 2. Then, (AI·6) may be written as

$$\rho^{(2)}(q_1, q_2; q_1', q_2'; t) = \int^{(N)}\cdots\int \boldsymbol{\phi}^*(q_1')\boldsymbol{\phi}^*(q_2')\boldsymbol{\phi}(q_2)\boldsymbol{\phi}(q_1)\rho^{(N)}(\overline{q}^N;\overline{q}^N;t)\,d\overline{q}^N, \tag{AI·8}$$

in which the quantized wave functions $\boldsymbol{\phi}$'s operate only on $\Psi^{(+)}_{\{n_t\}}(q^N)$ concerning the first index q^N of $\rho^{(N)}(q^N;q'^N;t)$. Eq. (AI·4) shows that the transformed operators of a_t^* and a_t in the N-representation should operate only on $C(\{n_t\},t)$ concerning the first index $\{n_t\}$ of $\rho^{(N)}(\{n_t\};\{n_t'\};t)$. Thus the expression (2·8) is obtained from (AI·5) and (AI·8).

Appendix II. The macroscopic electric current

From (5·3) and (5·2), the macroscopic electric current density is given by

$$\boldsymbol{J} = (c\hbar/2miD)\sum_{\mu}\int(\overline{\boldsymbol{\phi}^{\mu*}\nabla\boldsymbol{\phi}^{\mu}} - \overline{\nabla\boldsymbol{\phi}^{\mu*}\boldsymbol{\phi}^{\mu}})\,d\boldsymbol{x}\,, \tag{AII·1}$$

where the bars represent the time averaging. Substituting (2·10), the expression for $\boldsymbol{\phi}^{\mu}$, into the integrand, we obtain

$$\boldsymbol{J} = (1/D)\sum_{\mu}\sum_{k,l}\overline{a_k^{\mu*}a_l^{\mu}}\,\boldsymbol{S}_{kl}\,, \tag{AII·2}$$

where

$$\boldsymbol{S}_{kl}=(e\hbar/2mi)\int(\varphi_k^*\nabla\varphi_l-\nabla\varphi_k^*\varphi_l)\,d\boldsymbol{x} \qquad \text{(AII·3)}$$

are the well-known stream-matrices. The diagonal elements of the stream-matrices play the decisive role in the theory of electric conduction, while the non-diagonal elements, which vanish for free particles and appear in the theory of the optical property of solid, do not contribute to $\boldsymbol{J}$ as we shall see later.

We obtain immediately

$$\begin{aligned}\boldsymbol{J}=&(e/D)\sum_{\mu}\sum_{n}\sum_{\boldsymbol{k}}\bar{g}^{\mu}(n\boldsymbol{k},\,t)\,\langle n\boldsymbol{k}\,|\boldsymbol{p}/m|\,n\boldsymbol{k}\rangle\\&+(e/D)\sum_{\mu}\sum_{n,\,l(>n)}\sum_{\boldsymbol{k}}\{\langle n\boldsymbol{k}\,|\boldsymbol{p}/m|\,l\boldsymbol{k}\rangle\,\overline{a^{\mu}_{nk}\,a^{\mu}_{lk}}+\text{c.c.}\},\end{aligned} \qquad \text{(AII·4)}$$

where $\boldsymbol{p}$ means the operator $(\hbar/i)\nabla$ and the following relation is used:

$$\langle n\boldsymbol{k}\,|\boldsymbol{p}|\,l\bar{\boldsymbol{k}}\rangle=\delta(\bar{\boldsymbol{k}}-\boldsymbol{k})\,\langle n\boldsymbol{k}\,|\boldsymbol{p}|\,l\boldsymbol{k}\rangle\,. \qquad \text{(AII·5)}$$

The second term of the expression (AII·4) corresponds to the various vertical transitions between the different zones in the reduced-zone scheme.

We obtain, from (4·9),

$$\begin{aligned}&\bar{\rho}(\boldsymbol{x};\cdots 1-n_k^{\mu},\;1-n_l^{\mu}\cdots;\,t)\\&\quad=i\hbar\,\langle\boldsymbol{x}\,|\,\boldsymbol{H}_e\,|\cdots 1-n_k^{\mu},\;1-n_l^{\mu}\cdots\rangle\,\{\omega(E_l-E_k)/\tau\}\\&\quad\times\{\rho(\boldsymbol{x};\,\boldsymbol{x};\,t)-\rho(\cdots 1-n_k^{\mu},\;1-n_l^{\mu}\cdots;\;''\;;\,t)\},\end{aligned} \qquad \text{(AII·6)}$$

if $n_k^{\mu}=0$ and $n_l^{\mu}=1$. Consequently, we obtain

$$\begin{aligned}\overline{a^{\mu}_{lk}\,a^{\mu}_{lk}}=&i\hbar\,\langle l\boldsymbol{k}\,|A|\,n\boldsymbol{k}\rangle\,\{\omega(E_l(\boldsymbol{k})-E_n(\boldsymbol{k}))/\tau\}\\&\times\sum\{n^{\mu}_{lk}(1-n^{\mu}_{nk})-(1-n^{\mu}_{lk})\,n^{\mu}_{nk}\}\,\rho(\boldsymbol{x},\,\boldsymbol{x};\,t),\end{aligned} \qquad \text{(AII·7)}$$

if we consider only the terms containing A in $\boldsymbol{H}_e$, (4·3), for the sake of simplicity. The expression (4·13) tells us that $a^{\mu*}_{nk}\,a^{\mu}_{lk}$ vanishes if $E_l(\boldsymbol{k})\neq E_n(\boldsymbol{k})$ for $l\neq n$, which is quite correct at least for the mono-valent metal. Thus we obtain the expression (5·4).

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Meson-deuteron Reaction and Nucleon Isobars

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The meso-disintegration of deuterons and the scattering of π-mesons by deuterons are discussed introducing the nucleon isobars rather phenomenologically. Agreements with experiments are satisfactory in the case of meso-disintegration, and the deviations from impulse approximation in the case of the scattering are in right direction.

§ 1. Introduction

The existence of nucleon isobars was formerly anticipated from the strong coupling meson theory, and seemed to be argued by the experiments on the photo-production of neutral π-mesons from nuclei for which the cross sections were comparable with those for charged π-mesons.[1] Experiments on the scattering of π-mesons by hydrogen also seem to suggest the existence of nucleon isobars.[2] To clarify the rôle of the nucleon isobars, it may be worthwhile to investigate the interaction of π-mesons with two nucleon systems, i.e., the deuterons.

The strong coupling theory is the only one at present that can predict the existence of the nucleon isobars consistently, but it is in many aspects ambiguous and imperfect (such as the lack of the relativistical invariancy, extremely large coupling constant, etc.) and its form is so complicated that it is very difficult to carry out the calculations in a good approximation. We therfore do not refer to the strong coupling theory in detail, but start with some assumptions which can be deduced from the strong coupling theory. They are :

a) The existence of the nucleon isobars with spin and isotopic spin both equal to $3/2$. Their energies and lives are to be suitably chosen to fit the experimental data.

b) A selection rule. When a nucleon absorbs (or emmits) a meson, it cannot stay in an ordinary nucleon state, but must be excited to an isobar state, or inversely, an isobar must fall into an ordinary nucleon state when it emmits (or absorbs) a meson.**

c) The predominant p-state interaction of the π-mesons with nucleons.[3]

d) The potential of the force between a nucleon and its isobar, which is, according to Pauli and Kusaka,[4] of the form

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**) In the strong coupling theory, the interaction Hamiltonian of nucleons and unbound mesons is proportional to $\langle \sigma_\alpha i \quad I_\alpha \rangle_{II}$ (written in Pauli and Dancoff's notation), from which this selection rule is derived.

$$V(r)=f(r)\,(\boldsymbol{e}_I^{\alpha}\boldsymbol{e}_{II}^{\alpha})+g(r)\Big(\frac{(\boldsymbol{e}_I^{\alpha}\boldsymbol{r})\,(\boldsymbol{e}_{II}^{\alpha}\boldsymbol{r})}{r^2}-\frac{1}{3}(\boldsymbol{e}_I^{\alpha}\boldsymbol{e}_{II}^{\alpha})\Big)\,. \tag{1}$$

e) The charge independence of the whole theory; i.e., the invariance under the rotation in the isotopic space. (see ref. 2 Anderson *et al.*)

f) The energy dependence of the matrix elements of the meson-nucleon interaction proportional to $p/\omega^{\frac{3}{2}}$. (p: momentum of incident mesons, ω: their energy) (This assumpton is not very essential in our discussions.)

With these assumptions, we discuss in Sec. II, the meso-disintegration of deuterons, and in Sec. III, the scattering of π-mesons by deuterons.

According to the assumptions (*b*) and (*f*), the meson-nucleon interaction Hamiltonian must be of the form

$$H=\frac{g}{\omega}\Psi^{*-\frac{3}{2}}O\varphi\psi^{-\frac{1}{2}}+\cdots \tag{2}$$

where g is the coupling constant, ω the energy of the incident mesons, $\Psi^{-\frac{3}{2}}$ the wave function of the isobars with charge $2e$ (the upper suffix means the charge number), O an operator which is linear in *grad* and contains the spin and isotopic spin matrices, φ the wave function of the meson field, and $\psi^{\frac{1}{2}}$ the wave function of the proton. Assumption (*e*), the hypothesis of charge independence and the hermitian property of the Hamiltonian completely determine the coefficients of the other terms such as $\Psi^{*-\frac{1}{2}}O\varphi\psi^{\frac{1}{2}}$ etc. (except for sign). Namely we have

$$\begin{aligned}H=\frac{g}{\omega}\Big\{&\Psi^{*-\frac{3}{2}}O\varphi\psi^{-\frac{1}{2}}+\frac{1}{\sqrt{3}}\Psi^{*-\frac{1}{2}}O(\varphi\psi^{\frac{1}{2}}+\sqrt{2}\,\varphi_3\psi^{-\frac{1}{2}})\\&+\frac{1}{\sqrt{3}}\Psi^{*\frac{1}{2}}O\,(\varphi^{*}\psi^{-\frac{1}{2}}+\sqrt{2}\,\varphi_3\psi^{\frac{1}{2}})+\Psi^{*\frac{3}{2}}\varphi^{*}\psi^{\frac{1}{2}}\Big\}+\text{herm. conj.}\end{aligned} \tag{2'}$$

Nucleons are always treated as non-relativistic particles in our discussions.

§ 2. The meso-disintegration of the deuterons

Experiments were performed by Durbin, Loar and Steinberger[5] using positive π-mesons with energies of 20～50Mev. The striking features of their results were:

i) almost $\cos^2\theta$ angular distributions of two protons and

ii) steep rise in cross sections with the energy of the incident mesons.

It has been shown that the usual weak coupling perturbation calculations[6] entirely fail to account for both i) and ii). To remedy these defect, Chew *et al*[7]. had to take the D-state of the deuteron into consideration that is far less than the S-state, and Matsuzaki and Sasaki[8] introduced an extraordinarily strong interaction between the two nucleons in the final state. Our theory is, howerver, in a satisfactory agreement with experiment without referring to any such assumptions.

1. Angular distribution

Following to the perturbation procedure, we treat the problem in two steps. The incident meson is first absorbed by one of the nucleons and this nucleon goes into an isobar state, (selection rule *b*) then under the influence of the nuclear force, the system of a nucleon plus an isobar goes into a two-proton state.

The two-nucleon state with total charge 2*e* interacting through nuclear forces (containing the tensor force), can be in either of the following states,

$$^1S'+{}^5\boldsymbol{D}+\cdots,\ 3p+\cdots,\ ^1D+{}^5\boldsymbol{S}+\cdots$$

and so on, where the black letters mean that one of the nncleens is excited to an isobar state. As the π-mesons can interact with nucleons only in p-state (assumption *c*), it is easily seen from the parity consideration that the $^3\boldsymbol{P}$ states are forbidden as the intermediate states. Since we are dealing with π-mesons with small momentum, (less than μc^2), and since the ground state of the deuteron consists mostly of the S-state, the intermediate states cannot be the $\boldsymbol{D}$-state. Therefore, we finally see that the intermediate state must be $^5\boldsymbol{S}$. The orbital angular momentum, L, of the gnal state must be 0 or 2 since the nuclear force of the type (1) can change the orbital angular momentum only by 0 or 2. The total isotopic apin, T, of the final state is obviously 1, there fore must be equal to 1 also in the intermediate states. The total spin, S, of the final state must be zero by Pauli principle, and accordingly, the total angular momentum J must be equal to 0 or 2. In the intermediate the value of J was 1 or 2, and since the nuclear force of type (1) cannot change the total angular momentum, the value of J in the final state must be 2. The values of S, L, T, and J are summarized in Table 1.

Table 1

	L	S	T	J
initial	0	1	0	1
intermediate	0	2	1	2
final	2	0	1	2

The orbital angular momentum, total spin angular momentom, total isotopic spin, and total angular momentum of our two-nucleon system.

The nuclear force cannot excite the deuteron in the ground state, a charge singlet state, into a state that contains an isobar, since it cannot change the value of T.

From the above considerations, we see the final state must be the ^1D state.

The state with $J=2$ can be constructed from the p-wave of the π-meson and the spin wave function of the deuteron as follows:

$$\begin{aligned} J_z=1,&\quad \frac{1}{\sqrt{2}}(\pi_0 d_1+\pi_1 d_0),\\ =0,&\quad \frac{1}{\sqrt{6}}(\pi_{-1} d_1+2\pi_0 d_0+\pi_1 d_{-1}),\\ =-1,&\quad \frac{1}{\sqrt{2}}(\pi_0 d_{-1}+\pi_{-1} d_0), \end{aligned} \tag{3}$$

where π's and d's are the wave function of the π-mesons in p-state and the spin wave

function of the deuteron respectively, and the suffices indicates the values of the z-component of the angular momenta. If we choose the direction of the incident mesons as the z-axis, the wave functions of the initial states are $\varphi_0\, d_m$, with $m=1$, 0, or -1, and we get from eqs. (3) the ratio of the final states with $J_z=1,0$, and -1 equal to $1/2 : 2/3 : 1/2$. We finally find the angular distribution proportional to

$$\frac{1}{2}\cdot(Y_1^2)^2+\frac{2}{3}\cdot(Y_0^2)^2+\frac{1}{2}\cdot(Y_1^2)^2\propto 3\cos^2\theta+1, \tag{4}$$

which is in good agreement with experimental data.

II. The excitation function

The excitation function can also get a better agreement with experimental data by introducing the nucleon isobars owing to the following reasons :

i) The energy denominator in the perturbation calculation $E-\omega$, where E is the energy of the isobars, decreases as the energy of the incident mesons increases.

ii) As our process occurs in two steps, the high momentum component of the wave function of the deuteron is not required to give a large contribution to the transition matrix elements.

Unfortunately, because of our ignorance of the detailed nature of the nuclear forces, especially the force between a nucleon and an isobar, we cannot draw any definite conclusion, but we can inspect some features of the excitation functions by assuming some suitable radial dependences of the nuclear force.

As the intermediate states and the final state have different values of orbital angular momentum, the central force does not contribute to our process. The tensor force, according to the meson theory (ps. pv.) of nuclear forces, has the well-known $1/r^3$ singularity, and does not give any meaningful result for our problem. We, therefore, aocording to Araki and Mori[9], replace it by

$$\frac{e^{-\mu r}}{r}b\left(\frac{(\boldsymbol{e}_I^\alpha \boldsymbol{r})(\boldsymbol{e}_{II}^\alpha \boldsymbol{r})}{r^2}-\frac{1}{3}(\boldsymbol{e}_I^\alpha \boldsymbol{e}_{II}^\alpha)\right),$$

where μ is the meson mass (we shall always refer to the unit system in which $\hbar=c=\mu=1$) and b is a constant which should be taken so as to give correct values for the magnetic moment and electric quadrupole moment of the deuteron. Araki and Mori gives $b=2.7\sim4.5$.

Following the usual perturbation procedure, one can easily obtain the differential cross section

$$d\sigma(\theta)=\frac{4}{9}\,b^2\frac{2}{27\pi}g^2\frac{M}{pk}\cdot 2\pi A^2\cdot\frac{1}{\omega^2}\,\frac{p^2}{|\Delta E|^2}f(k)\;\;(3\cos^2\theta+1), \tag{5}$$

where p and ω are the momentum and energy of the incident mesons respectively, k the relative momentum of the nucleons in the final state, k the relative momentum of the nucleons in the final state, M the mass of the nucleon, $\Delta E=$

$E-\omega$ and $Af(k)=k\int_0^\infty j_2(kr)\varphi_a(r)\frac{e^{-r}}{r}r^2\,dr$, φ_a is the wave function of the deuteron, which we have taken to be of Hulthen type, i.e., $\varphi_a=A\frac{e^{-\alpha r}-e^{-\beta r}}{r}$ with $\beta=7\alpha$, A the normalizing constant. Accordingly, we have

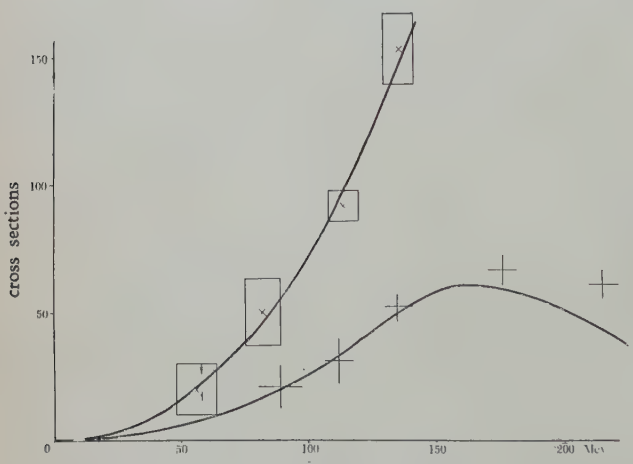
$$f(k)=\frac{1}{2}\left(1+3\left(\frac{\alpha+1}{k}\right)^2\right)\tan^{-1}\frac{k}{\alpha+1}-\frac{3}{2}\frac{\alpha+1}{k}$$

$$-\frac{1}{2}\left(1+3\left(\frac{\beta+1}{k}\right)^2\right)\tan^{-1}\frac{k}{\beta+1}+\frac{3}{2}\frac{\beta+1}{k}$$

and

$$A=\frac{1}{\beta-\alpha}\sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi}}.$$

To get the best agreement with the experimental data of the meson-nucleon scattering, we put the value of E to be 2.0 with imaginary part (half width) equal to 50 Mev., corresponding to $g^2=6.86$ (see Fig. 1). The best agreement is obtained by putting $b^2=3.6$ which is slightly smaller than the values required by the problem of magnetic dipole and electric quadrupole moment of the deuteron. The total cross sections, both calculated and experimental are plotted in Fig. 2.



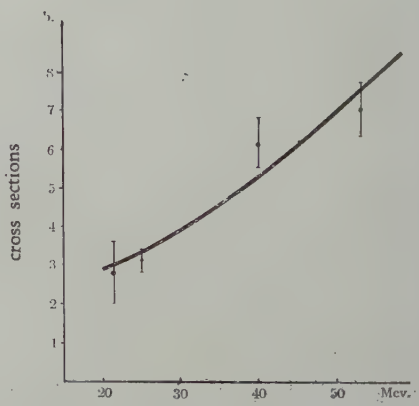
energies of the incident mesons

Fig. 1.

Meson nucleon scattering cross sections calculated from our theory putting $E=2\mu c^2$ and $\Gamma/2=50$ Mev at the resonance.

× : π^+H given by ref. 2. + : π^-H given by ref. 2.

× : π^+H given by ref. 10. • : π^-H given by ref. 10



energies of the incident mesons

Fig. 2.

Cross sections for the meso-disintegration of the deuteron, calculated and experimental. Experimntal values are due to ref. 5.

§ 3. Scattering of π-mesons by deuterons

The experiment performed by Isaacs, Sachs and Steinberger have shown that for the kinetic energy of the incident mesons of 58 Mev., π^+-D and π^--D cross sections are nearly the same in accord with the hypothesis of charge independence, and less than π^+-H plus π^--H by about 15 percent. It was shown by themselves that the calculation based on the weak coupling theories could not account for these experimental results. On the other hand, according to Anderson *et al.*,[11] the difference between the cross sections of $\pi^\pm$-D and π^+-H plus π^--H is not so large in the low energy region, and becomes striking only in the high energy region (higher than 115 Mev.). If this effect were due to interference, the smaller it must be the higher the energy of the incident meson becomes. One of the most reasonable way to account for this effect is, therefore, to consider the rôle of the nuclear force between two nucleons during the interaction with π-mesons.

Treating the interaction Hamiltonian as a perturbation, we can immediately get the second order transition matrix elements, i.e.,

$$A=A'+A'',$$

where

$$A'=\sum_j{}' \frac{(f|H|j)(j|H|i)}{E_i-E_j} \tag{8}$$

and

$$A''=\sum_{j'}{}' \frac{(f|H|j')(j'|H|i)}{E_i-E_{j'}}.$$

A' is the matrix element corresponding to the process that the deuteron first absorbs a meson and then emmits a meson, and A'' corresponding to the process that the deuteron first emmits a meson and then absorbs a meson. We have, accordingly,

$$E_i=\omega+\langle\frac{q^2}{M}+V\rangle_i, \tag{9}$$

$$E_j=E+\frac{p^2}{2M}+\langle\frac{q^2}{M}+V\rangle_j$$

and

$$E_i-E_j=(\omega-E-\frac{p^2}{2M})+\left\{\langle\frac{q^2}{M}+V\rangle_i-\langle\frac{q^2}{M}+V\rangle_j+p^2/4M\right\}, \tag{10}$$

where q is the relative momentum of the nucleons that constitute the deuteron. Expanding (8) with respect to the quantity in the second bracket of (10), and retaining only the first term we have

$$A'=\sum_j{}' \frac{(f|H|j)(j|H|i)}{\omega-E-p^2/2M}$$

$$-\sum_j{}' \frac{(f|H|j)(j|H|i)\{(\langle V\rangle_i-\langle V\rangle_j)+\frac{1}{M}(\langle q^2\rangle_i-\langle q^2\rangle_j+\frac{p^2}{4})\}}{(\omega-E-p^2/2M)^2}. \tag{11}$$

Since we finally make use of the closure approximation, and due to the even parity of the wave

function of the deuteron, an operator linear in q does not contribute to the matrix element, and therefore, $\langle q^2\rangle_i - \langle q^2\rangle_f + \frac{p^2}{4}$ makes no contribution to the matrix element. We finally get

$$A^I = \sum_j \frac{(f|H|j)(j|H|i)}{\omega - E - p^2/2M} - \sum_j \frac{(f|H|j)(j|[H,V]|i)}{(\omega - E - p^2/2M)^2}, \tag{12}$$

$$A^{II} = -\sum_j \frac{(f|H|j')(j'|H|i)}{\omega + E + p^2/2M} - \sum_j \frac{(f|H|j')(j')(j'|[H,V]|i)}{(\omega + E + p^2/2M)^2}. \tag{12'}$$

The first terms in (12) and (12′) are the terms that can be obtained by the impulse approximation,[12] and the second terms are the corrections due the nuclear force in the intermediate states.

If there were no nucleon isobar, $E=0$, the denominators of the second terms are always very small compared with the first terms. In our treatment, however, assuming the existence of nucleon isobars, the second term in (12) becomes appreciably large when the energy of the incident meson is near the energy of the isobar.

Neglecting the nucleon recoil in the final states, we obtain the differential cross section,

$$d\sigma = \frac{\omega^2}{4\pi} \sum_f \left\{ \left| \sum_j \frac{(f|H|j)(j|H|i)}{\omega - E - p^2/2M} - \sum_{j'} \frac{(f|H|j')(j'|H|i)}{\omega + E + p^2/2M} \right|^2 \right. \tag{13}$$

$$\left. - \mathrm{Re} \sum_j \frac{(i|H|j)(j|H|f)}{(\omega - E - p^2/2M)^*} \sum_{j'} \frac{(f|H|j')(j'|[H,V]|i)}{(\omega - E - p^2/2M)^2} \right\}.$$

In calculating (13), terms small compared with the second term in the curly bracket are neglected, since they are smaller than the second term by a factor 20 or more. The firsm term in the curly bracket is the term that can be obtained by the impulse approximation. As we are neglecting the effect of interference, for they are known to be very small in the high energy region, this term gives $\sigma(\pi^+ - H)$ plus $\sigma(\pi^- - H)$. From Hamiltonian (2′), we have

$$\sigma_I = \sigma(\pi^+ - H) + \sigma(\pi^- - H)$$

$$= \frac{1}{12\pi} g^4 \frac{p^4}{\omega^4} \left\{ \frac{2}{|\omega - E - p^2/2M|^2} + \frac{2}{|\omega + E + p^2/2M|^2} \right.$$

$$\left. - \frac{4}{9} \mathrm{Re} \frac{1}{(\omega - E - p^2/2M)^*} \frac{1}{(p + \omega + p^2/2M)} \right\}. \tag{14}$$

In evaluating the second term, the correction due to the nuclear force, we must now evaluate the matrix elements of V. The intermediate state consisting of a nucleon and an isobar must have the total spin S and total isotopic spin $T=1$ or 2. As the ground state of the deuteron is the charge singlet state, T in the intermediate state must be equal to 1. Therefore, we have only to evaluate the matrix elements of V for the states $T=1$, $S=1$ or 2.

The tensor force only appears in combination with the D-state of the deuteron which is far less than the S-state, and is very complicated to manipulate, so we neglect the

effect of the tensor force. The expectation value of $(e_j^\alpha e_{II}^\alpha)$ is $-\frac{1}{3}$ for the state $S=2$, $\frac{5}{9}$ for the state $S=1$, and $-\frac{1}{3}$ for the ground state of the deuteron. Therefore the matrix element $[H, V]$ vanishes for j with $S=2$, (strictly speaking, after summing up with respect to j and f), and we must calculate the rario of the matrix elements corresponding to the processes that occur through 3-let and 5-let intermediate states. This ratio can easily be evaluated from the invariant property of the Hamiltonian under the rotation in space. If we take the direction of the incident mesons as z-axis, 1/4 of the total processes occur through 3-let state when the z-component of the angular momentum of the deuteron is 1 or -1, and zero when the z-component of the angular momentum of the deuteron is zero. We get finally, averaging with respect to initial spin,

$$\sigma(D)-\{\sigma(\pi^+-H)+\sigma(\pi^--H)\} \tag{15}$$

$$=-g^4/4\left(-\frac{8}{3}\langle V\rangle_{deut\ rm}\right)\frac{2}{3}\cdot\frac{4\pi}{3}\cdot\frac{\omega^2}{4\pi^2}\cdot\frac{\not{p}^4}{\omega^6}\cdot 2\mathfrak{Re}\left(\frac{1}{(\omega-E-\not{p}^2/2M)^*}\times\right.$$

$$\left.\times\frac{1}{(\omega-E-\not{p}^2/M)^2}\right)\left\{\frac{1}{4}(1+\frac{1}{9}+\frac{1}{2}+\frac{1}{18})+\frac{1}{4}(\frac{2}{9}+\frac{1}{9})\right\}.$$

We have taken the wave function of the deuteron to be of Hulthen type as in Sec. II. This choice of the wave function gives the value of $\langle V\rangle_{deut.}$ 17.9 Mev. Numerical values are plotted in Fig. 3. Agreement with experiment of Anderon *et al.* is fairly good in the region 100∼130 Mev. of the incident meson energy.

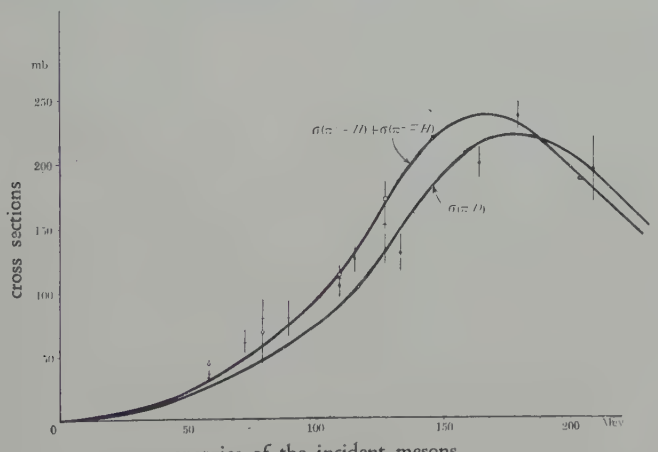


Fig. 3.

Cross sections for the the scattering, $\sigma(\pi^+H)+\sigma(\pi^-H)$ and $\sigma(\pi D)$. Experimental values are due to ref. 11 and ref. 10.

§ 4. Summary and conclusion

We have discussed the meson-deuteron reaction by assuming *a*) the existence of nucleon isobars with spin and isotopic spin both equal to 2/3, *b*) a selection rule, *c*) the p-state

interaction of π-mesons with nucleons, and a few other assumptions. Following the second order perturbation procedure, we have obtained the cross sections and the angular distributions of two protons for the meso-disintegration, and the difference of the scattering cross sections between π^+-H plus π^--H and $\pi^\pm$-D. Agreement with experiment is satisfactory in the case of the meso-disintegration, and right in sign and order of magnitude in the case of the scattering.

We believe that we have revealed some evidence of the existence of the nucleon isobars by extending our discussion from the one-nucleon systems to two-nucleon systems. More strong evidence will be got when the experiments on the meson scattering and photomeson production proceed into more high energy region. The authors thank to Professor Yamanouchi for his kind encouragement. They are also indebted to the Yukawa-Yomiuri Fellowship for the financial aid supplied to one of them (S.M.).

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and other papers of their group.

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The General Theory of the Interaction Representation, II

— General Fields and Interactions —

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In the case when the interaction contains the higher derivatives than free part the method of previous paper is extended. It is shown that our formalism is perfectly equivalent to the Heisenberg-Pauli's method, but not to the Ostrogradski's one. Lastly, the example of the quantization of non-local interaction is given.

§ 1. The structure of the theory of the interaction representation

In this section we briefly discuss the structure of the theory of the interaction representation for the system of the arbitrary fields and interactions with higher derivatives, the detailed investigation of which has been given in the previous paper I[1] for the case of simple interactions. Moreover, in this discussion some confusions in the usual investigations for the interaction representation are removed.

The theory of the interaction representation (I.R.) is based on the relativistic field equation for the field quantities $Q_\alpha(x)$ $(\alpha=1, 2, \cdots)$, the covariant commutation relations for $Q_\alpha(x)$ and the Schrödinger equation for the state vector $\Psi(\sigma)$.

In the interaction representation, $Q_\alpha(x)$ satisfies the relativistic field equation for the free field, and so it is necessary to pick up the free Lagrangian L^0 from the total Lagrangian. If we regard the every term, which is composed of the operators of the same field, as the part of the free Lagrangian, the physical image of the free field becomes very complex. As the simple example of such a case we have λU^3-term for the neutral scaler field, which is required as the counter term in the renormalization theory. Therefore, we will define the free Lagrangian in the following paragraphs according to the simple physical property of the free particle.

As is well-known, for the energy-momentum k_μ of the free particle with the mass $\varkappa$, we have the simple relation

$$k_\mu k_\mu + \varkappa^2 = 0. \tag{1·1}$$

(1·1) requires that $Q_\alpha(x)$ satisfies the equation of the simple harmonic oscillators (S.H.O.).

In general the equation of S.H.O. has the following form:

$$F(\Box)\, Q_\alpha(x) \equiv \prod_{i=1}^{n} (\Box - \varkappa_i^2)\, Q_\alpha(x) = 0. \tag{1·2′}$$

The simplest case of $(1\cdot2')$ is:

$$(\Box-\varkappa^2)Q_\alpha(x)=0. \tag{1·2}$$

Therefore, we proceed with our discussions according to the "requirement of S.H.O." that the state of the system of the free fields in the quantum field theory corresponds to that of S.H.O. in the quantum theory with the canonical formalism.

Then, the relativistic field equation for $Q_\alpha(x)$ in the interaction representation has the following form:*1)

$$\Lambda_{\alpha\beta}(\partial)Q_\beta(x)=0 \tag{1·3}$$

and we have the following requirement for the derivation operator $\Lambda_{\alpha\beta}(\partial)$:*2) *3)

(I) $(1\cdot2)$ (or the more general relation $(1\cdot2')$) can be derived from $(1\cdot3)$.

Then free Lagrangian is picked up so that it gives the field equation satisfying the requirement (I).

The commutation relation for $Q_\alpha(x)$ is also determined by the requirement of S-H.O., the relativistic invariancy of the theory and taking into account the fact that the velocity of the propagation of the disturbance can not be larger than that of light:

$$[Q_\alpha(x), Q_\beta(x')]_\pm = iR_{\alpha\beta}(\partial)\Delta(x-x'), \tag{1·4}$$

$$\Lambda_{\alpha\beta}(\partial)R_{\beta\gamma}(\partial)=(\Box-\varkappa^2)\delta_{\alpha\gamma}. \tag{1·5}$$

In $(1\cdot4)$ $(-)$ and $(+)$ correspond to the fields with the integer- and the half-integer-spin, respectively.*4) The physical meaning of $(1\cdot5)$ is as follows:

When we transform $Q_\alpha(x)$ into the principal axes $q_r(k)$ $(r=1, 2, \cdots)$ of S.H.O. in the momentum space, the quantum theory of Q_α-field based on the field equation $(1\cdot3)$ (c.f. Appendix III.) and the commutation relation $(1\cdot4)$ is equivalent to the quantum theory of S.H.O. which is founded on the well-known Lagrangian with principal axes $q_r(k)$ and the commutation relations of S.H.O.*5) *6) *7)

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*1 Throughout this paper we adopt the same notations as those in I. $(1\cdot3)$ is the linear equation, because $(1\cdot2)$ and $(1\cdot2')$ are linear.

*2 In this paper the field equation means just one including sign and constants which is obtained from the suitable Lagrangian by means of the variation principle.

*3 Usually there is also the requirement (II): the relativistic field equation is differential equation of the first degree. However, in our following discussions this requirement (II) is not studied. Although (II) is required in the usual wave mechanics of the particles as the necessary condition for the existence of the probability function, its physical meaning is rather vague in the quantum field theory. The general theory of the relativistic wave equations based on the requirements (I) and (II) has been studied in detail by Bhabha and Chandra, but the complete result has not yet been obtained. Recently, the interest suggestion on this problem was given by Hepner[2].

*4 As is well-known, this fact comes from the requirement of the invariancy of the theory for the Lorentz transformation including the time reflection[3].

*5 The detailed accounts on this fact have been given in I. Strictly speaking, only the commutation relation of $(-)$-type corresponds to that of S.H.O., because only two values $(n=0,1)$ are allowed for the principal number n in the case of the commutation relation of $(+)$-type. However, for the sake of convenience, two cases of $(\pm)$-types are en bloc called the commutation relation of S.H.O.

*6 The star * means the Hermite conjugate quantity.

*7 We shall improve some errata of paper I in Appendix III. Moreover we shall note that we must omit the footnotes *** of p. 22 and **** of p. 23 in paper I, because it is needless to assume the adiabatically switching in and off of interactions in the theory of present paper and Q_{out} and Q_{in} are defined as $Q(x,\infty)$ and $Q(x,-\infty)$, respectively.

Then it is clear that the energy-momentum four vector T_μ can be rewritten as follows:

$$T_\mu{}^0 = -i \sum_k \sum_{\alpha=\pm} \sum_r k_\mu n_r^\alpha(k) + C\text{-number}. \tag{1·6}$$

$n_r^\pm(k)$ is the well-known number operators of positively and negatively charged particles respectively and $T_\mu{}^0$ is derived from the Lagrangian L^0 of the free field by the usual rule as follows:

$$\left.\begin{aligned} T_\mu{}^0 &= -i \int_\sigma T_{\mu\nu}{}^0 d\sigma_\nu , \\ T_{\mu\nu}{}^0 &= -\frac{\partial L^0}{\partial Q_{\alpha;\nu}} Q_{\alpha;\mu} + L^0 \delta_{\mu\nu} , \\ Q_{\alpha\mu} &= \partial_\mu Q_\alpha . \end{aligned}\right\} \tag{1·7}$$

Using (1·6), we can rewrite the field equation (1·3) in the following form:

$$-\partial_\mu F(x) = [F(x), T_\mu{}^0], \tag{1·3′}$$

where $F(x)$ is functional of $Q_\alpha(x)$.

The derivation operator $R_{\alpha\beta}(\partial)$ generally contains the term which is proportional to $(\Box - \varkappa^2)$:

$$R_{\alpha\beta}(\partial) = R_{\alpha\beta}'(\partial) + (\Box - \varkappa^2) R_{\alpha\beta}''(\partial). \tag{1·8}$$

Therefore, the commutation relation (1·4) can be rewritten as follows: *)

$$[Q_\alpha(x), Q_\beta(x')]_\pm = i R_{\alpha\beta}'(\partial) \Delta(x-x'). \tag{1·4′}$$

However it is convenient to write the commutation relation in the form (1·4), because (1·5) gives the simple general rule to obtain the commutation relation and explicitly shows the intimate connection between the commutation relation and the Green function.

Since we have the theorem**:

The highest degree b of the derivation operator in $R_{\alpha\beta}(\partial)$ for the field with the spin s and mass $\varkappa$ is

$$b = \begin{cases} 2s & \text{for } \varkappa \neq 0 \\ 1 & \text{for } \varkappa = 0, \quad s = \text{half-integer}, \\ 0 & \text{for } \varkappa = 0, \quad s = \text{integer}. \end{cases}$$

Expressing $R_{\alpha\beta}(\partial)$ with the sum of all the possible derivation operators with the degree $\leq b$, the same transformation property as $Q_\alpha \times Q_\beta$ and the undeterminate constants and determining the latter constants according to the relation (1·5), we can easily obtain $R_{\alpha\beta}(\partial)$ and the covariant commutation relation (1·4). The examples for determining $R_{\alpha\beta}(\partial)$ of various fields were given in I.

By way of (1·5), the commutation relation is intimately connected with the Green function. When $\Delta_G(x)$ satisfies the relation $(\Box - \varkappa^2)\Delta_G(x) = \delta(x)$, for example:

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* When we determine the commutation relation so that it is compatible with the field equation, (1·4′) is directly obtained.

** The proof and the detailed account of this theorem were given in I.

$$\Delta_G(x)=\begin{cases} -\Delta(x) \\ \dfrac{1}{2i}\Delta_F(x) \\ \Delta^{\mathrm{ret}}(x) \\ \Delta^{\mathrm{adv}}(x). \end{cases} \tag{1·9}$$

Then $R_{\alpha\beta}(\partial)\Delta_G(x)$ is the Green function of the field equation (1·3):

$$\Lambda_{\alpha\beta}(\partial)R_{\beta\gamma}(\partial)\Delta_G(x)=\delta(x)\delta_{\alpha\gamma}. \tag{1·10}$$

In other words, we have the following theorem: The commutation relation with the derivation operator, which appears in the Green function of the field equation (1·3), satisfies the requirement of S.H.O..

It must be noted that, when we write the commutation relation in the form of (1·4′), the inverse of above theorem does not always hold true.

Now we discuss the Schrödinger equation. The Schrödinger equation can be obtained when we have the interaction Hamiltonian $H(x:n)$ (n is the normal unit vector of the space-like surface σ). However the general rule for determining the interaction Hamiltonians has been still unknown and so they have been tentatively constructed by use of the integrability condition[4]:

$$[H(x':n),\ H(x'':n)]-i\frac{\partial H(x'':n)}{\partial\sigma(x')}+i\frac{\partial H(x':n)}{\partial\sigma(x'')}=0. \tag{1·11}$$

However, in fact, we can find many interactions satisfying the integrability condition, and most of them are not the correct interaction Hamiltonian. Although in the particularly simple case the most simple $H(x:n)$ which satisfied (1·11) may be the true interaction Hamiltonian, in general the interaction Hamiltonian can not be determined by use of the integrability condition and the principle of simplicity. Such the circumstances were concretely explained in § 5 of I by use of the electromagnetic interaction in the Duffin-Kemmers' meson theory. Therefore it becomes necessary to find the general rule for determining the interaction Hamiltonian. We have shown in I that the interaction Hamiltonian can be uniquely determined by use of the unitary transformation, which combines the Heisenberg representation (H.R.) with the interaction representation (I.R.). In the following we shall show that this fact holds true for arbitrary interaction. Then, we can uniquely construct the total Hamiltonian in H.R. and the interaction Hamiltonian in I.R. and so, obtain the canonical formalism even for the interactions with higher derivatives.

Now we briefly summarize the rule for determining the interaction Hamiltonian, rewriting it into the convenient form to extend to the case of interactions with higher derivatives. When we denote the quantities in H.R. by the underline, the field equations in H.R. are as follows:

$$\Lambda_{\alpha\beta}(\partial)\underline{Q}_\beta(x)=\underline{J}_\alpha(x), \tag{1·12}$$

$$\begin{aligned}
\underline{J}_\alpha(x) &\equiv \sum_{m=0}^{\infty} D_{\mu_1\cdots\mu_m} \underline{j}_{\alpha:\mu_1\cdots\mu_m}(x) \\
&\equiv D_a \underline{j}_{\alpha:a}, \\
D_a &\equiv D_{\mu_1,\cdots,\mu_m} \equiv (-1)^{m+1} \partial_{\mu_1}\cdots\partial_{\mu_m}, \\
\underline{j}_{\alpha:a} &\equiv \underline{j}_{\alpha:\mu_1\cdots\mu_m} \equiv \partial \underline{L}'/\partial \underline{Q}_{\alpha:\mu_1\cdots\mu_m}, \\
\underline{Q}_{\alpha:\mu_1\cdots\mu_m} &\equiv \partial_{\mu_1}\cdots\partial_{\mu_m} \underline{Q}_\alpha,
\end{aligned} \tag{1·13}$$

where $\underline{L}'$ is the interaction Lagrangian.

Using (1·10) we can rewrite (1·12) into the integral form:

$$\underline{Q}_\alpha(x) = Q_\alpha(x) + \int_{-\infty}^{\infty} R_{\alpha\beta}(\partial) D_a \Delta_G(x-x') \underline{j}_{\beta:a}(x') dx'. \tag{1·14'}$$

$Q_\alpha(x)$ satisfies the free field equation (1·3) and so we regard $Q_\alpha(x)$ as the quantity in I.R.. Various $\Delta_G(x)$-functions give the various interaction representation. Since we can equally well treat even if we take any Δ_G-function satisfying (1·10), let us adopt $\Delta^{\rm ret}(x)$ as the Δ_G-function, i.e.,

$$\underline{Q}_\alpha(x) = Q_\alpha(x) + \int_{-\infty}^{\infty} R_{\alpha\beta}(\partial) D_a \Delta^{\rm ret}(x-x') \underline{j}_{\beta:a}(x') dx'. \tag{1·14}$$

In the course of introducing the unitary transformation $U(\sigma)$ connecting H.R. with I.R., we can not define it as the transformation combining $\underline{Q}_\alpha(x)$ and $Q_\alpha(x)$, because in general $\underline{Q}_\alpha(x)$ $(\alpha=1, 2, \cdots)$ contain not only the canonical independent but also dependent variables.

Therefore, in order to define the unitary transformation connecting I.R. with H.R., we introduce the field operators $Q_\alpha(x, \sigma)$ (x does not always lie on σ), which serve as the independent canonical variables in H.R.. Strictly speaking, $Q_\alpha(x, \sigma)$ is the free field, the independent canonical variables of which agree with those in the Heisenberg representation at x/σ.*) Then, $Q_\alpha(x, \sigma)$ must satisfy the following requirements:

(i) $Q_\alpha(x, \sigma)$ is connected with $Q_\alpha(x)$ by the unitary transformation $U(\sigma)$ which depends only on the space-like surface σ:

$$Q_\alpha(x, \sigma) = U^{-1}(\sigma) Q_\alpha(x) U(\sigma), \qquad U(-\infty) = 1, \tag{1·15}$$

(ii) $\underline{Q}_\alpha(x)$ is the functional of $Q_\alpha(x, \sigma)$ at the same point x:**)

$$\underline{Q}_\alpha(x) = F(Q_\alpha(x, \sigma))|_{x/\sigma}. \tag{1·15'}$$

Since $\underline{Q}_\alpha(x)$ $(\alpha=1, 2, \cdots)$ are not the canonical independent variables, it becomes very tedious to apply the Heisenberg-Pauli's quantization method to the system with the interactions containing higher derivatives. However it is exemplified in § 2 that even in this case we can find the canonical independent variables to carry out the Heisenberg-Pauli's method. Since these canonical independent variables evidently satisfy the requirements (i)

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* x/σ means x laid on σ.

** This functional contains the normal vector n_μ and the derivation operator ∂_μ operating on $Q_\alpha(x, \sigma)$.

and (ii), we can develop the covariant quantization theory, which is equivalent to the Heisenberg-Pauli's method, by use of these requirements (i) and (ii).* In fact it is exemplified in § 2 that $Q_\alpha(x,\sigma)$ is equivalent to the canonical independent variables of the Heisenberg-Pauli's canonical formalism.**

In the functional (1·15′) the degree of time derivatives can be reduced to the first degree by means of the free field equation for $Q_\alpha(x,\sigma)$. It can be easily understood that this fact corresponds to the elimination of time derivatives in the canonical formalism.

From the requirement (i) $Q_\alpha(x,\sigma)$ satisfies the field equation and the commutation relation of the free field for the fixed σ:

$$\left.\begin{aligned}&[Q_\alpha(x,\sigma),\, Q_\beta(x',\sigma)]_\pm = iR_{\alpha\beta}(\partial)\Delta(x-x'),\\ &\Lambda_{\alpha\beta}(\partial)\,Q_\beta(x,\sigma)=0.\end{aligned}\right\} \qquad (1\cdot16)$$

From (1·15′) we have the relation between $Q_\alpha(x,\sigma)$ and $\underset{-}{Q}_\alpha(x)$:

$$\left.\begin{aligned}&Q_\alpha(x)=U^{-1}(\sigma)\,Q_\alpha(x)\,U(\sigma)\big|_{x/\sigma},\\ &\underset{-}{Q}_\alpha(x)\equiv F(Q_\alpha(x)),\end{aligned}\right\} \qquad (1\cdot15'')^{***}$$

and so we can also describe the $\underset{-}{Q}_\alpha(x)$-field on the surface σ by means of $Q_\alpha(x,\sigma)$.

From (1·14), the requirements (i) and (ii), we can easily show that $Q_\alpha(x,\sigma)$ has the following form:

$$\left.\begin{aligned}&Q_\alpha(x,\sigma)=Q_\alpha(x)+\int_{-\infty}^{\sigma} D_\alpha R_{\alpha\beta}(\partial)\Delta(x-x')\,j_{\beta/\alpha}(x')\,dx'+g^2\int_{-\infty}^{\sigma} G_\alpha(x,x')\,dx',\\ &\Lambda_{\alpha\beta}(\partial)\,G_\beta(x,x')=0.\end{aligned}\right\} \qquad (1\cdot17)$$

The third term in the right side of (1·17) for the case x/σ must be the function which is independent of the field quantities in the past because of the requirement (ii), and so this has the following form:

$$G(x,x')=\partial_\mu h_\mu(x,x'), \qquad (1\cdot17')$$

where $h_\mu(x,x')$ is the certain function of x, x' and n_μ.

It can be seen from (1·14) and (1·17) that $Q_\alpha(x)$ $(=Q_\alpha(x,-\infty))$ does not always agree with $Q_\alpha(x)$ even for $\sigma(x)=-\infty$ because of the interaction at $\sigma=-\infty$****. It is not necessary to assume the adiabatically switching in and off in order to make $Q_\alpha(x,\sigma)$ the canonical independent variable as $\sigma=\pm\infty$. It is the well-known fact that similar circumstances are found for the vector field with the simple interaction with no

* It must be noticed that the commutation relation for the canonical quantities in the Heisenberg-Pauli's quantization method is equivalent to (1·14), because they are described in the free field representation. This fact corresponds to the situation in which $Q_\alpha(x,\sigma)$ satisfies the field equations and commutation relations of the free field. (i) and (ii) are the complete conditions for the equivalence between our theory and the Heisenberg-Pauli's formalism.

** This fact clarifies the meaning of Dirac-Wentzel potential in quantum electrodynamics[5], because $Q_\alpha(x,\sigma)$ can be regarded as its generalization.

*** cf. (4·19) in I.

**** $\sigma(x)$ means the surface containing the point x.

derivatives. The state of the field on σ can be described as the state of S.H.O., because $Q_\alpha(x, \sigma)$ behaves as the system of S.H.O.

Now we discuss the necessary and sufficient condition for the requirement (i). Using the interaction Hamiltonian $H(x, n)$ which is defined as follows*):

$$\begin{aligned} i\frac{\delta U(\sigma)}{\delta\sigma(x)} &= H(x:n)U(\sigma)|_{x/\sigma} \\ &= U(\sigma)H(x/\sigma:n), \end{aligned} \tag{1·18}$$

we introduce the Schrödinger equation

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)} = H(x:n)\Psi(\sigma). \tag{1·19}$$

As shown in I, the complete condition for the existence of the unitary transformation of the requirement (i) is equivalent to the existence of $H(x:n)$ satisfying the following relation[1]:

$$[Q_\alpha(x,\sigma), H(x'/\sigma:n)] = iR_{\alpha\beta}(\partial)D_\alpha\Delta(x-x')\underline{j}_{\beta\alpha}(x') + ig^2G(x,x'), \tag{1·20}$$

where g is the coupling constant.

Therefore, we must determine $G(x, x')$ so that $H(x:n)$ satisfying (1·19) exists. Then $G(x, x')$ can be uniquely determined in the form of power series of the coupling constant g by using (1·19) and the following relation which is derived from (1·17) and (1·14):

$$\begin{aligned} M(\partial)\underline{Q}_\alpha(x) &= M(\partial)Q_\alpha(x,\sigma)|_{x/\sigma} \\ &+ \frac{1}{2}\int[M(\partial)D_\alpha R_{\alpha\beta}(\partial), \varepsilon(x-x')]\Delta(x-x')\underline{j}_{\beta\alpha}(x')dx' \\ &- g^2\int_{-\infty}^{\sigma} M(\partial)G(x,x')dx', \end{aligned} \tag{1·21}$$

where $M(\partial)$ is an arbitrary derivation operator. The term of the first order in g for $M(\partial)\underline{Q}_\alpha(x) - M(\partial)Q_\alpha(x,\sigma)|_{x/\sigma}$ is determined by the interaction Lagrangian $\underline{L}'$. Substituting (1·21) into $\underline{j}_{\beta\alpha}(x')$ of (1·20), we obtain the g^2-term in the first term in (1·20). Then from (1·20) we can determine the g-term in $H(x:n)$ and the g^2-term in $g^2G(x,x')$. We see from (1·20) and (1·16) that the g-term of $H(x:n)$ is $-\underline{L}'(x)$**) ***).

It is very difficult to develop concretely and generally this method for determining $H(x:n)$ because of the q-number-property of Q_α. Therefore in order to see roughly the procedure

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* When $f(x)$ is a functional of $Q_\alpha(x)$, the notation $f(x,\sigma)$ means the same functional but with $Q_\alpha(x,\sigma)$ replacing $Q_\alpha(x)$. For example: $f(x) = \partial_\mu Q_\alpha(x)$, $f(x,\sigma) = \partial_\mu Q_\alpha(x,\sigma)$ and $f(x/\sigma) = \partial_\mu Q_\alpha(x,\sigma)|_{x/\sigma} \neq \partial_\mu Q_\alpha(x/\sigma)$.

** $\underline{L}'(x)$ is defined from $\underline{L}'(x)$ by substituting $Q_\alpha(x)$ into $\underline{Q}_\alpha(x)$.

*** Therefore, the existence of the interaction Hamiltonian is partially based on the intimate connection (1·10) between the commutation relation (1·16) and the Green function.

for determining $H(x:n)$, we proceed with our discussions regardless of the detailed order of field operators and in the g^2-approximation. Then, g^2-term in (1·20) has the following relation (see (A·17) in I):

$$[Q_\alpha(x,\sigma),\, W(x'/\sigma:n)]=\frac{i}{2}R_{\alpha\beta}(\partial)D_a\Delta(x-x')\int_{-\infty}^{\infty}dx''\Big\{\frac{\partial j_{\rho:b}}{\partial Q_{\beta:a}}j_{\tau:c}(x'')\times$$

$$[D_b(-\partial')D_c(\partial')R_{\rho\tau}(\partial'),\,\varepsilon(x'-x'')]\,\Delta(x'-x'')\Big\}+g^2G(x,x'),\qquad(1\cdot22)$$

where $W(x:n)$ is the g^2-term in $H(x:n)$:

$$H(x:n)=-L'(x)+W(x:n)+O(g^3).\qquad(1\cdot23)$$

Since we have the relation

$$\int_{-\infty}^{\infty}dx'\,f(x)[D_b(-\partial)D_cR_{\rho\tau}(\partial),\,\varepsilon(x-x')]\,\Delta(x-x')=D_{\rho\tau:bc}(\partial)f(x),\qquad(1\cdot24)$$

where $f(x)$ is an arbitrary function and $D_{\rho\tau:bc}(\partial)$ is the certain derivation operator. From (1·24) we have*)

$$D_{\rho\tau:bc}(\partial)=D_{\tau\rho:cb}(-\partial).\qquad(1\cdot25)$$

Therefore, there is $W(x:n)$ satisfying (1·22) only when we take

$$g^2G(x,x')=-\frac{i}{4}R_{\alpha\beta}(\partial)D_a\Big[\Delta(x-x')\frac{\partial j_{\rho:b}(x')}{\partial Q_{\beta:a}}D_{\rho\tau:bc}(\partial')j_{\tau:c}(x')$$

$$-j_{\rho:b}(x')D_{\rho\tau:bc}(\partial')\Big\{\Delta(x-x')\frac{\partial j_{\tau:c}(x')}{\partial Q_{\beta:a}}\Big\}\Big]+O(g^3).\qquad(1\cdot26)$$

Then we obtain

$$W(x:n)=\frac{1}{4}j_{\rho:b}(x)D_{\rho\tau:bc}(\partial)j_{\tau:c}(x),\qquad(1\cdot27)$$

and so

$$\int W(x':n)dx'=\frac{1}{4}\iint j_{\alpha:a}(x')j_{\beta:b}(x'')$$

$$\times[R_{\alpha\beta}(\partial)D_a(\partial')D_b(-\partial'),\,\varepsilon(x'-x'')]\,\Delta(x'-x'')dx'dx'',\qquad(1\cdot28)$$

which again agrees with (A·19) in I.

Since it is seen from (1·24) that $D_{\rho\tau:bc}(\partial)j_{\tau:c}(x)$ has the same transformation property for the Hermite transformation as that of $Q_{\rho:b}(x)$, (1·27) shows that $W(x:n)$ is the Hermitic operator and so $U(\sigma)$ is the unitary matrix. By means of this interaction Hamiltonian $U(\sigma)$ can be written in the following form:

$$U(\sigma)=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_m\,P(H(x_1:n),\cdots,H(x_m:n)).\qquad(1\cdot29)$$

* Moreover we take into account the following relation:

$$R_{\alpha\beta}(\partial)=R_{\beta\alpha}(-\partial),$$

which can be easily derived from (2·9) and (2·7) in I.

Replacing (A·17) in I by (1·25) and (1·28), we can derive the following relation for $U(\infty)$:*)

$$U(\infty)=\sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_m P^*(L'(x_1),\cdots,L'(x_m)). \qquad (1\cdot29')$$

P^* means the following operation: Firstly, we treat P^*-bracket as if it is Dyson's P-bracket without rewriting $L'(x)$ by means of the field equation (1·3), and secondly we make the following replacement:

$$\langle P(M(\partial)Q_\alpha(x), N(\partial')Q_\beta(x'))\rangle_0 \to \frac{1}{2}M(\partial)N(\partial')R_{\alpha\beta}(\partial)\Delta_F(x-x'). \qquad (1\cdot30)$$

In other words P^* is the operation in which all the derivation operators including $R_{\alpha\beta}(\partial)$ are taken out of the bracket and then the bracket is treated as the Dyson's P-symbol. This fact holds true also for the expectation value of any physical quantities on the surface of the finite time as was proved in the previous paper. For example, the expectation value of the functional $f(Q_\alpha)$ is:

$$\langle f(Q_\alpha(x))\rangle=\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_m\langle P(f(Q_\alpha(x)), H(x_1:u),\cdots,H(x_m:u))\rangle$$

$$=\sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_m\langle P^*(f(Q_\alpha(x)),\cdots,L'(x_m))\rangle. \qquad (1\cdot29'')$$

From (1·25) and (1·26) we have:

$$\int_{-\infty}^{\sigma(x)}G(x,x')dx'=\frac{i}{8}\int_{-\infty}^{\infty}dx' j_{\rho:b}(x')[D_{\rho\tau:bc}(\partial'),\varepsilon(x-x')]$$

$$\times R_{\alpha\beta}(\partial)D_a\Delta(x-x')\frac{\partial j_{\tau:c}(x')}{\partial Q_{\beta:a}}. \qquad (1\cdot31)^{**)}$$

Thus we can see that $G(x,x')$ has the form (1·17′) and so satisfies the requirement (ii). In (1·31) the derivation operator $D_{\rho\tau:bc}(\partial')$ operates not only on $\varepsilon(x-x')$ but also on $\Delta(x-x')\partial j_{\tau:c}(x')/\partial Q_{\beta:a}$. When we derive (1·31) we make the partial integration and use (1·25).

Thus, *the unitary transformation* $U(\sigma)$ *and the interaction Hamiltonian are obtained by use of* (1·19).

We have shown in I that $H(x:u)$ *obtained from* (1·19) *satisfies the integrability condition* (1·9)[1].

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* Thus, we have now proved that (1·29′), which has been frequently adopted for the interactions of general types, gives the correct S-matrix[1),5)].

** In (1·31) we neglect the term, which comes from the lower limit $(\sigma(-\infty))$, in the course of the partial integration because it does not depend on $\sigma(x)$. In fact, this term, which may be denoted as $C(x,-\infty)$, can be removed when we use the following form for $Q_\alpha(x,\sigma)$ instead of (1·17):

$$Q_\alpha(x,\sigma)=Q_\alpha(x)+\int_{-\infty}^{\sigma}D_a R_{\alpha\beta}(\partial)\Delta(x-x')j_{\beta:a}(x')dx'+g^2\int_{-\infty}^{\sigma}G(x,x')dx'-C(x,-\infty), \qquad (1\cdot17'')$$

and we can derive (1·19) also from (1·17″).

It must be noted that the derivation operator in (1·23) is not $R_{\alpha\beta}'(\partial)$ in (1·4′) but $R_{\alpha\beta}(\partial)$ in (1·5).

Although we can obtain $H(x:n)$ and $Q_\alpha(x,n)$ as the power series of g, it is very difficult to determine their region of convergence. Therefore we assume the convergence for $H(x:n)$ and $Q_\alpha(x,\sigma)$ in what follows (assumption of convergence).

When we introduce the energy-momentum four vector T_μ in the Heisenberg representation according to relations

$$\left.\begin{aligned} \underline{T}_\mu &\equiv -i\int_\sigma \underline{T}_{\mu\nu}(x')\,d\sigma_\nu', \\ T_{\mu\nu}(x) &\equiv U^{-1}(\sigma)\,\{T_{\mu\nu}^{0}(x)-\delta_{\mu\nu}H(x:n)\}\,U(\sigma), \end{aligned}\right\} \tag{1·32}$$

where $T_{\mu\nu}^{0}$ is the energy-momentum tensor of the free field in the interaction representation, then, T_μ *satisfies the conservation law and has the role of the displacement operator for the Heisenberg operator*:

$$-\partial_\mu \underline{F}(x)=[\underline{F}(x),\,\underline{T}_\mu]\,. \tag{1·33}$$

The proof of (1·32) was given in Appendix III of I.

Thus we see that $(-iT_4)$ is the total Hamiltonian in the Heisenberg representation, which, as well be exemplified in § 3, agrees with the total Hamiltonian obtained by Heisenberg-Pauli's method.

When $H(x:n)-(-L'(x))$ contains the normal independent term, the usual method for determining $H(x:n)$ based on the integrability condition is incomplete. We can see from (1·20) and (1·19) that, $H(x:n)-(-L'(x))$ contains the normal independent terms, when the derivation operator $\Box$ appears in $R_{\alpha\beta}(\partial)$ or in the interaction Lagrangian. The example of the former case was given in § 5 of I for the Duffin-Kemmer's meson under the electromagnetic interaction and the latter case is concretely exemplified in § 3.

For the case, in which the derivation operator in $\underline{L}'(x)$ is at most of the first order*) and the spin of the field is at most 1,

$$G(x,x')=0.$$

When $L'(x)$ contains the derivation operators of the higher degree or the field operators with higher spin $(s>1)$, $H(x:n)$ and $Q_\alpha(x,\sigma)$ are the infinite series of terms containing the higher derivatives. Then, the higher derivatives sometimes appear companying with the increase of power with respect to g in the power series of $G(x,x')$. Then, it may happen that $H(x:n)$ and $Q_\alpha(x)$ of (1·15′) are not the local but the non-local quantities on the space-like surface σ through x because of higher derivatives. However, the non-localizability in the time-like direction does not appear, because the time derivatives with high degrees can be eliminated by means of the free field equation for $Q_\alpha(x,\sigma)$.

As will be exemplified in § 3, when $\underline{L}'(x)$ is the non-local quantity spreading over the region with the radius l, the each term of the power series $H(x:n)$ is the similar

* The more complete condition for the requirement that $H(x:n)$ and $Q_\alpha(x,\sigma)$ are the quantities of the finite power of g, was shown in detail by Y. Katayama (private communication).

non-local quantity. Moreover, in such case $F(Q_\alpha(x,\sigma))$ in $(1\cdot15')$ is also the non-localizable functional of the local quantity $Q_\alpha(x,\sigma)$. That is, in the case of the interaction Lagrangian with higher derivatives, the property of the non-localizability appear in the relation between Q_α and the canonical independent variables.*) However, it must be noted that the non-localizability in the time-like direction does not appear in this case, too.

Lastly we note that the somewhat curious fact, that the higher derivatives in Lagrangian do not induce the increase of number of the canonical independent variables in our theory, can be understood by taking into account the situation, that the formalism based on the canonical independent variables $Q_\alpha(x,\sigma)$ corresponds to the theory of $Q_\alpha(x)$ with the field equations $(1\cdot12)$ and the auxiliary conditions $(1\cdot21)$ for higher derivatives $M(\partial)=\partial_\mu, \partial_\mu\partial_\nu\cdots$.

Hereupon, we have three methods to treat the system with higher derivatives, i.e.,

(i) We regard the part to be reduced to the Klein-Gordon equation as the free one and make use of the method discussed in this section.

(ii) We regard the part to be reduced to the equation of multiple harmonic oscillator, if it exists, as the free one and make use of the relation $(A\cdot3)$ which will be verified in Appendix I.

(iii) We treat from the beginning the system by the method of Heisenberg-Pauli[10] which will be illustrated in § 2.

(i) and (iii) are the equivalent formalisms. (ii) is the different method from (i). In (ii) the energy is not always positive definite**).

§ 2. Some examples of the quantization

This section is devoted to discuss the relation between the Heisenberg-Pauli's method and ours.

(i) Firstly we consider the system, for which Lagrangian is given as follows:

$$\underline{L}=-\underline{\bar{\psi}}(\gamma\partial+\varkappa)\underline{\psi}+\lambda\Box\underline{\bar{\psi}}\cdot\Box\underline{\psi}\,, \tag{2\cdot1}$$

where ψ is the spinor field operator.

In this case, according to regard whether the first term or all as the free part, we have two methods of quantization, and so we examine the relation of those methods and show the equivalency between the former and the Heisenberg-Pauli's methods.

From $(2\cdot1)$, one obtains:

$$\{-(\gamma\partial+\varkappa)+\lambda\Box^2\}\underline{\psi}(x)=0. \tag{2\cdot2}$$

We firstly consider the first term in $(2\cdot1)$ as the Lagrangian of the free field. In this formulation the commutation law is obtained from $(1\cdot4)$ as follows:

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* This fact suggests that the field $Q_\alpha(x,\sigma)$ describes only the motion of the center independent of the detailed internal structure expressed by Q_α-field.

** We shall note that we can develop the theory, in which the interaction representation is treated by the method (ii), along the similar line in this paper by introducing $Q_\alpha(x,\sigma)$, which satisfies (i) and (ii) in *P.* 5.

$$\{\psi(x), \bar{\psi}(x')\} = \frac{1}{i} S(x-x'), \tag{2·3}$$

where $\psi(x)$ satisfies

$$-(\gamma\partial + \varkappa)\psi(x) = 0 \tag{2·4}$$

and we calculate the interaction Hamiltonian from the stand point that we combine (2·2) with (2·4) by means of an unitary transformation. From (2·2), one finds

$$\underline{\psi}(x) = \psi(x) + \lambda \int_{-\infty}^{\infty} \square S^{\text{ret}}(x-x') \cdot \square' \underline{\psi}(x')\, dx', \tag{2·5}$$

and we define the quantity

$$\psi(x, \sigma) = \psi(x) + \lambda \int_{-\infty}^{\sigma} \square S(x-x') \cdot \square' \underline{\psi}(x')\, dx', \tag{2·6}^{*)}$$

which satisfies the same relations as (2·3) and (2·4). Therefore, we obtain the equation to determine the interaction Hamiltonian,

$$[\psi(x, \sigma), H(x'/\sigma : n)] = i\lambda \square S(x-x') \cdot \square' \underline{\psi}(x'), \tag{2·7}$$

and by successive use of the equation

$$\begin{aligned} \underline{\psi}(x) &= \psi(x/\sigma) + \frac{\lambda}{2} \int_{-\infty}^{\infty} [\square, \varepsilon(x-x')]\, S(x-x') \cdot \square' \underline{\psi}(x')\, dx' \\ &= \psi(x/\sigma) + \lambda \square (\gamma\partial - \varkappa) \underline{\psi}(x), \end{aligned} \tag{2·8}$$

the interaction Hamiltonian is determined in form of power series as follows:

$$H(x : n) = -\lambda \square \bar{\psi}(x) \cdot \square \psi(x) + 2\lambda^2 \varkappa^2 \bar{\psi}(x) \psi(x) - \lambda^3 \varkappa^{10} \bar{\psi}(x) \psi(x) + \cdots . \tag{2·9}^{**)}$$

When we express successively $\underline{\psi}(x)$ by $\psi(x, \sigma)$ by use of (2·8), then we have

$$\underline{\psi}(x) = f(\lambda \varkappa^3) \psi(x, \sigma)|_{x/\sigma}, \tag{2·10}$$

where we use the relation

$$(\gamma\partial + \varkappa)\psi(x, \sigma) = 0 \tag{2·11}$$

and $f(\lambda\varkappa^3)$ is the function of $\lambda\varkappa^3$ which tends to 1 for $\lambda = 0$. The commutation relation of $\underline{\psi}(x)$ is obtained by means of the commutation relation of $\psi(x, \sigma)$ as follows:

$$\begin{aligned} [\psi_\alpha(\boldsymbol{r}, t), \psi_\beta^*(\boldsymbol{r}', t)]_+ &= f(\lambda\varkappa^3) f^*(\lambda\varkappa^3) [\psi_\alpha(x, \sigma), \psi_\beta^*(x', \sigma)]_{+, x, x'/\sigma(t)} \\ &= \frac{1}{i} f(\lambda\varkappa^3) f^*(\lambda\varkappa^3) \{\gamma_4 S_{\alpha\beta}(x-x')\}_{t=t'} \\ &= f(\lambda\varkappa^3) f^*(\lambda\varkappa^3) \delta(\boldsymbol{r}-\boldsymbol{r}'). \end{aligned} \tag{2·12}$$

Since ff^* is always positive, as seen in (2·12), we have no negative energy.

* In this case we can take $G(x, x') = 0$ in (1·17), as will be shown later.

** The first term of (2·9) can be rewritten in the form which has not $\square$ by use of the equation of motion, but we adopt above form in order to compare with the interaction Lagrangian.

Now, let us prove the equivalence between the Heisenberg-Pauli's method and ours. Firstly, we must decide the suitable independent canonical variable of this system, because we can not eliminate the time derivatives by using $\underline{\psi}$ as an independent canonical variable in the formalism of Heisenberg-Pauli. For simplicity we denote $\psi(x/\sigma)$ by $\varphi(x)$, i.e.,

$$\varphi(x)=\underline{\psi}(x)-\lambda\Box(\lambda\partial-\varkappa)\underline{\psi}(x) \tag{2·13}$$

$$=\underline{\psi}(x)-\frac{\lambda}{2}([\Box+\varkappa^2)(\gamma\partial-\varkappa)\underline{\psi}(x)+O(\lambda^2). \tag{2·14}$$

By using (2·14) we can rewrite the Lagrangian (2·1) in the following form :

$$\underline{L}=\lambda\Box\overline{\underline{\psi}}\cdot\Box\underline{\psi}-\overline{\underline{\psi}}(\gamma\partial+\varkappa)\underline{\psi}$$

$$=-\overline{\varphi}(\gamma\partial+\varkappa)\varphi+\lambda\varkappa^4\overline{\varphi}\varphi+O(\lambda^2) \tag{2·15}$$

$$=-\overline{\varphi}(\gamma\partial+\mu)\varphi\,. \tag{2·16}$$*)

(2·15) perfectly coincides with (2·16) up to the second term, when μ is the constant

$$\mu=\varkappa-\lambda\varkappa^4+\frac{9}{2}\lambda^2\varkappa^7+\cdots\,. \tag{2·17}$$

In fact we can expect this result through the following discussion. From (2·2) we can see that $\underline{\psi}$ satisfies the equation

$$(\gamma\partial+\mu)\underline{\psi}=0, \tag{2·18}$$

$$\lambda\mu^4+\mu-\varkappa=0. \tag{2·19}$$

According to the general theory of the 4th order equation, the solution of (2·19) is easily obtained. However, since we adopt the solution which tends to $\varkappa$ for $\lambda\to 0$, we shall discuss the first several terms of the Taylor series expanded around $\lambda=0$.

If we differentiate (2·17) with respect to λ and substitute $(\lambda=0, \mu=\varkappa)$ into it, then we obtain

$$\mu'|_{\lambda=0}=-\varkappa^4,$$

$$\mu''|_{\lambda=0}=9\varkappa^7, \cdots\,.$$

Thus we find that μ agrees with (2·17). Since (2·13) and (2·18) give the equation $(\gamma\partial+\mu)\varphi=0$, we can expect the equivalence of (2·15) and (2·16). The series (2·17) have a region of convergence for λ and $\varkappa$, because the equation (2·19) has the real root, in which case μ can be expressed compactly as a function of λ and $\varkappa$. We can easily seen from (2·15) and (2·16) that no time derivative appears in the total Hamiltonian $\varphi^*\varphi-L$ and that its interaction part $(-\lambda\varkappa^4\overline{\varphi}\varphi+\cdots)$ has the same form as (2·9). Thus, (2·16) shows that our formalism coincides with the Heisenberg-Pauli's method, in which φ is used as an independent canonical variable.

Besides the above consideration, we have another method of quantization of (3·1), i.e., Pais-Uhlenbeck's (or Ostrogradski's) method, which will be discussed in detail in

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* $\overline{\varphi}\equiv\varphi^*\gamma_4$

Appendix I. In this method, by factorizing (2·2) we have

$$\{-(\gamma\partial+\varkappa)+\lambda\Box^2\}\underline{\psi}(x)=\lambda\prod_{i=1}^{4}(\gamma\partial+a_i)\underline{\psi}(x)=0. \tag{2·20}$$

In order that (2·20) is to be reduced to (1·2′) with $n=4$, the relation $\lambda\neq 0$ is necessary. If $\lambda\neq 0$, we have the commutation relation

$$[\underline{\psi}_\alpha(x),\overline{\underline{\psi}}_\beta(x')]_+=i\frac{1}{\lambda}\sum_{i=1}^{4}a_i S_{i\alpha\beta}(x-x'), \tag{2·21}$$

where

$$S_i(x)=-\frac{i}{(2\pi)^3}\int(i\gamma k-a_i)\frac{1}{2K_i}[e^{i(k_l x_l-K_i t)}-e^{-i(k_l x_l-K_i t)}]\,d^3k. \tag{2·22}^{*)}$$

If we put $t=t'$, then we have

$$[\underline{\psi}_\alpha(\boldsymbol{r},t),\overline{\underline{\psi}}(\boldsymbol{r}',t)]_+=-\frac{1}{\lambda}\sum_{i=1}^{4}a_i\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2·23}$$

Since a_i are the coefficients of partial fraction of $\{\lambda\Box^2-(\gamma\partial+\varkappa)\}$, $\sum_{i=1}^{4}a_i$ is clearly zero. Therefore the commutation relation is

$$[\underline{\psi}_\alpha(\boldsymbol{r},t),\overline{\underline{\psi}}_\beta(\boldsymbol{r}',t)]_+=0. \tag{2·24}$$

As shown here, the method, by which the first term of Lagrangian (2·1) is regarded as a free part and is quantized through the interaction representation, is quite different from the method, by which the total of Lagrangian is quantized as a whole, i.e., the Ostrogradski's one.

(ii) As the somewhat complicated example we adopt the following Lagrangian:

$$\underline{L}=-\overline{\underline{\psi}}(\gamma\partial+\varkappa)\underline{\psi}+g\{\overline{\underline{\psi}}\partial_\mu\underline{\psi}\cdot B_\mu-(\partial_\mu\overline{\underline{\psi}})\cdot\underline{\psi}B_\mu\}. \tag{2·25}$$

In (2·25) $\underline{\psi}$ is the lepton field and B_μ is the sum of terms of products of two heavy particle field operators with no derivatives**).

Firstly we shall apply the Heisenberg-Pauli's method in (2·25). Then we must find the canonical independent variables (φ, Π) so that (i) $\partial_4\varphi$ and $\partial_4\Pi$ do not appear in the total Hamiltonian and (ii) they satisfy the constraint $i\Pi=-\varphi^*$ which is required for the quantization of the spinor field. We cannot use $\underline{\psi}$ as the canonical independent variable φ, because in such formalism we cannot eliminate the $\partial_4\overline{\underline{\psi}}$ of the second term of the interaction Lagrangian in (2·25). We can eliminate it by using the following $\varphi(x)$ as the canonical independent variable:

$$\varphi=\underline{\psi}-g\gamma_4\underline{\psi}B_4-\frac{1}{2}g^2\underline{\psi}B_4^2+O(g^3), \tag{2·26}$$

---

* $K_i\equiv(k_l k_l+a_i^2)^{1/2}$.

** For the simplicity we consider the case in which B_μ is made to be the Hermite operator, although our discussion in this section can be extended in more general cases, for example, the Konopinski-Uhlenbeck's case.

and so

$$\underline{\psi}=\varphi+g\gamma_4\varphi B_4+\frac{3}{2}g^2\varphi B_4^2+O(g^3). \tag{2·27}$$

In fact, substituting (2·27) into (2·25), we have

$$\underline{L}=-\overline{\underline{\psi}}\gamma_4\partial_4\varphi+g\overline{\underline{\psi}}\partial_4\varphi\cdot B_4+\frac{g^2}{2}\overline{\underline{\psi}}\gamma_4\partial_4\varphi\cdot B_4^2-\overline{\underline{\psi}}(\gamma_k\partial_k+\varkappa)\varphi+g\{\overline{\underline{\psi}}\partial_k\varphi-\partial_k\overline{\underline{\psi}}\cdot\varphi\}B_k$$

$$-g\overline{\underline{\psi}}(\gamma_k\partial_k+\varkappa)\gamma_4(\varphi\cdot B_4)-\frac{3}{2}g^2\overline{\underline{\psi}}(\gamma_k\partial_k+\varkappa)\varphi\cdot B_4^2-g^2\overline{\underline{\psi}}\gamma_k\varphi\cdot B_4\partial_k B_4$$

$$+g^2[\overline{\underline{\psi}}\gamma_4\partial_k\varphi-\partial_k\overline{\underline{\psi}}\cdot\gamma_4\varphi]B_4B_k+O(g^3), \tag{2·28}*$$

besides the terms of completely divergent form which have no physical effects. Then, using the canonical conjugate quantity

$$i\Pi=\partial\underline{L}/\partial\partial_4\varphi=-\overline{\underline{\psi}}\gamma_4+g\overline{\underline{\psi}}B_4+\frac{g^2}{2}\overline{\underline{\psi}}\gamma_4B_4^2+O(g^3), \tag{2·29}$$

we can easily show that the Hamiltonian

$$H=i\Pi\partial_4\varphi-\underline{L} \tag{2·30}$$

contains no time derivatives. From (2·29) we have

$$i\Pi=-\varphi^*. \tag{2·31}$$

Therefore, (φ, Π) are the canonical independent variables in the Heisenberg representation, and so we can introduce the commutation relation

$$[\varphi(\boldsymbol{\nu}, t), \Pi(\boldsymbol{\nu}', t)]_+=i\delta(\boldsymbol{\nu}-\boldsymbol{\nu}'). \tag{2·32}$$

As is well-known the total Hamiltonian H is the time-displacement operator, i.e., we have

$$[\underline{F}(x), \int Hd^3x]=-\partial_4F(x). \tag{2·33}$$

Now we shall apply our quantization method to (2·25). From (2·25) and (1·17) we have

$$\underline{\psi}(x)=\psi(x/\sigma)+g\gamma_\nu\underline{\psi}B_\mu n_\mu n_\nu+g^2\int_{-\infty}^{\sigma(x)}\underline{G}(x, x')dx', \tag{2·34}$$

$$\partial_\mu\underline{\psi}(x)=\partial_\mu\psi(x, \sigma)|_{x/\sigma}-\varkappa g\psi B_\nu n_\mu n_\nu+g(\partial_\rho\psi)B_\rho\gamma_\nu n_\nu n_\mu$$

$$+g\partial_\mu(\gamma_\nu\underline{\psi}B_\rho)n_\rho n_\nu+g\partial_\rho(\gamma_\nu\underline{\psi}B_\rho)n_\mu n_\nu$$

$$+g\partial_\nu(\gamma_\nu\underline{\psi}B_\rho)n_\rho n_\mu+2g\partial_\alpha(\gamma_\nu\underline{\psi}B_\rho)n_\alpha n_\rho n_\nu n_\mu. \tag{2·35}$$

From (1·19) we have,

* $\mu\equiv(1, 2, 3, 4)$: $k\equiv(1, 2, 3)$.

$$[\psi(x,\sigma), H(x'/\sigma:n)] = \frac{1}{i} g S(x-x') \partial_\mu' \underline{\psi}' \cdot B_\mu'$$

$$+ \frac{g}{i} \partial_\mu S(x-x') \underline{\psi}' B_\mu' + i g^2 G(x,x'). \qquad (2\cdot36)^{*)}$$

Substituting (2·34) and (2·35) into (2·36) we can find that $H(x':n)$ satisfying (2·36) exists only when we take $G(x,x')$ as follows:

$$ig^2 G(x,x') = -\frac{1}{2i} g^2 [S(x-x') B_\mu' \gamma_\nu (\partial_\mu' \psi(x',\sigma) B_\rho') n_\rho n_\nu$$
$$+ (\partial_\mu' S(x-x') B_\mu') \gamma_\nu \psi' B_\rho' n_\rho n_\nu + S(x-x') B_\mu' \gamma_\nu (\partial_\rho' \psi(x',\sigma) B_\rho') n_\mu n_\nu$$
$$+ (\partial_\rho' S(x-x') B_\mu') \gamma_\nu \psi' B_\rho' n_\mu n_\nu + S(x-x') B_\mu' \gamma_\nu (\partial_\nu' \psi(x',\sigma) B_\rho') n_\mu n_\rho$$
$$+ (\partial_\nu' S(x-x') B_\mu') \gamma_\nu \psi' B_\rho' n_\mu n_\rho + 2S(x-x') B_\mu' \gamma_\nu (\partial_\alpha' \psi(x',\sigma) B_\rho') n_\alpha n_\rho n_\nu n_\mu$$
$$+ 2(\partial_\alpha' S(x-x') B_\mu') \gamma_\nu \psi' B_\rho' n_\alpha n_\rho n_\nu n_\mu]_{x'/\sigma} + O(g^3) \qquad (2\cdot37)$$

and then we obtain

$$H(x:n) = g[\bar{\psi}(\partial_\mu \psi) \cdot B_\mu - (\partial_\mu \bar{\psi}) \cdot \psi B_\mu] + g^2 (\partial_\mu \bar{\psi}) \cdot \gamma_\nu \psi B_\rho B_\mu n_\rho n_\nu$$
$$+ g^2 \bar{\psi} \gamma_\nu (\partial_\mu \psi) \cdot B_\rho B_\mu n_\rho n_\nu - \varkappa g^2 \bar{\psi} \psi B_\nu B_\mu n_\nu n_\mu$$
$$+ \frac{g^2}{2} [\bar{\psi} B_\mu \gamma_\nu (\partial_\mu \psi B_\rho) - (\partial_\mu \bar{\psi} B_\rho) \cdot \gamma_\nu \psi B_\mu] n_\rho n_\nu$$
$$+ \frac{g^2}{2} [\bar{\psi} B_\mu \gamma_\nu (\partial_\rho \psi B_\rho) - (\partial_\rho \bar{\psi} B_\rho) \gamma_\nu \psi B_\mu] n_\mu n_\nu$$
$$+ \frac{g^2}{2} [\bar{\psi} B_\mu \gamma_\nu (\partial_\nu \psi B_\rho) - (\partial_\nu \bar{\psi} B_\rho) \gamma_\nu \psi B_\mu] n_\mu n_\rho$$
$$+ g^2 [\bar{\psi} \gamma_\nu B_\mu (\partial_\alpha \psi B_\rho) - (\partial_\alpha \bar{\psi} B_\rho) \gamma_\nu \psi B_\mu] n_\alpha n_\mu n_\nu n_\rho$$
$$+ O(g^3). \qquad (2\cdot38)$$

We have from (2·37)

$$g^2 \int_{-\infty}^{\sigma(x)} G(x,x') dx' = -\frac{1}{2} g^2 B_\mu \gamma_\sigma \gamma_\nu \psi(x/\sigma) B_\rho n_\mu n_\sigma n_\rho n_\nu + O(g^3), \qquad (2\cdot39)$$

and so

$$\underline{\psi}(x) = \psi(x/\sigma) + g \gamma_\nu \underline{\psi} B_\mu n_\mu n_\nu$$
$$- \frac{1}{2} g^2 \gamma_\sigma \gamma_\nu \psi(x/\sigma) B_\mu B_\rho n_\mu n_\sigma n_\rho n_\nu + O(g^3). \qquad (2\cdot40)$$

Similarily we obtain

---

* $\underline{\psi}' \equiv \underline{\psi}(x')$, $B_\mu' \equiv B_\mu(x')$.

$$\underline{\bar{\psi}}(x)=\bar{\psi}(x/\sigma)+g\underline{\bar{\psi}}\gamma_\nu B_\mu n_\mu n_\nu$$
$$-\frac{1}{2}g^2\bar{\psi}(x/\sigma)\gamma_\sigma\gamma_\nu B_\mu B_\rho n_\mu n_\sigma n_\rho n_\nu+O(g^3). \tag{2·41}$$

Comparing (2·40) and (2·41) with (2·26) and (2·29) respectively, we find

$$\left.\begin{aligned}&\psi(x/\sigma)=\varphi(x),\\&\bar{\psi}(x/\sigma)=\bar{\varphi}(x)=-iH\gamma_4,\end{aligned}\right\} \tag{2·42}$$

when σ is the flat surface. Thus we see that $(\psi(x,\sigma),\bar{\psi}(x,\sigma))_{x/\sigma}$ are the canonical independent variables in the Heisenberg representation and so our formalism is equivalent to Heisenberg-Pauli's quantization method. Moreover, we can of course show that T_4 defined in (1·32) agrees with $\int H d^3x$. Then (2·33) is an example for (1·33).

§ 3. An example of the non-local interaction

In the present section we take the following interaction as the example of the non-local one :

$$\underline{L'}=\frac{\lambda}{2}\{\underline{\bar{\psi}}\gamma_\mu\underline{\psi}\cdot\Box e^{-l^4\Box^2}\underline{A_\mu}+H.C.\}. \tag{3·1}^{*)}$$

(3·1) is the non-local quantity spreading over the region with the radius l. ψ and A_μ are the field operators of the electron and the electromagnetic fields, respectively.

The interaction Hamiltonian, $\psi(x,\sigma)$ and $A_\mu(x,\sigma)$ are determined by the following relations**):

$$\left.\begin{aligned}\psi(x,\sigma)=\psi(x)+\frac{\lambda}{2}\int_{-\infty}^{\sigma}S(x-x')\{\gamma_\mu\underline{\psi'}\Box' e^{-l^4\Box'^2}\underline{A_\mu'}\\+\Box' e^{-l^4\Box'^2}\underline{A_\mu'}\cdot\gamma_\mu\underline{\psi'}\}+\lambda^2\int_{-\infty}^{\sigma}G_1(x,x')d^4x',\\A_\mu(x,\sigma)=A_\mu(x)+\lambda^2\int_{-\infty}^{\sigma}G_2(x,x')d^4x'.\end{aligned}\right\} \tag{3·2}$$

$$[\psi(x,\sigma),H(x'/\sigma:n)]=\frac{\lambda}{2}S(x-x')\{\gamma_\mu\underline{\psi'}\Box' e^{-l^4\Box'^2}\underline{A_\mu'}\}+i\lambda^2G_1(x,x'), \tag{3·3}$$
$$[A_\mu(x,\sigma),H(x'/\sigma:n)]=\lambda^2G_2(x,x').$$

$G_1(x,x')$ and $G_2(x,x')$ are determined by the method of the successive approximation as follows :

$$G_1(x,x')=\frac{1}{4}S(x-x')\gamma_\mu\psi'\Box' e^{-2l^4\Box'^2}(\bar{\psi}'\gamma_\mu\psi')$$

---

* *H.C.* means the Hermite conjugate quantity.

** $\underline{A_\mu'}\equiv\underline{A_\mu}(x')$, $\underline{\psi'}\equiv\underline{\psi}(x')$

$$+\frac{1}{4}\{[\Box' e^{-\lambda^2\Box\Box'^2}(\bar{\psi}'\gamma_\mu\psi')\}\cdot S(x-x')\gamma_\mu\psi'$$

$$-\frac{1}{4}\{[\Box' e^{-\lambda^2\Box\Box'^2}S(x-x')\gamma_\mu\psi'\}(\bar{\psi}'\gamma_\mu\psi')$$

$$-\frac{1}{4}(\bar{\psi}'\gamma_\mu\psi')\{[\Box' e^{-\lambda^2\Box\Box'^2}S(x-x')\gamma_\mu\psi'\}+O(\lambda), \tag{3·4}$$

$$G^2(x,x')=0. \tag{3·5}$$

Therefore we have

$$\lambda^2\int_{-\infty}^{\sigma(x)}G_1(x,x')dx'=\frac{\lambda^2}{8}\int_{-\infty}^{\infty}(\bar{\psi}'\gamma_\mu\psi')\{[\Box' e^{-\lambda^2\Box\Box'^2},\varepsilon(x-x')]S(x-x')\gamma_\mu\psi'\}d^4x'$$

$$+\frac{\lambda^2}{8}\int_{-\infty}^{\infty}\{[\Box' e^{-\lambda^2\Box\Box'^2},\varepsilon(x-x')]S(x-x')\gamma_\mu\psi'\}\cdot(\bar{\psi}'\gamma_\mu\psi')d^4x'$$

$$+O(\lambda^3). \tag{3·6}$$

In (3·4) and (3·6) the derivation is operated on all the quantities in the bracket $\{\}$. (3·6) has the form (1·17′). However in this case (3·6) is the non-local quantity spreading over the region with the radius $l/2^{1/4}$ on the space-like surface.

From (3·5), (3·4) and (3·2) we have

$$\left.\begin{aligned}A_\mu(x/\sigma)&=A_\mu(x)-\lambda e^{-l^4\Box^2}(\underline{\psi}(x)\gamma_\mu\underline{\psi}(x)),\\ \psi(x/\sigma)&=\underline{\psi}(x)+\lambda^2\int_{-\infty}^{\sigma(x)}G_1(x,x')dx',\end{aligned}\right\} \tag{3·7}$$

and so from (3·3) we obtain

$$H(x:n)=-\frac{\lambda}{2}\{\bar{\psi}\gamma_\mu\psi\Box e^{-l^4\Box^2}A_\mu+H.C.\}$$

$$+\frac{\lambda^2}{4}[(\bar{\psi}\gamma_\mu\psi)\Box e^{-2l^4\Box^2}(\bar{\psi}\gamma_\mu\psi)+H.C.]+O(\lambda^4). \tag{3·8}$$

The first term in (3·8) of course agrees with $-L'(x)$, which is zero in the present case.

Although $L'(x)$ $(=0)$ satisfies the integrability condition, it is seen that the interaction Hamiltonian is not $(-L'(x))$ but (3·8). However, the various physical quantities can be easily calculated by means of (1·29′) and (1·29″). In the course of this calculation it must be noticed that we should not use the equation $\Box A_\mu=0$ (and so $L'=0$) before the derivation operators are taken out of the P^*-bracket.

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Appendix I

Interaction representation in Pais-Uhlenbeck's theory of non-local action

As was already mentioned in the previous section, when we take (1·2′) as the equation of the simple harmonic oscillator, we can also carry out the quantization by regarding the term which is contained in the bilinear form with the higher derivative in the Lagrangian as a free part. With regard to the quantization of the bilinear term with higher derivative, there are well-known methods, namely, the method of Ostrogradski and the harmonic method of Pais-Uhlenbeck[8], and both methods are equivalent as shown by Pais-Uhlenbeck. We shall, in this section, confirm that the relation $\Lambda(\partial)R(\partial)=(\Box-\varkappa^2)$, which is valid in the case of the usual local field can be extended as follows for the Pais-Uhlenbeck's case of the field equation containing higher derivatives (i.e., even in the case of the non-local action):

$$\Lambda(\partial)R(\partial)=F(\Box) \tag{A·1}$$

$$F(\Box)=\prod_{i=1}^{n}(\Box-\varkappa_i^2), \tag{A·2}$$

where all $\varkappa_i^2$ are assumed to be real and distinct in what follows.

In order to prove (A·1) we write the commutation relation in the following form:

$$[Q(x),Q(x')]_{\pm}=iR(\partial)\Delta(x-x'). \tag{A·3}$$

In (A·3) $\Delta(x)$ is the modified delta-function, the definition of which is given in Appendix II.

Using the method in 2 of the paper I or Appendix of the present paper we can prove that (A·1), (A·2) and (A·3) give the commutation relation satisfying the requirement of S.H.O. with the principal axes

$$q_i(k)\equiv\{C_i/R(k)\}^{1/2}Q(k). \tag{A·4}$$

Now we shall show that the Pais-Uhlenbeck's method is equivalent to the quantization method based on (A·3). We shall introduce $Q_i(x)$ according to Pais-Uhlenbeck's method as follows:

$$\begin{aligned}Q(x)&=\frac{1}{(2\pi)^3}\int Q(k)\delta(F(-k^2))e^{ikx}d^4k \\ &=\sum_{i=1}^{n}\eta_i C_i Q_i(x),\end{aligned} \tag{A·5}$$

where

$$Q_i(x)=\frac{1}{(2\pi)^3}\int Q(k)\delta(k^2+\varkappa_i^2)e^{ikx}d^4k. \tag{A·5′}$$

We have from (A·3) and (A·5)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\eta_i C_i\eta_j C_j[Q_i(x),Q_j(x')]_{\pm}=i\sum_{i=1}^{n}\eta_i' C_i R(\partial)\Delta_i(x-x'). \tag{A·6}$$

Now consider only the term of $i \neq j$ in (A·6), so this term is always zero, because, since

$$[Q_i(x), Q_j(x')]_{\pm}$$

must be the function of $x-x'$, only the part $-k=k'$ has non-vanishing value, in which k and k' are the Fourier momentum of $Q_i(x)$ and $Q_j(x')$, respectively, i.e.,

$$\delta(k^2+\varkappa_i^2)\,\delta(k'^2+\varkappa_j^2)=0 \qquad \text{for } \varkappa_i \neq \varkappa_j . \tag{A·7}$$

Therefore we have

$$[Q_i(x), Q_j(x')]=i\eta_i'\,\delta_{ij}\frac{R(\partial)}{C_i}\Delta_i(x-x'). \tag{A·8}$$

We can easily show by using (A·4) that Pais-Uhlenbeck's method is based on this commutation relation.

Appendix II
Δ-function in non-local action

From the relation

$$F(\Box)\bar{\Delta}(x)=-\delta(x) \tag{A·2·1}$$

we have

$$\bar{\Delta}(x)=\frac{(-1)^{N-1}}{(2\pi)^4}P\int\frac{e^{ikx}}{\prod_{i=1}^{N}(k^2+\varkappa_i^2)}d^4k . \tag{A·2·2}$$

By use of the relation

$$\bar{\Delta}(x)=-\frac{1}{2}\,\varepsilon(x)\Delta(x) \tag{A·2·3}$$

we obtain

$$\begin{aligned}\Delta(x)&=-\frac{i(-1)^{N-1}}{(2\pi)^3}\int\delta\Big(\prod_{i=1}^{N}(k^2+\varkappa_i^2)\Big)e^{ikx}\varepsilon(k_0)\,d^4k\\ &\equiv\Delta^+(x)+\Delta^-(x),\end{aligned} \tag{A·2·4}$$

because, from the relations, which are easily proved,

$$\prod_{i=1}^{N}(k^2+\varkappa_i^2)^{-1}=\sum_{i=1}^{N}\eta_i C_i(k^2+\varkappa_i^2)^{-1}, \tag{A·2·5}$$

$$\delta\Big(\prod_{i=1}^{N}(k^2+\varkappa_i^2)\Big)=\sum_{i=1}^{N}\eta_i C_i\,\delta(k^2+\varkappa_i^2), \tag{A·2·6}$$*)

where

* These relations are already derived by W. Thirring. Phys. Rev. **77** (1950), 570.

$$\left.\begin{aligned}\eta_i C_i &= \prod_{j\neq i}(\varkappa_j^2-\varkappa_i^2)^{-1}\\ C_i &= \prod_{j\neq i}|\varkappa_j^2-\varkappa_i^2|^{-1}\end{aligned}\right\} \tag{A·2·7}$$

$$-\eta_i=\eta_{i+1}\,, \tag{A·2·8}$$

we have

$$\left.\begin{aligned}\bar{\Delta}(x)&=\sum_i \eta_i' C_i \bar{\Delta}_i(x)\\ \Delta(x)&=\sum_i \eta_i' C_i \Delta_i(x)\end{aligned}\right\}, \tag{A·2·9}$$

in which we use the notation

$$\eta_i'=(-1)^{N-1}\eta_i \tag{A·2·10}$$

and $\bar{\Delta}_i(x)$ and $\Delta_i(x)$ are the ordinary $\bar{\Delta}$ and Δ functions with mass $\varkappa_i$, respectively.

Since η_i (or η_i') alternatively changes sign for $i=1, 2, \cdots$, as easily seen from (A·2·8), we shall normalize as $\eta_i'=1$ for $i=1$.

Appendix III

The improvement of the proof for (1, 5)

We shall improve some defects in paper I concerning the proof of (1, 5). In order that the requirements of S.H.O. are satisfied it is necessary to show that the action integral of the Q_α-field agrees with that of S.H.O. not only on the path of real motion ($k_\mu^2+\varkappa^2=0$) but also on the all possible paths. Such the equivalence was somewhat implicitly proved in paper I because we did not use the relation $(k_\mu^2+\varkappa^2)\delta(k_\mu^2+\varkappa^2)=0$, although we explicitly introduced $\delta(k_\mu^2+\varkappa^2)$ in action integral. Now we shall show the equivalence more explicitly.

The action integral of S.H.O. is

$$\int L\,dt=\frac{1}{2}T^2V\int d^3x\,dt\,q_r(x)(\Box-\varkappa^2)q_r(x) \tag{A·3·1}$$

$$=-\frac{1}{2}T^2V\int\frac{d^4k}{(2\pi)^4}(k_\mu^2+\varkappa^2)(q_r^-(k)q_r^+(k)+q_r^+(k)q_r^-(k))\,, \tag{A·3·2}$$

where

$$q_r(x)=\frac{1}{(2\pi)^4}\int d^4k\,q_r(k)e^{ik_\mu x_\mu},$$

and $q_r^+(k)$, $q_r^-(k)$ are the positive and negative frequency parts of $q_r(k)$, respectively.

Constants V and T are volume of whole space and the length of time in the world respectively and defined as follows:

$$\left.\begin{aligned}V&=\left[\int e^{ik_1r_1}d^3x\right]_{k=0}=(2\pi\,\delta(0))^3,\\ T&=\left[\int e^{ik_0t}dt\right]_{k_0=0}=2\pi\,\delta(0).\end{aligned}\right\} \tag{A·3·3}$$

We have the well-known calculation rule

$$\left.\begin{aligned}\frac{1}{V}\sum_k f(k)&=\frac{1}{(2\pi)^3}\int d^3k\, f(k),\\ \frac{1}{T}\sum_{k_0} f(k_0)&=\frac{1}{2\pi}\int dk_0 f(k_0).\end{aligned}\right\} \qquad (A\cdot3\cdot4)$$

Since canonical conjugate quantity of $q_r(x)$ is $T^2V\partial q_r(x)/\partial t$ we have the following commutation relation of S.H.O.:

$$T^2V[q_r(x_i,t),\dot{q}_s(x_i',t)]_{\pm}=i\delta_{rs}\delta^3(x_i-x_i'). \qquad (A\cdot3\cdot5)$$

(A·3·5) can be derived from the following relation by using (A·3·4):

$$\left.\begin{aligned}k_0[q_r^+(k),q_s^-(k')]_{\pm}&=\delta_{kk'},\\ [q_r^+(k),q_s^+(k')]_{\pm}&=[q_r^-(k),q_s^-(k')]_{\pm}=0.\end{aligned}\right\} \qquad (A\cdot3\cdot6)$$

The equivalence of (A·3·2) and the action integral of Q_α-field

$$\int L\,dt=T^2\int\frac{d^4k}{(2\pi)^4}[Q_\alpha^-(k)\Lambda_{\alpha\beta}(k)Q_\beta^+(k)+Q_\alpha^+(k)\Lambda_{\alpha\beta}(-k)Q_\beta^-(k)] \qquad (A\cdot3\cdot7)$$

gives the following relation

$$\Lambda_{\alpha\beta}(\partial)\overline{R}(r:\partial)U_{\alpha r}{}^*U_{\beta s}=(\Box-\varkappa^2)\delta_{rs}, \qquad (A\cdot3\cdot8)$$

where we transform $q_r(k)$ into $Q_\alpha(k)$ by means of suitable quantity $\overline{R}(r:k)$ and unitary transformation $[U_{\alpha r}]$:

$$\sum_r(\overline{R}(r:k)V/2)^{1/2}U_{\alpha r}{}^*q_r^+(k)=Q_\alpha^+(k), \quad \text{etc.}. \qquad (A\cdot3\cdot9)$$

(A·3·8) gives the relation

$$\Lambda_{\alpha\beta}(\partial)R_{\beta\gamma}(\partial)=(\Box-\varkappa^2)\delta_{\alpha\gamma}, \qquad (A\cdot3\cdot10)$$

where

$$R_{\alpha\beta}(k)=\sum_r U_{\alpha r}U_{\beta r}{}^*\overline{R}(r:k). \qquad (A\cdot3\cdot11)$$

Hitherto we have not used the relation $k_\mu{}^2+\varkappa^2=0$ and so (A·3·10) is the complete condition for the requirement of S.H.O.. Now we can show that (A·3·6) gives the commutation relation (1·4) on the path of real motion ($k_\mu{}^2+\varkappa^2=0$). In fact, we have from (A·3·6) and (A·3·9)

$$[Q_\alpha^+(k),Q_\beta^-(k')]_{\pm}=R_{\alpha\beta}(k)\delta^3(k_i-k_i')/2k_0. \qquad (i=1,2,3) \qquad (A\cdot3\cdot12)$$

$Q_\alpha^+(k)$, $Q_\alpha^-(k)$ are the positive and negative frequency parts of $Q_\alpha(k)$ which is defined as follows*):

$$Q_\alpha(x)\equiv\frac{T}{(2\pi)^4}\int d^4k\,Q_\alpha(k)e^{ik_\mu x_\mu}$$

---

* $Q_\alpha(k)$ corresponds to $Q_\alpha(k)/2k_0$ in paper I.

(A·3·12) gives the commutation relation (1·4) when k_μ (and k_μ') satisfies the relation $k_\mu^2+\varkappa^2=0$.

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Note added in proof:

It is easily seen that $Q_\alpha(x,\infty)$ and S-matrix (1·30) agree with those of Møller-Kristensens' (Dansk. Vidensk. Sels. **27** (1952), No. 7) and Bloch's (Dansk. Vidensk. Sels. **27** (1952), No. 8) theories under the assumption of the adiabatically switching in and off (at $\sigma=\infty$) of the interactions. Moreover, although in our theory the non-local interactions are written in the differential form, it is expected that we can easily rewrite our theory by means of the integral form of the non local interactions.

The proof of the integrability condition in Appendix II of Paper I is also valid for the non-local interaction because $Q_\alpha(x)$ is the functions only on x and σ.

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The Theory of V-particles

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In order to explain the inconsistency between the production and the decay probabilities of V^0, it is proposed to assume that V^0 has an unknown degeneracy. From the lifetime and the production rate of V^0, the state density of V^0 is estimated to be of the order $\sim 10^{10}\, m_\pi c^2$, and from the dispersion of the Q-values for the decay of V^0, the total number of the internal states of V^0 is estimated to be very roughly of the order $\sim 10^{10}$. Using these results, the cross section for the production of V^0 by the nucleon-nucleon collision is calculated.

§ 1. General discussions

In the few years since V-particles were first observed in cosmic-ray penetrating showers[1], more than four hundreds of V-particles have been photographed, and we may think that the following principal properties of V-particles are definitely established: (1) V-particles result from the impact of mesons and probably also of nucleons, upon nuclei. (2) V-particles are generally produced singly and not in pairs. (3) In the great majority of cases V-particles decay by the production of a proton and a negative pi meson. (4) The data are consistent with the assumption of a two-body decay for a majority of V-particles. (5) On the assumption of a decay into a proton and a negative pi meson, the Q values range from about 10 Mev to about 100 Mev. The distribution curve of the Q-values has two maximums, a great maxima in the neighbourhood of 35 Mev, and a small maximum in the neighbourhood of 75 Mev. (6) The rest mean lifetime of V-particles is about 2.5×10^{-10} sec[2-9].

The rate of existence of V-particles in cosmic-ray penetrating showers is of the order of a few percent comparing with that of pi mesons[2]. So in these high energy collisions, they seem to be produced with reasonable probability. Therefore, in these events some reasonably strong interactions must be involved. Considering the possibility that V-particles decay in tracing back the way in which they were created, the question arises as to how they can last so long.

Three alternative explanations have been proposed which may be schematized as (1) hypothesis of indirect production with introduction of one or several intermediary particles, (2) hypothesis that V-particles are not produced singly, but produced accompanying the heavy brother, and (3) isobaric states of nuclei with very high angular momenta[10].

The first turned out to be the correct explanation of the mu meson. This, however, does not work well for V-particles. Let the hypothetical parent of V^0 be called x and be strongly coupled to nucleons. If the decay of x into V^0 were extremely rapid, the

latter could recreate x virtually, and since it is strongly coupled through nucleons to still lighter particles, it would decay preferentially to these, and the heavy V-particles would not in fact be produced. If, on the other hand, the decay of x into V^0 were very slow, then x itself through its interaction with nucleons could again decay into still lighter particles than V^0.

The second explanation assumes that a heavy mate is associated in the production of V-particles, and there is therefore not enough energy to return through this state to the unstable nucleon system. It is true that there is little experimental evidence for the production of pairs of these particles, but, considering the fact that different heavy unstable particles are produced with almost the same rate, we must think it is still an open question.

Isobaric states of nucleon with very high angular momenta should be metastable both against radiation and meson decay, and may account for the long lifetimes. Although it depends a little on what we take as the range of the interaction, angular momentum of the order 6 to $8\hbar$ will guarantee the lifetime of 10^{-10} sec for the metastable states of nucleons which are predicted by the so-called strong coupling theory. However, the same strong coupling theory then also predicts the existence of the singly or doubly charged particles which in other points have almost the same properties as V-particles.

It seems to me that we must accept the reality as it is. The stability of V-particles shows that V^0 occupies a large volume in phase space, or that V^0 has a large number of accessible internal states, and that those states are so similar as cannot in practice be separated from each other. On this idea, I should like to postulate that V^0 has an unknown state density ρ. Using this quantity as an arbitrary parameter, we may be able to explain the properties of V^0 consistently. In the following section some examples of the calculations of this sort are treated to make clear the situation concerning with this hypothetical degeneracy, which, for a while, we have to think as a mere phenomenological concept.

§ 2. Production and decay of V^0

Experimentally V^0 decays spontaneously with a lifetime of about 2.5×10^{-10} seconds. In a great majority of cases we may think it decays into a proton and a negative pi meson. Therefore, we shall postulate the reactions

$$V^0 \rightleftarrows p + \pi^- .$$

Assuming V^0 to be a particle with $1/\cdot 2\hbar$ spin and obeying the Dirac equations, and the pion described by a pseudoscalar field with a pseudoscalar coupling, the interaction energy suitable to produce the reactions would therefore be

$$g\int (\Pi \tilde{P} \beta \alpha_1 \alpha_2 \alpha_3 V + \Pi^* \tilde{V} \beta \alpha_1 \alpha_2 \alpha_3 P) d\Omega ,$$

where g is a suitable interaction constant having the dimensions of an electric charge, Π and Π^* denote the amplitude of the pion field and its complex conjugate, P and V are

the amplitude of the proton field and that of the V° field, $\tilde{P}$ and $\tilde{V}$ their Hermite conjugates. Using Heavyside units for the pion field, Π and Π^* will be

$$\Pi=\sum_s \frac{\hbar c}{\sqrt{2\Omega E_s}}(a_s-b_s^*)e^{+(i/\hbar)p_s\cdot r}$$

$$\Pi^*=\sum_s \frac{\hbar c}{\sqrt{2\Omega E_s}}(a_s^*-b_s)e^{-(i/\hbar)p_s\cdot r}$$

where Ω denotes the normalization volume a_s and b_s are the annihilation operators for the positive and the negative pions of energy E_s and momentum p_s or $-p_s$ respectively.

The calculation of the lifetime for the decay mode $V^{\circ}\to p+\pi^-$ is quite elementary. Using the perturbation theory, one will have

$$\frac{1}{\tau}=\frac{1}{4\pi}\left(\frac{g^2}{\hbar c}\right)\frac{cp}{\hbar}\frac{m_P}{m_V}[\gamma-1]$$

where p and γ are the momentum and γ of the secondary proton:

$$\gamma=\frac{1}{\sqrt{1-(v_P/c)^2}}=\frac{E_P}{m_Pc^2}.$$

In order to obtain the numerical value of τ from this equation, we must know the Q-value for this reaction. We shall use $Q=35$ Mev which corresponds to the larger maximum of the distribution of the empirical Q-values. Using this value for Q, one will have

$$\frac{1}{\tau}=\left(\frac{g^2}{\hbar c}\right)\times 5.18\times 10^{19}\ \text{sec}^{-1}.$$

Inserting in this equation $\tau=2.5\times 10^{-10}$ sec, we have

$$\left(\frac{g^2}{\hbar c}\right)=0.77\times 10^{-10}.$$

The production of V° by the impact of a negative pion upon a proton we shall consider as the reverse process of the disintegration of V°. The cross section σ for this process in the laboratory system is

$$\sigma=\frac{\pi}{2}\left(\frac{\hbar}{m_\pi c}\right)^2\frac{g^2}{\hbar c}\left(\frac{m_\pi c^2}{\beta\gamma}\right)\left[1-\frac{m_Vc^2}{E_V}\right]\rho,$$

where β and γ are those for the projected pion,

$$\beta=\frac{v_\pi}{c},\qquad \gamma=\frac{1}{\sqrt{1-\beta^2}}$$

and ρ is the state density of the internal states of V°. Inserting in this equation the suitable numerical values, we have

$$\sigma=m_\pi c^2\rho\times 2.8\times 10^{-30}\ \text{cm}^2.$$

Through his experiments on the artificial production of V° by bombarding negative pions upon a carbon target, Schein estimated the magnitude of σ to be of the order $\sim 10^{-28}$ cm^2 [11]. Using this value, we can very roughly estimate the order of magnitude of the state density ρ. One will have

$$m_\pi c^2 \rho \sim 10^{10}.$$

In this equation, for convenience' sake, $m_\pi c^2$ is multiplied on ρ to have a dimensionless quantity. However, $m_\pi c^2 \rho$ may have some physical meanings. Let us consider the total number of the internal states of V°. This quantity may be estimated to be of the order ρ times ΔE, the breadth of the dispersion of the Q-values. We shall put ΔE equal to the difference of the two maximums of the distribution of the Q-values.

$$\Delta E \sim 75 - 35 = 40\, Mev.$$

This value, however, is very roughly of the order of $m_\pi c^2 = 140$ Mev. Therefore, we may think that the total number of the internal states of V° is of the order $\sim 10^{10}$.

The production of V° by the impact of a proton upon a neutron may be one of the most interesting problems for the physicists who have a high-energy proton accelerator in their laboratory. Let us calculate the cross section for this reaction. In the center of mass system, one will have

$$d\sigma = \frac{1}{16\pi}\left(\frac{\hbar}{m_\pi c}\right)^2 \left(\frac{e_2 f}{\hbar c}\right)^2 \left(\frac{m_\pi c^2}{E_0}\right)^2 \frac{P_2}{P_1} F(E_0, \theta)\, \Delta E \rho \frac{d\omega}{4\pi},$$

where e_2 is the interaction constant between the pions and the nucleons, P_1 and P_2 are the momenta of the initial proton and the final V°, E_0 is the total energy, $\Delta E \rho$ the total number of the internal states of V°, and

$$F(E_0, \theta) = \frac{E_V E_P - c^2 P_1 P_2 \cos\theta - m_V m_P c^4}{E_V E_P - c^2 P_1 P_2 \cos\theta + 1/2(m^2_\pi - m^2_V - m^2_P)c^4} \times \frac{E'_P E_N - c^2 P_1 P_2 \cos\theta - m_P m_N c^4}{E'_P E_N - c^2 P_1 P_2 \cos\theta + 1/2(m^2_\pi - m^2_P - m^2_N)c^4},$$

where E_P and E_N are the energies of the initial proton and the neutron, E'_P and E_V are those of the final proton and V°, θ is the angle between the initial proton and the final V°. Inserting in it the suitable numerical values, we have in two extreme cases

$$\sigma \sim \left(\frac{m_\pi c^2}{E_0}\right)^2 \times 10^{-27}\text{cm}^2, \text{ for } P_2 \gg m_\pi c$$

and

$$\sigma \sim \left(\frac{\epsilon}{m_\pi c^2}\right)^{1/2} \times 10^{-27}\text{ cm}^2, \text{ for } P_2 \ll m_\pi c$$

where ϵ is the excess of E_0 over the threshold energy of this reaction.

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On the Structure of the Interaction of the Elementary Pasticles, IV

—— *On the Interaction of the Second Kind** ——

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By extending the method of the renormatization theory and making use of the similar idea as the radiation damping a possibility is examined of establishing the non-singular theory of interactions of the second kind. When using the propagation functions modified by the field reaction in place of the usual one we can obtain, in general, less singular theories. It is shown that for a given set of interactions, some of which belong to the second kind, we can always construct the non-singular theory if we further introduce, when necessary, auxiliary interactions of the second kind. Several examples of the non-singular theory obtained in this way are given and some remarks are further added in connection with the characteristic features of this method.

§ 1. Introduction and summary

Based on the use of the relativistically covariant formalism of the quantum theory of wave fields the renormalization theory aims at rewriting the theory in its equivalent non-singular form**. In this theory the effects of the reaction of self-fields are divided into two parts, i.e., the observable and the unobservable parts, respectively, and divergences appearing in the latter are removed by the procedure of renormalization to give definite finite results to be compared with the reality. As usually said, the validity of this method is justified in view of the great success in the field of quantum electrodynamics. Hence, there arises, in this connection, an important problem as to whether the renormalization procedure would give the non-singular form not only for quantum electrodynamics, but also for the theory which describes the behavior of all the elementary particles including, of course, the system of electron and electromagnetic fields.

As has previously been investigated in detail[1], when only the interactions of the first kind exist in the nature the renormalization prescription is successful in giving the closed non-singular theory, while it is no longer so if there exist interactions of the second kind. Concerning the renormalization method when applied to the most general system of the interactions of the first kind the detailed considerations have been given in (III), where it was found that although there are several possible rules for the separation

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* A preliminary report of this work has appeared in Prog. Theor. Phys. 8 (1952), 675.

** We shall call a theory "closed" or "completely convergent" according as all divergences thereof can be removed by the finite number of counter interactions or any divergence does not appear at all. Further, we call these two cases "non-singular" theories.

of divergences, which correspondingly lead to different finite answers, the unique finite results, however, can be expected under the consistent use of well-defined rules.

On the other hand, as was observed in (II) there are some evidences in favour of the existence of interactions of the second kind in connection with the phenomena of β-decay and multiple production of mesons. Furthermore, the interaction between nucleon and π-meson seems to favour this kind, too. Thus, we would be faced with the question whether it is possible to establish the closed theory of interactions of this kind in some way, say, by extending the renormalization method so as to be applicable to this case.

In the renormalization theory, however, we know that there are some inherent arbitrarinesses. In this theory, some parts of the secondary effect appearing in the higher order perturbation approximation are renormalized into the primary interactions. Since nothing is known at present about the laws that prescribe the actul forms of interactions permissible in the nature, we are left with a freedom in that we can choose the primary interactions so as not to contradict with the experimental facts*). By utilizing such circumstances we can expect that the non-singular theory may also be obtained for the case of interactions of the second kind.

If we have the theory in which the unitarity of the S-matrix is guaranteed, the strong singularity appearing in the lower order perturbation calculations may altogether be compensated by the strong damping effect coming from the higher order terms. Such a consideration was just the starting-point of Heitler[2] and Wilson[3] in their theory of radiation damping. As is easily seen from the dimensional consideration the field reaction due to the interactions of the second kind gives the large damping effect[1], and so we can expect that the non-singular theory may be obtained if we appropriately take into account the correspondingly strong field reaction due to the interaction of this kind**). Of course, as shown in (I) the renormalization method based on the usual perturbation approximation is invalid, so that in order to construct the closed theory for the interactions of this kind it will become necessary to adopt other approximation methods.

In this paper we have investigated such a possibility. For a given set of interactions of the first and the second kinds we have not yet found a method for establishing a non-singular theory. We can prove, however, that for a given set of interactions we can always get the non-singular theory if we further assume, when necessary, that the supplementary interactions of the second kinds (called in what follows "*auxiliary interaction*"), whose coupling constants are of finite values but of dimensions of large power of length, coexist together with the former interactions***). Furthermore, remarkable is the fact that in some cases the completely convergent theory can be obtained, in which it is unnecessary

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* In quantum electodynamics the primary interactions such as the current type interaction and the mass term have the correspondence theoretic basis, while terms such as $\lambda\phi^3$ and $\lambda\phi^4$ in the renormalizable meson theory have no such theoretical grounds.

** Thus, we can consider a rather paradoxical possibility that the existence of interactions of the second kind may better the convergence of the whole theory, which is divergent in the presence of interactions of the first kind only.

*** In some cases, the auxiliary interaction may be one of interactions which have primarily been given.

to introduce the counter terms with infinite coupling constants and any ultra-violet divegence does not appear. The secondary introduction of the strong interaction of the second kind may be regarded as corresponding to the use of counter terms in the renormalization theory and further the use of the auxiliary fields may also be considered as the original idea of the mixed field theory. As a result of the arbitrariness concerning the choice of the primary interactions our method leaves various possibilities in constructing the non-singular theories.

In our method to be developed in the followoing some parts of the field reaction due to the interaction of the second kind are beforehand taken into account and then the remaining parts are considered. Perhaps there may be many possible ways for such prescriptions*), but in this paper we shall consider the method in which the corrected propagatarion functions are used instead of the usual uncorrected ones**).

Let $\sum^{\alpha}$ be the proper self-energy part of the α-field in the m^{α}-th order perturbation approximation due to the auxiliary interaction, which may in general involves diverging parts. In order to extract the finite part $\sum_c^{\alpha}$ from it we can use the counter terms, which we beforehand introduce into the Lagrangian. Taking into account all the iteration processes of $\sum^{\alpha}$ the propagation function Δ_F^{α} of α-field is transformed into the modified one $\Delta_F^{\alpha'}$. Between these two functions there exists the following integral equation

$$\Delta_F^{\alpha'} = \Delta_F^{\alpha} + \Delta_F^{\alpha} \sum_c^{\alpha} \Delta_F^{\alpha'}, \tag{1·1}$$

from which we can obtain the formal solution

$$\Delta_F^{\alpha'} = \Delta_F^{\alpha} / (1 - \Delta_F^{\alpha} \sum_c^{\alpha}). \tag{1·2}$$

Corresponding to this modification, the external line U_i^{α} is to be replaced by

$$U_i^{\alpha'} = (1 + \Delta_F^{\alpha'} \sum_c^{\alpha}) U_i^{\alpha}. \tag{1·3}$$

Now, it is our proposal to calculate the S-matrix elements in terms of $\Delta_F^{\alpha'}$ as far as possible. In this case the calculations are made almost in the same way as usual, but we must set up some further rules for picking up the necessary graphs, lest we should overlook the contribution of some Feynman graphs of usual perturbation theory or count them more than once (§ 2, V). Further, we can always choose the auxiliary interactions such that the counter terms, which has been introduced in order to cancel the diverging parts of $\sum^{\alpha}$, plays the role of this cancellation only, so that these terms can be left out of consideration as far as we use the propagation functions $\Delta_F^{\alpha'}$. When the special types of interactions of the second kind are adopted as the auxiliary interaction, which contain two field quantities (henceforth callled "*two-vertex interaction*"), $\sum^{\alpha}$ does not diverge and so the corresponding counter terms become unnecessary.

In § 2 it is proved that when we use the m^{α}-th order self-energy parts for the correction of Δ_F^{α}, we can obtain the non-singular theory. In this case it is further supposed

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* For instance, higher order effects may be taken into account by using the corrected vertex parts. The formulation of this method seems, however, to be somewhat difficult. In this connection, see the forthcoming paper by one of the authors (S.K.).

** This method is to be regarded as a generalization of the proposal made by N. Hu, reference 4.

that our procedure also improves the degree of convergence of the perturbation series, since the higher order terms in this case give rise to very small effects in the high energy region*). For the case $m^\alpha > 2$ the above mentioned calculation rule becomes very complicated. Thus, it will further be shown that by adopting suitable auxiliary interactions we can get the non-singular theory even for the case $m^\alpha = 2$. In this case we can obtain in some cases the non-singular theory such that apart from the counter terms, which has been introduced to compeneate the term $(\sum^r - \sum_c^\alpha)$, any infinite quantities do not appear at all.

When the usual interactions with field quantities more than two are adopted as auxiliary interactions, we shall call this prescription in the following the "*method A*". On the other hand, when we use the two-vertex interaction for the correction we shall call the "*method B*". This method is further classified into those of (B_1) and (B_2) according as two field quantities contained in auxiliary interactions belong to one field or two different fields (i.e., $fF(\partial)U^\alpha U^\beta$). In the case (B_2) we easily find that $m^\alpha = 2$ and can show that in some cases the completely convergent theory exists. In the case (B_1) we see that $m^\alpha = 1$ and it is generally proved that we can obtain, for neutral fields, the completely convergent theory and, in general, the closed theory involving a finite number of primitive divergences. Contrary to the case (A), the methods (B_1) and (B_2) do not require the introduction of counter terms, which cancel the divergences in $\sum^\alpha$, since $\sum^\alpha$ is convergent.

In § 3, some examples of non-singular theories are given, which can be obtained by our method. We can in fact make up a completely covergent quantum electrodynamics, a non-singular meson theory and a non-singular theory for β-disintegration. According to the types of auxiliary interactions used for the field correction of propagation functions, that is, the method (A), (B_1) and (B_2) and kinds of the fields introduced into those interactions, the various types of non-singular theories are obtained. We may say that such an arbitrariness corresponds to the similar status of the renormalization theory in the choice of primary interactions and may be regarded as an enlarged form of the latter situation. In fact, when we use the interactions of the first kind as the auxiliary ones, nothing appears essentially different from the usual theories (§ 2, (ii)). Hence, the usual scheme of the renormalization theory for interactions of the first kind may merely be considered as a certain special case among several possibilities of our non-singular theories. In this respect, it also becomes necessary to investigate, in comparison with experimental facts, which type of non-singular theories may really correspond to the actual world.

In § 4, the characteristic features of our method and its relationships with the usual theory are briefly explained. Since the effect of the reaction of self-fields are being taken into account to some extent, the present theory sometimes gives the similar results as the resonance in the strong-coupling theory (for example, in the ratio $\sigma(\pi^+)/\sigma(\pi^-)$ of the

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* This is evident from a simple dimensional consideration. For instance, the quantity, which involves logarithmic divergences, has the finite part of the form $\log(k^2/\varkappa^2)$, say. On the other hand, the converging quantity has the form of positive power of $(1/k^2)$.

meson-nucleon scattering[5]). Some mathematical remarks are further given in the connection with the definition of poles of the functions $\Delta_F^{\alpha\prime}$. As the auxiliary interactions we frequently use the interactions having higher derivatives than the usual ones, which, as well-known, give rise to several peculiarities. In particular, when we regard the auxiliary two-vertex interaction as a part of the free Lagrangian, the theory reduces to that of non-local action[6]. In this connection, theoretical foundations are given for justifying our adoption of the definition of S-matrix (I, 2·5) and it is shown that our method is based on the quantization method different from that of the theory of non-local action and the difficulty concerning the appearance of the negative energy, say, does not appear at all*). Finally, a remark is given on the gauge-invariance of the theory.

In what follows, the same notations and definitions shall be used as in the previous papers (I), (II) and (III) and the equation (2·9) in (I), say, will be referred to as (I, 2·9).

§ 2. General considerations

i) Suppose that there are several fields α, β,... in interaction with each other via the total interaction $\sum_l H_l$ (primary interaction), in which the coupling constant of H_l is of the dimension of η_l-th power of a length and that the propagation function Δ_F^{α}, Δ_F^{β}... and the free operator U^{α}, U^{β},..., of the fields α, β,... are transformed into $\Delta_F^{\alpha\prime}$, $\Delta_F^{\beta\prime}$,... and $U^{\alpha\prime}$, $U^{\beta\prime}$,..., respectively by means of the auxiliary interactions H_{l_α}, H_{l_β},.... The auxiliary interaction $H_{l\alpha}$, which, of course, includes U^{α}, is to be either picked up from among the primary interactions or ad hoc introduced into the primary interaction as such. In the following we shall use the notation $H=\sum_l H_l$ for the primary interactions including H_{l_α}, H_{l_β},.... It must be noted that the interaction H'_{l_α}, as obtained by replacing ∂_μ, the derivation operator acting on the charged field quantities in $H_{l\alpha}$, by $\pm ie\,A_\mu$ must be introduced together with H_{l_α} from the requirement of the gauge-invariance of the theory. Taking into account the definitions (I, 2·13) and (I, 3·3) we find

$$\eta_{l_\alpha}=\eta'_{l_\alpha}. \tag{2·1}$$

In the following we shall employ (I, 2·5) as the S-matrix, the physical meaning of which is discussed in the last section. In this section we shall prove that by use of the suitable auxiliary interactions we can obtain the non-singular theories.

ii) Now, we shall show that in order to obtain the damping factor in the denominator of $\Delta_F^{\alpha\prime}$, H_{l_α} must be the interaction of the second kind, i.e., $\eta_{l_\alpha}\neq 0$.

Using the representation in the momentum space, we have

$$\begin{aligned}\Delta_F^{\alpha}(x) &= \Gamma_{ij}^{\alpha}(\partial_\mu)\,\Delta_F(x)\\ &= \text{const.}\int d^4k\,\frac{\Gamma_{ij}^{\alpha}(-ik_\mu)}{k_\mu^2+(x^\alpha)^2-i\varepsilon}\,e^{-ik_\mu x_\mu}.\end{aligned} \tag{I, 2·9}$$

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* The detailed discussions are given in reference 10.

As proved in the separate paper, the derivation operator $\Gamma^{\alpha}_{ij}(\partial_{\mu})$ satisfies the relation

$$\sum_{j} \Lambda^{\alpha}_{ij}(\partial)\, \Gamma^{\alpha}_{jk}(\partial) = (\Box - (\varkappa^{\alpha})^2)\,\delta_{ik}\,, \tag{2·2}$$

where $\Lambda^{\alpha}_{ij}(\partial)$ is the derivation operator appearing in the field equation of the α-field:

$$\sum_{j} \Lambda^{\alpha}_{ij}(\partial)\, U^{\alpha}_{j} = 0. \tag{2·2'}$$

When we assume that the normalization of the operator Γ^{α} is determined as in (I), Γ^{α}_{ij} takes the following form:

$$\Gamma^{\alpha}_{ij} = a^{\alpha(0)}_{ij}\,\partial^{b^{\alpha}} + a^{\alpha(1)}_{ij}\,\varkappa^{\alpha}\partial^{b^{\alpha}-1} + \cdots + a^{\alpha(b^{\alpha})}_{ij}\,(\varkappa^{\alpha})^{b^{\alpha}}\,, \tag{I, 3·1}$$

where $a^{\alpha(0)}_{ij}$, $a^{\alpha(1)}_{ij}$,... are dimensionless constants. Then, it is easily seen by means of the dimensional consideration that the mass term in the free Lagrangian of the α-field is of the form

$$(\varkappa^{\alpha})^{2-b^{\alpha}}\, U^{\alpha} U^{\alpha}. \tag{2·3}$$

The dimension of the proper self-energy part $\sum^{\alpha}$ due to the interaction $H_{l_{\alpha}}$ must be the same as that of the constant factor in (2·3), and so we have

$$[\textstyle\sum^{\alpha}] = [\text{length}^{b^{\alpha}-2}]. \tag{2·3'}$$

Thus, we find that $\sum^{\alpha}_{c}$, the finite remainder of the m^{α}-th order self-energy part due to $H_{l_{\alpha}}$, turns out to be of the asymptotic form for large k,

$$\textstyle\sum^{\alpha}_{c}(k) \sim \sum_{n=1}^{m^{\alpha}} c_n f^{n}_{l_{\alpha}} k^{2-b^{\alpha}+n\eta_{\alpha}}\,, \tag{2·4}$$

and so we find

$$\Delta'^{\alpha}_{F}(k) = \frac{\Delta^{\alpha}_{F}(k)}{1-\sum^{\alpha}_{c}(k)\Delta^{\alpha}_{F}(k)} \sim \begin{cases} k^{b^{\alpha}-2} & \text{for } \eta_{l_{\alpha}} \leqq 0, \\ k^{b^{\alpha}-2-m^{\alpha}\eta_{\alpha}} & \text{for } \eta_{l_{\alpha}} > 0. \end{cases} \tag{2·5}$$

Therefore, only if $\eta_{l_{\alpha}}$ is positive, Δ'^{α}_{F} obtains the damping factor $\sim k^{m^{\alpha}\eta_{\alpha}}$ in the denominator and so b^{α} changes itself into $b^{\alpha'}$, the corresponding constant for Δ'^{α}_{F} according to the relation

$$b^{\alpha'} = b^{\alpha} - m^{\alpha}\eta_{l_{\alpha}}\,. \tag{2·6}$$

iii) In this sub-section, we shall give a sufficient (but not necessary) condition for the closure of the theory.

Taking into account (2·6) and (I, 2·20), the graph $G(E^{\alpha}, V^{\alpha}, M^{\alpha})$, in which internal and external lines of the fields α, β,... are represented by Δ'^{α}_{F}, Δ'^{β}_{F},... and $U^{\alpha'}$, $U^{\beta'}$,..., respectively, is divergent only when the following condition is satisfied,

$$\sum_{\alpha}\left(d^{\alpha} + \frac{1}{2}b^{\alpha} - \frac{1}{2}m^{\alpha}\eta_{l_{\alpha}} + 1\right)E^{\alpha} - \sum_{i} n_i \eta'_i \leq 4 - D_G, \tag{2·7}$$

where

$$\eta_l' \equiv \eta_l - \frac{1}{2}\sum_\alpha m^\alpha \lambda_l^\alpha \eta_{l_\alpha}, \tag{2·8}$$

$$D_G \equiv \sum_\alpha (N^\alpha + M^\alpha) - \sum_\alpha E^\alpha d^\alpha, \tag{2·9}$$

and d^α is the minimum degree of the derivation operators acting on U^α in $H = \sum_l H_l$. D_G is, therefore, a characteristic constant of the graph under consideration and η_l' is to be regarded as the constant η_l of the interaction in the revised sense as above mentioned.

If there is at least one interaction H_l with $\eta_l' > 0$, any $G(E^\alpha, N^\alpha, M^\alpha)$ will always diverge for a sufficiently large value of n_l (i.e., in the higher order approximation of the perturbation calculation based on the modified functions $\Delta_F^{\alpha'}$s). Hence, we have the following necessary condition for obtaining the non-singular theory;

$$\eta_l' \leq 0, \tag{2·10}$$

that is,

$$\eta_l \leq \frac{1}{2}\sum_\alpha m^\alpha \lambda_l^\alpha \eta_{l_\alpha}. \tag{2·10'}$$

Introducing the characteristic constants e^α and h^α for the α-field defined by

$$e^\alpha \equiv d^\alpha + \frac{1}{2}b^\alpha + 1, \tag{2·11}$$

$$h^\alpha \equiv \frac{1}{2}m^\alpha \eta_{l_\alpha} - e_\alpha, \tag{2·12}$$

we then rewrite (2·7) in the form

$$-\sum_\alpha h^\alpha (E^\alpha - \sum_l \lambda_l^\alpha n_l) \leq 4 - D_G + \sum_l n_l (\eta_l - \sum_\alpha \lambda_l^\alpha e^\alpha). \tag{2·7'}$$

Now, we shall prove in the following a sufficient condition for the closure of the theory is

$$h \equiv \mathrm{Min}(h^\alpha) > \frac{1}{2}\{\eta_l - \sum_\alpha \lambda_l^\alpha e^\alpha\} + \varepsilon, \tag{I†}$$

where $\mathrm{Min}(h^\alpha)$ means the minimum value among the numbers h^α, h^β,... For, in general, there is the relation

$$\sum_\alpha (\sum_l n_l \lambda_l^\alpha - E^\alpha) \geq 2\sum_l n_l, \tag{2·13}$$

and so

$$-\sum_\alpha h^\alpha (E^\alpha - \sum_l \lambda_l^\alpha n_l) \geq 2h \sum_l n_l. \tag{2·14}$$

From (2·14) and [I†] we see that the relation

$$-\sum_\alpha h^\alpha (E^\alpha - \sum_l \lambda_l^\alpha \eta_l) > 4 + \sum_l n_l \{\eta_l - \sum_\alpha \lambda_l^\alpha e^\alpha\} - D_G \tag{2·15}$$

holds, if

$$\varepsilon \sum_l n_l > 2. \tag{2·16}$$

In view of the condition (2·7) we can thus conclude that the higher order graphs such as at least satisfy the above inequality (i.e., graphs of higher order than $2/\varepsilon$) do not give rise to divergences. In particular, if $\varepsilon=2$, that is, if the condition

$$h > \frac{1}{2}\{\eta_l - \sum_\alpha \lambda_l^\alpha e^\alpha\} + 2 \qquad [\mathrm{I}]$$

is satisfied, all graphs converge as far as they are calculated in terms of $\Delta_F^{1\prime}$'s.

Since (2·10) is the necessary condition, it must be satisfied when [I] or [I$^+$] holds. In fact, this is seen in the following way: If, for instance, the condition [I] holds, we have, for all l's and α's,

$$\frac{1}{2} m^\alpha \eta_{l_\alpha} - e^\alpha \quad \frac{1}{2}\{\eta_l - \sum_\alpha \lambda_l^\alpha e^\alpha\} + 2. \qquad [\mathrm{I}']$$

Multiplying both sides by λ_l^α and summing over α, we get the relation

$$\frac{1}{2}\sum_\beta m^\alpha \lambda_l^\alpha \eta_{l_\alpha} - X > \eta_l, \qquad (2\cdot17)$$

where

$$X \equiv 4 + \left(\frac{c_l}{2} - 1\right)\tilde{A}_l, \qquad (2\cdot17')$$

$$\tilde{A}_l \equiv \{A_l - \sum_\alpha \lambda_l^\alpha d^\alpha\}. \qquad (2\cdot17'')$$

Since $\tilde{A}_l \geq 0$ from the definition of A_l (I, 2·10) and $C_l \geq 2$, we find that $X > 0$. Thus, we see that (2·10) can be derived from the condition [I].

In this respect, it should be noticed that contrary to the result in the paper (I) (2·10) is not necessary but sufficient condition for the closure of the theory, because $b^{\alpha\prime}$ may sometimes be negative as seen from (2·6). Since [I] or [I$^+$] is a sufficient condition, being rather too stringent ones for our present purpose, we can frequently find some example of non-singular theories, for which this condition is not satisfied.

iv) Now, we shall show that it is always possible to assume the suitable auxiliary interactions H_{l_α} in such a way as to satisfy [I] or [I$^+$].

[I$'$] is further rewritten in the form with the use of (2·17$''$):

$$\frac{1}{2} m^\alpha \eta_{l_\alpha} - e^\alpha > \frac{1}{2}\tilde{A}. \qquad [\mathrm{I}'']$$

When $m^\alpha \geq 2$, considering that the left hand side $> \eta_{l_\alpha} - \sum_\alpha \lambda_{l_\alpha}^\alpha e^\alpha$, the above condition is satisfied if the relation

$$\tilde{A}_{l_\alpha} - \frac{1}{2}\tilde{A}_l > 4 \quad (\text{for } m^\alpha \geq 2) \qquad (2\cdot18)$$

holds.

If $\tilde{A}_l$ belongs to $H_l \fallingdotseq H_{l_\alpha}, H_{l_\beta}, \cdots$, (2·18) can be satisfied by taking H_{l_α} having an appropriately large value of $\tilde{A}_{l_\alpha}$. When $\tilde{A}_l$ is coincident with $\tilde{A}_{l_\beta}$ belonging to $H_{l_\beta}(\fallingdotseq H_{l_\alpha})$, it is easily seen that we can always take a set of values $\tilde{A}_{l_\alpha}, \tilde{A}_{l_\beta}, \cdots$ so as to satisfy (2·18) (for example, we can take the values such that $\tilde{A}_{l_\alpha}=\tilde{A}_{l_\beta}=\cdots$). Therefore, when $m^\alpha \geq 2$, we can always construct the interactions $H_{l_\alpha}, H_{l_\beta}, \cdots$ to obtain the theory with the finite number of primitive divergences. In this case the contributions of all graphs do not diverge so far as they are calculated in terms of $\Delta_F^{\alpha\prime}$'s.

When $m^\alpha=1$ (method B_1) [I'] becomes

$$\tilde{A}_{l_\alpha}-4+\sum_\alpha(\lambda_{l_\alpha}^\alpha-2)e^\alpha > \tilde{A}_l\,,$$

for which, corresponding to (2·12) we have

$$\tilde{A}_{l_\alpha}-\tilde{A}_l>4 \qquad (m^\alpha=1). \qquad [\mathrm{I}''']$$

It should be noted that the interaction H_{l_α} need not be considered in the course of calculation if use is made of $\Delta_F^{\alpha\prime}$'s, for in the case $m^\alpha=1$, H_{l_α}, being of the form $f\Gamma(\partial)(\Gamma^\alpha(\Gamma^\alpha$, only plays the role of modifying Δ_F^α into $\Delta_F^{\alpha\prime}$. Therefore, A_l in [I''] does not take the suffix l_α (or l_β) and so we can always construct the interaction H_{l_α} such as satisfies [I'''']. However, when the field α is the charged one we must take into account the interactions H_{l_α}''s to be introduced because of the requirement of the gauge-invariance. From (2·1) and the definition of $\tilde{A}_l$, $\tilde{A}_l'$ is at most $(\tilde{A}_{l_\alpha}-1)$*. Therefore, the condition [I'''] does not always hold for $l=l'_\alpha$. Even in this case we can construct the interactions H_{l_α}, $H_{l_\beta}, \cdots$ so that [I$^+$] with $\varepsilon < \frac{1}{2}$ is satisfied. This is in fact achieved if

$$\tilde{A}_{l_\alpha} > \tilde{A}_l+2\varepsilon \qquad (\varepsilon<\tfrac{1}{2}). \qquad [\mathrm{I}^{+\prime}]$$

The last condition is seen to be possible for $l=l'_\alpha$ if one remembers the relation

$$\tilde{A}_{l_\alpha} > \tilde{A}_{l_\alpha}-(1-2\varepsilon) \geq \tilde{A}_{l'_\alpha}+2\varepsilon. \qquad (2\cdot19)$$

Similarly, we see that [I$^{+\prime}$] holds for $l=l'_\beta(l_\beta \neq l'_\alpha)$ provided that we take, for example, $\tilde{A}_{l_\alpha}=\tilde{A}_{l_\beta}=\cdots$. In this case, therefore, we can again find the possibilities of the non-singular theory. In more general cases, where we adopt the both cases ($m^\alpha=1$, $m^\beta \geq 2$) at the same time, it is sufficient for our present purpose if we use $H_{l_\alpha}, H_{l_\beta}, \cdots$, which satisfy [I$^{+\prime}$] or (2·18) respectively (for instance, if we assume $\tilde{A}_{l_\alpha}=\tilde{A}_{l_\beta}=\cdots$).

The success of the method (B_2) is easily understood for the neutral α-field. For example, we can see from (2·5) that a non-singular theory is obtained by means of $H_{l_\alpha}=f\Gamma(\partial)(\Gamma^\alpha\Gamma$, where Γ is another neutral field and $F(\partial)$ some derivation operator. When Γ does not appear in all the other interactions than H_{l_α}, each matrix element, when calculated by means of $\Delta_F^{\alpha\prime}$, does not contain any internal line of Γ-field. Moreover, if Γ-field has a very large mass as compared to the energy in the usual region, there is

* It is easily seen from (2·17'') that $\tilde{A}_{l'_\alpha}$ is smaller than $\tilde{A}_{l_\alpha}$ because the degree of derivation operators decreases for the replacement $\partial_\mu \to ieA_\mu$.

no external line of U'-field, either. Thus, we find that the U'-field only plays the role of making the theory non-singular. If U'^{α} is a charged field, H_{I_α} requires the introduction of H_{I_α}' as seen above. Thus, this consideration does no longer apply and the above, rather tedious, discussion becomes necessary.

v) Hitherto, we have assumed that divergent terms in $\sum'^{\alpha}(k)$ can be removed by the subtraction method as familiar in the renormalization theory. In this respect we shall note, in the following, the characteristics of the respective method (A), (B_1) and (B_2).

In the case of the method (A), where we adopt, as the auxiliary interactions, the familiar ones having field quantities more than two. First, we shall suppose H_{I_α} to be a three-vertex interaction. As $\gamma_{I_\alpha} > 0$ we see that the larger m^{α} becomes, the more divergent terms appear in the self-energy part $\sum'^{\alpha}$. These divergent terms can be removed by the analogous method as the usual renormalization based on the usage of counter terms beforehand introduced in the Lagrangian. In the case $m^{\alpha}=2$, these counter terms are of the form of the local mass terms $I'(\partial)U'^{\alpha}U'^{\alpha}$, which have not any other effects than the cancellation of divergent terms to transform $\sum'^{\alpha}$ into $\sum_c'^{\alpha}$ and which, therefore, can be left out of consideration in the succeeding calculation based on the use of $\Delta_F'^{\alpha'}$.

On the other hand, in the case $m^{\alpha} > 2$ the cancellation of divergent terms in $\sum'^{\alpha}$ requires interactions containing field quantities more than two*) and these interactions have not only the role of making $\sum'^{\alpha}$ finite, but also have other effects on the further calculation based on $\Delta_F'^{\alpha'}$. But, instead of successively removing those divergent terms, which correspond to respective sub-integrations, we can also bring all divergent terms in $\sum'^{\alpha}$ to the form of a non-local mass term such as $F(\partial)U'^{\alpha}U'^{\alpha}$ and then remove it by using the non-local counter interaction $-F(\partial)U'^{\alpha}U'^{\alpha}$. Similar circumstances arise in the case in which $m^{\alpha}=2$ but H_{I_α} contains the field quantities more than three.

Hence, the most simple prescription of the method (A) is to adopt the three-vertex interaction H_{I_α} but with many derivation operators and take $m^{\alpha}=2$. The success of this method can be seen from (2·18), because $\tilde{A}_{I_\alpha}$ becomes larger according as the degree of derivation operators in H_{I_α} increases. It is easily shown that the counter terms cancelling the divergent terms in $\sum'^{\alpha}$ have the forms $\partial^{2l^{\alpha}+x}U'^{\alpha}U'^{\alpha}(0 \leqq x \leqq 4-2\iota^{\alpha})$. The appearance of diverging terms in the Lagrangian is an essential difference between (A) and (B_1) or (B_2).

One needs no divergent counter terms in the Lagrangian in the case of the method (B_1) or (B_2). As was proved in (iv), we can obtain, in these cases, the non-singular theory by use of the suitable interaction H_{I_α} and moreover, in the case of (B_2) we can obtain the completely covergent theory provided that the condition [I] is satisfied.

Throughout all cases those divergences, which appear in the second stage of the calculation, that is, in calculating matrix elements by use of the modified fiunctions $\Delta_F'^{\alpha''}$s, are regarded as 'primitive' and can be removed in just the same way as the usual renormalization theory.

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* For example, we need the counter terms such as cancel the divergent terms of the vertex type when we are concerned with the self-energy of the higher order than two.

vi) Lastly, we shall discuss an important problem, that is, whether the S-matrix can be exclusively described in terms of $\Delta_F^{\alpha'}$. (i.e., without using Δ_F^{α} anywhere). Suppose the case in which we take $H=H_{l_\alpha}=fF(\partial)U^\alpha U^\alpha U^\beta$ and $H_{l_\beta}=0$, or in other words we calculate the matrix elements by use of $\Delta_F^{\alpha'}$, modified by H, and Δ_F^{β}. Under the use of the method (A) with $m^\alpha=2$, the contributions from the second order self-energy of the α-field due to H are all involved in the function $\Delta_F^{\alpha'}$.

Then, it is easily seen from the equation $(1\cdot1)$, which is graphically represented in Fig. 1, that as far as $\Delta_F^{\alpha'}$ is used, not all the graphs such as shown in Fig. 2 are to be taken into account, because there is more in Fig. 2 than the graphs we require, namely, those of Fig. 3, which are graphs just given by the usual perturbation calculation by use of Δ_F^{α}. Therefore, a part of the graphs in Fig. 2 should be excluded. For example, as a part of contributions from Fig. 2b is already involved in Fig. 2a, Fig. 2b, as a whole, must be omitted. Then, we must take into account Fig. 2c, for it contains the contributions from Fig. 3b. The first and second terms on the right hand side of Fig. 2c are equivalent to Fig. 3c and Fig. 3d, respectively. But, the third term thereof is superfluous, which must, therefore, be rejected. Thus, we can not take all the terms in Fig. 2c. The third term in Fig. 2c can not be described without using Δ_F^{α}, that is, the S-matrix can not be described in terms of $\Delta_F^{\alpha'}$, only. In this case, however, we can show in the following that even if we use both functions Δ_F^{α} and $\Delta_F^{\alpha'}$ at the same time there still remains a possibility to construct the non-singular theory.

The lines ——, ——, and ····· correspond to $\Delta_F{}^{\alpha}$, $\Delta_F{}^{\alpha'}$ and $\Delta_F{}^{\beta}$, respectively.

Fig. 1.

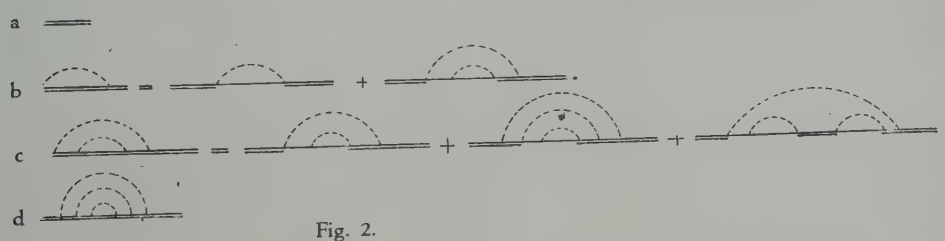


Fig. 2.

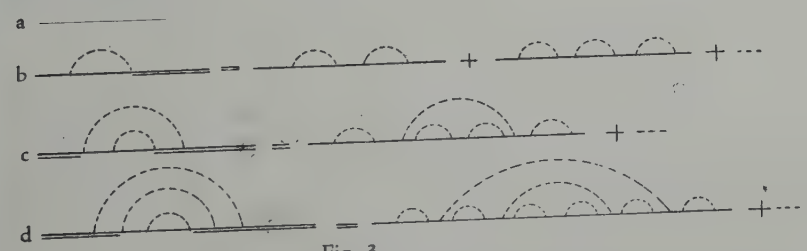


Fig. 3.

Fig. 4.

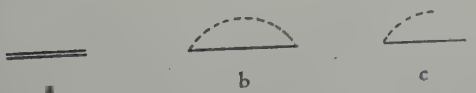
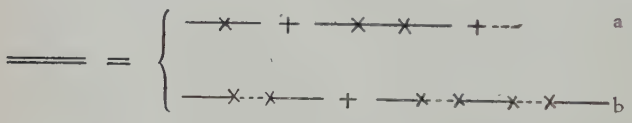


Fig. 5.

The above circumstance is due to the fact that the Figs. 4a and 4b correspond to an identical Feynman graph. More complicated cases, which appear in general in the higher order self-energy graphs, can also be reduced to the above case. When remembering however that the asymptotic forms for large k of Figs. 5a, 5b and 5c are, as seen from (2·4) and (2·5), of the forms equivalent to $1/\sum_c^{\alpha}(k)$, $\sum_c^{\alpha}(k)$ and $\sum_c^{\alpha}(k)$, respectively, we can find that the contributions from Fig. 4a and Fig. 4b are of the same power of k as $1/\sum'^{\alpha}(k)$ and so they lead to the same order of damping factor as $\Delta_F^{\alpha\prime}$. Of course, in the calculation of Fig. 5a, say, there appear the divergent terms of the two-vertex form, but they are all cancelled by the counter terms introduced so as to make $\sum'^{\alpha}$ finite in the calculation of $\Delta_F^{\alpha\prime}$ and so they do not require the further introduction of the new kind of counter terms. Thus, we can see that although Δ_F^{α} is partially necessary in the calculation of S-matrix, the analysis given in the preceding sections is still valid.

On the contrary, if we adopt Δ_F^{z} and $\Delta_F^{z\prime}$ in the above example, such a tremendous circumstance does not occur. This is also the case when the auxiliary interaction is of the form $fF(\partial)U^{\alpha}U^{\beta}U^{\gamma}$ $(\alpha\neq\beta\neq\gamma\neq\alpha)$. This circumstance is also considered as one of the essential differences between (A) and (B_1) or (B_2).

In the case of the methods (B_1) or (B_2) we must take account of all graphs except those corresponding to the iteration of Fig. 6a for (B_1) or Fig. 6b for (B_2). Moreover, in the case of (B_1) we can take away H_{I_α} from the further calculation under the use of $\Delta_F^{\alpha\prime}$, for the effects of H_{I_α} are all contained in the graph of Fig. 6a, that is, $\Delta_F^{\alpha\prime}$. Through both cases the S-matrix can be written in terms of $\Delta_F^{\alpha\prime}$ only.



The lines —— and ····· represent $\Delta_F{}^{\alpha}$ and $\Delta_F{}^{\beta}$, respectively

Fig. 6.

§ 3. Some examples of non-singular theories

In this section the method developed in the previous section will be applied to some practical cases.

i) *Convergent quantum electrodynamics*

Our method is also applied to the theory of interactions of the first kind. Of course, by use of the usual renormalization procedure we can obtain the closed non-singular theory. But, when our method is applied, there exist the possibilities that the theory becomes less singular or further completely convergent. Here, we shall be concerned with the quantum electrodynamics.

First, we shall apply the method (A). As the interaction Hamiltonian we take

$$H=H_1+H_2, \qquad (3\cdot1)$$

$$H_1=-ie\bar{\psi}\gamma_\mu\psi A_\mu, \quad H_2=if\bar{\psi}\gamma_5\gamma_\mu\psi\partial_\mu\phi,$$

where ϕ is an auxiliary field (C-meson field) introduced to modify the propagation function S_F of the electron field ψ. When we modify S_F by use of the finite part of

the second order self-energy part $\sum_c^e(k)$ due to the interaction H_2, we have from (2·4)

$$S_F'(k) \sim k^{-3}, \tag{3·2}$$

for f is of the dimension of a length. Or directly from (2·5) we get

$$b^{e\prime} = -1. \tag{3·2'}$$

In order to extract the finite part $\sum_c^e(k)$ from $\sum^e(k)$ we have beforehand to introduce into the original Lagrangian the counter terms $\lambda_1\bar{\psi}(\gamma\partial+\varkappa)^3\psi$, $\lambda\bar{\psi}(\gamma\partial+\varkappa)^2\psi, \cdots, \lambda_4\bar{\psi}\psi$, since $\sum^e(k)$ is quadratically divergent. Thus, when we calculate the S-matrix exclusively in terms of S_F' the diverging graphs are restricted to those satisfying the condition

$$\frac{1}{2}E^e + E^{ph} + 2E^m + 2n_1 + n_2 \leq 4, \tag{3·3}$$

where E^e, E^{ph} and E^m are numbers of external lines of electron, photon and meson, respectively and n_1 and n_2 numbers of vertices corresponding to H_1 and H_2. As easily seen, the only possible diverging graph is the one with $E^e=2$, $n_2=2$, which is however the one to be excluded in the further calculation. Therefore, apart from the exceptional cases stated in § 2, vi) divergences do not appear in the further calculation. But, it is evident that this model destroys the gauge-invariance of the theory. This is partly due to the fact that the counter terms used are not gauge invariant.

Next, we shall apply the method (B_1) by assuming the following interaction Hamiltonian

$$\begin{aligned} H &= \sum_{i=0}^{3} H_i, \\ H_0 &= \lambda(\partial_\mu\bar{\psi}\partial_\mu\psi + \varkappa^2\bar{\psi}\psi) + \frac{\lambda'}{4}\partial_\lambda F_{\mu\nu}\partial_\lambda F_{\mu\nu}, \\ H_1 &= -ie\bar{\psi}\gamma_\mu\psi A_\mu, \quad H_2 = ie(\bar{\psi}\partial_\mu\psi - \partial_\mu\bar{\psi}\cdot\psi), \\ H_3 &= \lambda e^2\bar{\psi}\psi A_\mu A_\mu, \end{aligned} \tag{3·4}$$

which are gauge invariant as a whole when combined with the free Lagrangian. Two-vertex interactions in H_0 are used here for the modification of propagation functions of electron and photon, respectively. First, we shall consider the propagation function of electron. As is easily seen, we have

$$\sum^e(k) = 2\pi i\lambda(k^2+\varkappa^2), \tag{3·5}^{*)}$$

and so from (1·2) we get

$$S_F'(k) = \frac{-1}{2\pi i}\frac{1}{ik\gamma+\varkappa+\lambda(k^2+\varkappa^2)}, \tag{3·6}^{**)}$$

---

* We have put $S_F'(k) = \frac{-1}{2\pi i}\frac{1}{ik\gamma+\varkappa}$ following the usual definition. The factor $(2\pi i)$ disappears if we take $S_F'(k) = \frac{-1}{ik\gamma+\varkappa}$.

** See the footnote on the next page.

For the propagation function of photon, we get similarly

$$\sum^{ph}(k) = -2\pi i\lambda'(k^2)^2, \tag{3·7}$$

from which we find

$$D'_F(k) = \frac{1}{2\pi i}\frac{1}{k^2(1+\lambda' k^2)}. \tag{3·8}^{**)}$$

We find, therefore,

$$b^{e\prime} = 0, \qquad b^{ph\prime} = -2. \tag{3·9}$$

Contrary to the previous model (A), the gauge invariance is preserved in the similar way as in the usual theory, i.e., in each stage of perturbation calculation with respect to e. The proof is easily carried through in the same way as in the ordinary quantum electrodynamics[7] by noticing the relation

$$S'_F(k+l)(-2\lambda kl - il\gamma)S'_F(k) = \frac{1}{2\pi i}\{S'_F(k) - S'_F(k+l)\}. \tag{3·10}$$

Now, since $\eta'_1 = -2$, $\eta'_2 = -1$, $\eta'_3 = -2$, the condition for the divergent graph (2·7) becomes in this case

$$E^e + 2n_1 + n_2 + 2n_3 \leq 4 - D_G. \tag{3·11}$$

When remembering the gauge invariance of the theory, the graphs satisfying this condition are those corresponding to $(E^{ph}=2,\ E^e=0,\ n_1=n_2=0,\ n_3=1)$, $(E^{ph}=2,\ E^e=0,\ n_1=n_3=0,\ n_2=2)$ and $(E^e=2,\ E^{ph}=0,\ n_1=n_2=0,\ n_3=1)$, $(E^e=2,\ E^{ph}=0,\ n_1-n_3=0,\ n_2=2)$. These graphs are photon's and electron's self-energy parts of the order $\sim e^2$, which give the logarithmically diverging charge and mass renormalization terms. The other graphs are all convergent.

In order to make the second order self-energy of the electron convergent it is sufficient to use, for instance, $\frac{\lambda'}{4}\Box F_{\mu\nu}\Box F_{\mu\nu}$ instead of the last term in H_0 of (3·4). On the other hand, to make the photon's self-energy part convergent it becomes necessary that $S'_F(k)$ is more strongly damped than the one given by (3·6). If, for this purpose, we introduce the term $\lambda\partial^n\bar{\psi}\psi$, say, into H_0, then $S'_F(k)$ becomes $\sim 1/k^n$, while accompanying the introduction of this term the gauge invariance of the theory requires the further terms such as $-ie\lambda A_\mu\partial^{n-1}\bar{\psi}\psi$, $\lambda e^2 A_\mu A_\nu\partial^{n-2}\bar{\psi}\psi$, $\cdots$, among which the second order effect of the first term and the second term give the quadratically diverging photon self-energy, and so lead to the logarithmically diverging charge renormalization term. Such a circumstance seems to be common to the charged field in general that the method (B_1) cannot remove some types of divergences, while for the neutral field it does not occur.

Now, we shall apply the method (B_2). As the interaction Hamiltonian we assume the following form

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** These functions are just equivalent to ones obtained when H_0 is considered as a part of the free Lagrangian.

$$H=\sum_{i=0}^{3} H_i\,,$$

$$H_0=\lambda(\partial_\mu\bar{\psi}\partial_\mu\varphi+\varkappa^2\bar{\psi}\varphi+\text{c.c.})+\frac{\lambda'}{4}\partial_\varkappa F_{\mu\nu}\partial_\lambda F^U_{\mu\nu}\,,$$

$$H_1=ie(\bar{\psi}\gamma_\mu\psi+\bar{\varphi}\gamma_\mu\varphi)A_\mu,$$

$$H_2=i\lambda e(\bar{\psi}\partial_\mu\varphi-\partial_\mu\bar{\psi}\varphi)A_\mu+\text{c.c.} \tag{3·12}$$

$$H_3=\lambda e^2\bar{\psi}\varphi A_\mu A_\mu+\text{c.c.},$$

where φ is an auxiliary charged spinor field (with mass $\varkappa'$ and spin 1/2) introduced for ψ, and $F^U_{\mu\nu}$ the anti-symmetric tensor composed of an auxiliary neutral vector field (with mass M) is introduced for A_μ.

The self-energy part of the electron becomes

$$\sum{}^{e}(k)=2\pi i\{\lambda(k^2+\varkappa^2)\}^2\frac{ik\gamma-\varkappa'}{k^2+\varkappa'^2}\,, \tag{3·13}$$

which gives the modified propagation function

$$S'_F(k)=\frac{-1}{2\pi i}\,\frac{1}{(ik\gamma+\varkappa)+\{\lambda(k^2+\varkappa^2)\}\dfrac{(ik\gamma-\varkappa')}{k^2+\varkappa'^2}}\,. \tag{3·14}$$

Similarly, the self-energy part of the photon

$$\sum{}^{ph}(k)=(2\pi i)\frac{\lambda'^2(k^2)^4}{k^2+M^2} \tag{3·15}$$

leads to the result

$$D'_F(k)=\frac{1}{2\pi i}\,\frac{1}{k^2-\lambda'^2\dfrac{(k^2)^4}{k^2+M^2}}\,. \tag{3·16}$$

Hence, we find

$$b'=-1,\quad l^{ph\prime}=-4,\quad \text{etc.}, \tag{3·17}$$

and

$$\eta_0'=-1,\quad \eta_1'=-4,\quad \eta_2'=-3,\quad \eta_3'=-5, \tag{3·18}$$

from which the condition (2·7) turns out to be

$$-\frac{1}{2}E^{sp}-E^{ph}+n_0+4n_1+3n_2+5n_3\leq 4-D_G, \tag{3·19}*$$

where E^{sp} means the numbers of external lines of spinor fields. In this case this inequality

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* n_0 is the number of vertices corresponding to the first term of H_0 in (3·12).

is satisfied for the set of values ($E^{ph}=2$, $E^{sp}=0$, $n_3=1$, $n_0=n_1=n_2=0$) and ($E^{ph}=2$, $E^{sp}=0$, $n_2=2$, $n_0=n_1=n_3=0$). But, when remembering the gauge-invariancy of the theory, which is proved in the same way as in (B_1), the corresponding graphs are seen to be convergent. Therefore, the present model do not involve any divergence. When constructing S'_F and D'_F there appear no divergences and so the method (B_2) gives the completely convergent theory*).

ii) *Meson theory*

As an example of the unrenormalizable meson theory we shall consider the neutral pseudo-scalar meson field in interaction with the nucleon field via the derivative coupling, which is described by the Hamiltonian

$$H=if\psi\gamma_5\gamma_\mu\phi\partial_\mu\phi. \tag{3·20}$$

When we make the correction to the ψ-field we are led to the same circumstances as in i) (the method A). That is, since $\eta'=-1$, and $b'=-1$, the condition (2·7) reads

$$\frac{1}{2}E^{\text{nucl.}}+2E^m+n\leq 4. \tag{3·21}$$

Thus, by use of the counter terms introduced to make $\sum^{\text{nucl.}}$ convergent we can also remove further divergences.

Next, we shall consider the case in which the meson propagation function is modified**). As $b^m=0$, it follows from (2·6)

$$b^{m\prime}=-2, \tag{3·22}$$

and from (2·8)

$$\eta'=0. \tag{3·23}$$

Thus, (2·7) gives the condition for divergent graphs

$$\frac{3}{2}E^{\text{nucl.}}+E^m\leq 4. \tag{3·24}$$

This case was previously suggested by Hu[1]. (3·24) is identical with the condition for the case of pseudo-scalar coupling. Here, we shall give some remarks on the meson self-energy graphs. At first sight it appears that in order to cancel the diverging parts thereof is necessary the counter term such as the one involving four derivation operators $\partial_\mu\partial_\nu\phi\partial_\mu\partial_\nu\phi$, since outside the self-energy integral there exist two derivation operators $\partial_\mu\partial_\nu$ from the outset and the graphs are quadratically divergent. The above counter terms can be constructed

* In view of the fact that the non-singular quantum electrodynamics can be obtained by our method as well as by the usual procedure of the renormalization and that if we take the constants λ and λ' to be sufficiently small the former leads to almost identical results as the latter, it seems for us that the ordinary renormalization theory is not necessarily the unique solution of the problem, but merely the one among several possibilities.

** When both propagation functions S_F' and D_F' are modified, the calculation rule becomes very complicated.

as neither the mass nor the charge renormalization. But, in general, two adjacent γ_5-factors in the closed loop of the meson self-energy graph can we combine by making pass through the intervening factor S_F, the result being then written in the form

$$\cdots\gamma_5\frac{ik\gamma-\varkappa}{k^2+\varkappa^2}\gamma_5\cdots\rightarrow\frac{ik\gamma-\varkappa}{k^2+\varkappa^2}+\frac{2\varkappa}{k^2+\varkappa^2}=S_F(k)+\frac{2\varkappa}{k^2+\varkappa^2}.$$

After eliminating all the γ_5's in this way the most highly diverging terms (quadratic divergence) are given by the cross term composed only of the factors S_F (i.e. not containing the factors $2\varkappa/(k^2+\varkappa^2)$). This term being of the same form as that of the photon self-energy graph in quantum electrodynamics, is thus taken to be vanishing. After all the meson self-energy becomes logarithmically divergent. Therefore, taking into account the two derivation operators stated above the required counter terms turn out to be

$$\delta\varkappa\phi\phi+B_2(\partial_\mu\phi\partial_\mu\phi+\varkappa^2\phi\phi).$$

Therefore, the required counter terms are all of the same form as those of pseudo-scalar coupling case. That the above counter terms can be interpreted as the mass and the charge renormalization may be evident from the consideration as suggested by Takeda[8].

iii) *Theory of β-decay*

Among the well-known interaction most troublesome are the direct interactions between four spinor fields such as those describing β-decay, decay and absorption of μ-meson and their coupling constants are of the dimension of the square of a length, i.e., $\eta=2$. And so, why the β-decay phenomenon, say, can well be interpreted by the usual Fermi theory, it seems one of the most incomprehensible problems in the presnt status of the renormalization theory[1),9)]. From our view-point however this problem is understood in the following way: Remembering the fact that there exists the strong interaction between nucleon and meson fields we can take $\delta^{\mathrm{mes.}}=-1$ (for example, by means of ii)) and then it follows

$$\eta'_{\beta\mathrm{decay}}=0. \qquad (3\cdot25)$$

This condition indicates that the β-interaction in this case can be treated as if it were the first kind and so the divergences appearing therein shall altogether be removed by a finite number of counter terms.

We can also apply the method (B_1) to this case. It is convenient to modify the neutron and the neutrino fields, respectively, since they have nothing to do with the requirement of gauge-invariance. For the closure of the theory it is sufficient if we assume the auxiliary interactions with $\eta_N=\eta_\nu=2$, for example $H_N=\lambda\partial_\mu\bar{\psi}_N\gamma_\mu\Box\psi_N$, $H_\nu=\lambda\partial_\mu\bar{\psi}_\nu\gamma_\mu\Box\psi_\nu$, since the condition $(2\cdot7)$ becomes then

$$3E_p+3E_e+E_N+E_\nu\leq 8. \qquad (3\cdot26)$$

Of course, if we are content with the lowest order approximation for the β-decay matrix elements because of the smallness of the coupling, we shall obtain the same results as the usual ones, for the external lines suffer no essential modification under the use of our method.

§ 4. Some remarks

In this section we shall give some remarks on the formulation and its physical meaning of the method developed in the preceding sections.

i) The constants η_{l_α} of auxiliary interactions H_{l_α}''s should be large enough to be possible of obtain the non-singular theory. Such interactions with large η's can be constructed either by using the derivation operators or by introducing the auxiliary fields with large spins (cf. II).

ii) In the previous and the present papers (I, II, III) we have adopted the S-matrix of the form

$$S=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n P^*(L'(x_1),\cdots,L'(x_n)), \tag{4·1}$$

where $L'(x)$ is the interaction Lagrangian and P^* means the operation, which behaves as the Dyson's P-operator except that the derivation operators involved in L''s act after the chronological ordering and so which effects, for the vacuum value, the following replacement:

$$\langle P^*(F(\partial)U_i^\alpha(x),F'(\partial')U_j^\alpha(x'))\rangle_0\to F(\partial)F'(\partial')\Gamma_{ij}^\alpha(\partial)\Delta_F(x-x'). \tag{4·2}$$

Therefore, it should be noted that $L'(x)$ must not be rewritten by use of the field equations of free fields. Let us consider, for example, the case in which $L'(x)$ contains the term $(\Box-(\varkappa^\alpha)^2)U^\alpha(x)$. This factor is zero in the interaction representation, but not in the Heisenberg representation. In the former representation, the interaction Hamiltonian $H(x)$ is non-vanishing contrary to $L'(x)$ and $\{H(x)-(-L'(x))\}$ contains the normal independent terms[10], which give in this representation the effects, equivalent to those of the interaction Lagrangian in the Heisenberg representation. What we can obtain by use of (4·1) is just these non-vanishing effects. Thus, in (4·1) $L'(x)$ must not be put equal to zero, since the P^*-operator is acting*.

The fact that (4·1) gives the correct S-matrix even for the interactions of quite general types was discussed in detail in the separate paper[10]. As shown there, we can introduce independent canonical variables, based on which the quantization can be accomplished by means of the canonical formalism. The field quantities then obeys the field equations and commutation relations for free fields with the definite mass $\varkappa$ in a suitable representation. (i.e. interaction representation). Moreover, it can be shown that this quantization method is equivalent to that of the Heisenberg-Pauli theory even in the case of interactions having higher derivatives. The field operator $U^\alpha(x)$ in the Heisenberg representation is obviously a certain functional of the independent canonical variables at the same point in this representation. In the case of the general interactions the interaction Hamiltonian and the functionals are the power series in the coupling constants and

* Thus, the interactions which we have termed so far the "Hamiltonian" are not the true Hamiltonians, but the negative of the interaction Lagrangians. Bearing in mind the above notice, however, we can still use this name for the latter.

so in the present status of the theory we can not but assume their convergence*). With this understanding, (4·1) is regarded as the S-matrix which connects the incoming and the outgoing waves of the independent canonical variables in the Heisenberg representation.

We shall give further remarks concerning the method (B_1) the auxiliary interaction of which is $H_{I_\alpha}=fF(\partial)U^\alpha F'(\partial')U^\alpha$. In this case we have from (1·2)

$$\Delta_F^{\alpha'}(k)=\frac{\Delta_F^{\alpha}(k)}{1-f\Delta_F^{\alpha}(k)F(-ik)F'(ik)}\,. \tag{4·3}$$

On the other hand, if we consider H_{I_α} not as an interaction term but as a term in the free Lagrangian, the field equation of the free α-field becomes

$$\{\Lambda^\alpha(\partial)-fF(-\partial)\}U^{\alpha'}=0 \tag{4·4}$$

instead of (2·2′). Then, it can easily be seen from (2·2) that (4·3) is the Green function of the equation (4·4).**)

As is well-known, in the theory of non-local action quantized in the familiar way[6] there appear several fields with respective masses, some of which are of negative values. These fields correspond to respective independent components introduced through the high derivatives. But, as mentioned above, by use of the other canonical variables, in which the high derivatives can be absorbed, we can obtain the quantum theory of the field with a single definite mass $\varkappa^\alpha$. From the view-point of the theory of non-local action our theory is considered as based on this quantization method.

In the case of (B_1) our method may be regarded as the quantum theory for field quantities U' which, for instance, satisfy a relation such as

$$\langle P(U^{\alpha'}(x),\,U^{\alpha'}(x'))\rangle^0=\Delta^{\alpha'}(x-x').$$

In the case of (B_2) and some cases of (A) we can not interpret the method in this way for the reason mentioned in § 2, vi).

iii) When we derive the function (1·2) from (1·1), we can not determine the property of $\Delta_F^{\alpha'}(k)$ at its poles especially those on the real axis of the k_0-plane, for the series expansion thereat does not converge. However, we can find this behavior by means of the well-known property of the S-matrix. When the instantaneous variation of the state is described by an unitary transformation $S(\sigma)$, the propagation of the signal between two points x and x' occurs according to the requirement of the causality, that is, the influences from the past and to the future are described by the converging and diverging waves, respectively, which, speaking in the language of momentum space, correspond to the negative frequency parts of the propagation function. This fact gives the property of $\Delta_F^{\alpha'}(x-x')$ so that the path of the integration of $\Delta_F^{\alpha'}(k)$ in the k_0-plane must make

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* We can, in fact, find some examples of interactions with higher derivatives, for which these series converge and can be written in the closed form.

** There appear similar circumstances in the case of the method (B_2) if we regard the interaction $fF(\partial)U^\alpha F'(\partial)U^\beta$ $(\alpha\neq\beta)$ as a part of the free Lagrangian and rewrite it into the diagonal form with respect to suffices α, β by a suitable transformation to the principal axis.

detours passing above or below the poles on the positive or negative real axis[11)*].

Corresponding to the alterations of internal lines, external lines are modified by the relation (1·3). As well known, the right hand side of (1·3), which eventually reduces to a numerical factor, involves some ambiguities, because $U^{\alpha\prime}$'s satisfy the equations of motion for free fields. This factor, however, is to be determined in connection with the unitarity of the S-matrix as was done in the determination of the charge renormalization constants in quantum electrodynamics. Since the determination in this case is consistently carried through under the assumption of adiabatic switching in and off of the interaction constant[12)], it is supposed that in our case this procedure may give right answers in agreement with the unitarity.

When poles are defined as above, the discussions regarding the coincidence of some poles (or displaced poles) are performed along a similar line as the usual theory, which, therefore, do not lead to any difficulty relating to the ultra-violet divergences.

The poles of $\Delta_F^{\alpha\prime}$ appear in quite a different manner from the usual ones and so there frequently arises the similar effect as the resonance in the final expression of the cross sections. For example, when the meson theory presented in § 3, ii) is applied to the treatment of the nucleon scattering[5)], the resonance effects appear and they strongly effect the steep form of the excitation function.

In this particular example, the connection of the present method with the usual damping theory can more clearly be understood. That is, if one neglects, in the denominator of S_F', the inductive (real) part and keeps only the resistive (imaginary) part of the field reaction, the results of the former becomes identical with the latter. Such a correspondence may perhaps be present in more general cases, too.

The new kind of poles also gives rise to the effects like the infra-red catastrophe of the usual perturbation theory, which seems, however, not to be essential difficulties of the method.

iv) In this paragraph we shall consider the gauge invariance of the theory. As already mentioned in connection with some examples of § 3, the method (A), when applied to the charged field, cannot lead to the gauge invariant results. This defect is due to the following two points; (1) that when constructing $\Delta_F^{\alpha\prime}$'s of charged fields one has used the gauge-dependent counter terms and (2) that the radiative corrections has been considered only for lines and not for the corresponding electromagnetic vertices. On the other hand the method (B_1) and (B_2), when starting with the gauge invariant Lagrangian, do not destroy the gauge invariancy. In the method (B_1), however, it becomes impossible from this requirement to construct interactions such as satisfy the condition [1] and so it seems very difficult to obtain the completely convergent theory, although in this case we can of course obtain the non-singular theory.

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* In fact, in the usual perturbation theory the path of the integration at the poles of $\Delta_F{}^{\alpha}$ satisfies the requirement of the causality, for

$$\Delta_F{}^{\alpha}(x-x') = \begin{cases} 2i\Delta^{+}(x-x'), & \sigma(x) > \sigma(x'), \\ -2i\Delta^{-}(x-x'), & \sigma(x) < \sigma(x'), \end{cases}$$

where $\sigma(x)$ indicates the space-like surface through the point x and $\sigma(x) > \sigma(x')$ means that $\sigma(x)$ is later than $\sigma(x')$.

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The Field Theory of Superconductivity

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Fröhlich's field theory of superconductivity is rewritten in a form in which the recent (perturbation) methods of writing down matrix elements can be readily applied.

Fröhlich (1950, 1952) has recently developed a theory of superconductivity in which the super-conducting behaviour of metals arises from an interaction of two fields, the field of "free" electrons in the metal and the field of the lattice vibrations. The general study of interacting fields has made great progress recently and it seems worth while to write Fröhlich's theory in a form in which the newer powerful methods can be applied to it. This seems desirable, particularly, since doubts have been raised as to the validity of the perturbation approach used by Fröhlich (Wentzel, 1951) and since a renormalization of the velocity of sound has proved necessary (Fröhlich 1952). These, precisely, are the problems which other field theories have to face. In this note we rewrite the Fröhlich theory without making any calculations. To these a subsequent paper will be devoted.

The electron field is characterised by the field functions $\psi_i(x)$ and $\psi_i^*(x)$, ($x=x$, t and $i=1, 2$ for the two spin states), which, provided no interaction is present, satisfy the equations

$$i\hbar\frac{\partial\psi_i}{\partial t}+\frac{\hbar^2}{2m}\Delta^2\psi_i=0, \tag{1}$$

$$-i\hbar\frac{\partial\psi_i^*}{\partial t}+\frac{\hbar^2}{2m}\Delta^2\psi_i^*=0. \tag{2}$$

The Lagrangian density can be written as

$$L_1=i\hbar\psi^*\dot{\psi}-\frac{\hbar^2}{2m}\ \text{grad}\ \psi^*\cdot\text{grad}\ \psi. \tag{3}$$

Write

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int a_k\, e^{ikx}\,dk \tag{4}$$

where

$$e^{ikx}=e^{i(kx-k_0t)},\quad k_0=\frac{E}{\hbar}=\frac{\hbar}{2m}k^2.$$

To satisfy Fermi statistics we postulate the anticommutation relationship

$$\{\psi_i(x,\ t),\ \psi_j^*(x',\ t)\}=\delta(x-x')\,\delta_{ij}, \tag{5}$$

all other anticommutators vanishing.

To characterise the vibrational field we use a field function $\phi(x)$,* defined as

$$\phi(x)=\sqrt{\frac{\hbar s}{(\pi)^3}}\int_{0<|\omega|<W}\frac{1}{\sqrt{2|\omega|}}\left[b_\omega\, e^{i(\omega x-\omega_0 t)}+b_\omega^*\, e^{-i(\omega x-\omega_0 t)}\right]d\omega \tag{6}$$

$\omega_0=s|\omega|$ and s is the observed velocity of sound. W is the maximum value of $|\omega|$ which in Debye approximation satisfies the relation

$$n(\text{the number of atoms/unit volume})=2W^3/3(2\pi)^2. \tag{7}$$

This definition ensures that $\phi(x)$ satisfies the equation

$$\left(\Delta^2-\frac{1}{s^2}\frac{\partial^2}{\partial t^2}\right)\phi(x)=0 \tag{8}$$

derivable from the Lagrangian density

$$L_2=\frac{1}{2}\dot{\phi}^2-\frac{s^2}{2}(\text{grad }\phi)^2. \tag{9}$$

The commutation relationship is postulated as

$$[b_\omega,\ b_{\omega'}^*]=\delta(\omega-\omega'). \tag{10}$$

So far we were dealing with free fields. The interaction Lagrangian density, as given by Bethe and others, can be introduced in the form

$$L_{int}=g\psi_i^*(x)\psi_i(x)\frac{\partial\phi(x)}{\partial t}. \tag{11}$$

The total Lagrangian density is therefore $L_1+L_2+L_{int}$.

An interaction representation can now be set up with the usual unitary transformation, such that the fields ψ and ϕ continue to satisfy the equations (1), (2) and (8), while the commutation relationships are given by (5) and (10). We shall later find that if in (9), s is to be interpreted as the observed sound velocity, the interaction Lagrangian (11) is incomplete and has to be supplemented by renormalization terms. In the subsequent discussion we shall always assume that the operators ψ, ψ^*, ϕ are interaction representation operators.

Let us denote by $|F\rangle$ the state in which electrons (of either spin) fill all energy levels upto the Fermi maximum $\hbar^2K^2/2m$. We are interested in calculating the self energy of this state $|F\rangle$. In the time dependent formulation of perturbation theory, this quantity is obtained by writing down the S-matrix and evaluating $\langle F|S|F\rangle$. The S-matrix, in a perturbation expansion, is given as

$$S=\sum\frac{1}{n!}(-i/\hbar)^n\int dx_1\cdots dx_n\ T(L_{int}(x_1)\ L_{int}(x_2)\cdots L_{int}(x_n)). \tag{12}$$

T is Wick's ordering operator (Wick, 1950). The S-matrix has been written in terms of the interaction Lagrangian, rather than the interaction Hamiltonian. The reason for this will be discussed later.

The problem, therefore, is one of the evaluation of the matrix element

$$\langle F|T(L_{int}(x_1)\ L_{int}(x_2)\cdots\ L_{int}(x_n))|F\rangle. \tag{13}$$

*) The definition of $\phi(x)$ actually depends on the detailed description of the lattice and its vibrations and the above definition is one of the simplest one could adopt.

For this we use Wick's theorem on the decomposition of a T-product into a sum of normal products. For boson operators we retain Wick's definition of normal product, but for fermion operators a change in the definition will facilitate the evaluation of (13).

Consider boson operators first. If N is the symbol for normal product (such that creation operators stand to the left of annihilation operators), it is easy to verify, using (6) and (10), that

$$T\phi(x_1)\phi(x_2) = N\phi(x_1)\phi(x_2) + d(x_1 - x_2), \tag{14}$$

where

$$d(x_1 - x_2) = \frac{\hbar}{2} \cdot \frac{2i}{(2\pi)^4} \int_{0 \leq \omega \leq \Omega} d\boldsymbol{\omega} \int_{-\infty}^{+\infty} d\omega_0 \frac{e^{iwx}}{\omega_0^2 - \omega^2 s^2}. \tag{15}$$

The use of $L_{int}(x)$ in the S-matrix ensures (Koba, 1950) that

$$\left\langle T \frac{\partial\phi(x_1)}{\partial t_1} \frac{\partial\phi(x_2)}{\partial t_2} \right\rangle_0 = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} d(x_1 - x_2), \tag{16}$$

where $\langle \quad \rangle_0$ stands for vacuum expectation value.

For the fermion operator, $\psi(x)$ is written as

$$\psi(x) = u_1(x) + u_2^*(x), \tag{17}$$

where

$$u_2^*(x) = \frac{1}{(2\pi)^{3/2}} \int_{0 \leq |\boldsymbol{k}| \leq K} d\boldsymbol{k} \; a_k e^{ikx}. \tag{18}$$

$u_2^*(x)$ annihilates the electrons in the state $|F\rangle$ and $u_2(x)$ creates these electrons. Thus

$$u_2(x)|F\rangle = 0, \qquad \langle F|u_2^*(x) = 0, \tag{19}$$

and

$$\{u_1(x_1),\ u_2(x_2)\} = 0.$$

This decomposition of $\psi(x)$ allows us to consider the Fermi state* $|F\rangle$ as the reference state, having in fact, the same role as the "vacuum state" in the relativistic theory of electrons and positrons. Thus $u_2^*(x)$ can be interpreted as creating a hole in the Fermi State, and $u_2(x)$ as annihilating a hole. If, now, we define normal product $N^*(U)$ corresponding to a product U of ψ, ψ^* operators, as a re-arrangement of U, after it has been expanded out in terms of u and u^* operators, such that all starred operators u^* stand to the left of un-starred operators u, with their proper signature, then

$$N^*\psi(x)\psi^*(y) = -u_1^*(y)u_1(x) + u_1(x)u_2(y) + u_2^*(x)u_1^*(y) + u_2^*(x)u_2(y) \tag{20}$$

and

$$T\psi(x)\psi^*(y) = \psi(x)\psi^*(y), \qquad t_x > t_y \tag{21}$$

$$= N^*\psi(x)\psi^*(y) + \{u_1^*(y)u_1(x)\}$$

$$= N^*\psi(x)\psi^*(y) + \{\psi^*(y), \psi(x)\} - \{u_2(y), u_2^*(x)\} \tag{22}$$

$$= -\psi^*(y)\psi(x), \qquad t_y > t_x$$

*) The idea of this type of modification is already in H. Miyazawa's work, H. Miyazawa, Prog. Theor. Phys. 6(1951), 788.

$$= N^*\psi(x)\psi^*(y) - \{u_2(y), u_2^*(x)\}. \tag{23}$$

Writing $\{\psi^*(y), \psi(x)\} = s(x-y)$

and $\{u_2(y), u_2^*(x)\} = \Delta s(x-y)$,

we have

$$T\psi_i(x)\ \psi_j^*(y) = N^*\psi_i(x)\ \psi_j^*(y) + \delta_{ij}[\theta(x-y)s(x-y) - \Delta s(x-y)]. \tag{24}$$

Here $\theta(x-y) = 1 \qquad t_x > t_y,$

$= 0 \qquad t_y > t_x$

and from the expansion (4) and relation (5), we have

$$\Delta s(x-y) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} dk_0 \int_{\kappa \le |k| \le K} d\boldsymbol{k}\ \delta(k_0 - \frac{\hbar \boldsymbol{k}^2}{2m}) \exp[ik(x-y)], \tag{25}$$

while one representation for $\theta(x-y)\ s(x-y)$ is

$$\theta(x-y)\ s(x-y) = -\frac{1}{2\pi i}\ \frac{1}{(2\pi)^3}\int_{C_+} \frac{d\boldsymbol{k}\ dk_0}{k_0 - (\hbar/2m)\boldsymbol{k}^2} \exp\,[ik(x-y)], \tag{26}$$

the $\boldsymbol{k}$ integrations being from $-\infty$ to $+\infty$ while for k_0 integration, C_+ is a path above the real k_0 axis.

Equation (24) is written equivalently as

$$\langle T_i\psi(x)\psi_j^*(y)\rangle_F = \delta_{ij}[\theta(x-y)s(x-y) - \Delta s(x-y)] \tag{27}$$

With the propagation functions d and $\theta s - \Delta s$, and with state $|F\rangle$ acting as the reference (vacuum) state, it is quite easy to state rules for writing down the S-matrix elements.

Thus to write down the matrix element for some given type of process, with specified number of particles emitted and absorbed (with reference to $|F\rangle$) draw all graphs which have the right set of external lines corresponding to the particles absorbed and emitted. Each vertex in each graph has three lines ending at it, one incoming electron line, one outgoing electron line and one phonon line. To each graph with n vertices corresponds one normal constituent S_G of S. Choosing a particular graph σ, we write down the expression

$$S_n = L_{int}(x_1)\cdots L_{int}(x_n) \tag{28}$$

and pair off the factors of S_n as indicated by the graph G. Replace each factor pair $\partial\phi(x_1)/\partial t_1\ \ \partial\phi(x_2)/\partial t_2$ by $\dfrac{\partial}{\partial t_1}\dfrac{\partial}{\partial t_2}\ d(x_1-x_2)$, each factor pair $\psi_i(x_1)\psi_j^*(x_2)$ by $\delta_{ij}[\theta(x_1-x_2)s(x_1-x_2) - \Delta s(x_1-x_2)]$, and write the remaining unpaired factors of S_n, $\psi_q(x_a)\cdots\psi^*(x_b)$ as an N^* product $N^*(\psi_p(x_a)\cdots\psi_q^*(x_b))$.

The whole expression is multiplied by ± 1 according as the signature of the factors appearing in the rearrangement

$$(\psi_i(x_1)\psi_j^*(x_2)(\quad)\cdots(\quad))(N^*\psi_p(x_a)\cdots\psi_q^*(x_b)) \tag{29}$$

is even or odd.

A substitution of the wave functions of the absorbed and emitted particles (with reference to $|F\rangle$) and an integration over $x_1x_2\cdots x_n$ finally gives the required contribution to the matrix element.

In actual evaluation each graph with m internal electron lines (and associated propagation function $\theta s-\Delta s$) can be replaced by $2m$ graphs containing electron lines with either of the two propagation functions θs or $(-\Delta s)$. It may be pointed out that a closed loop containing electron lines with associated propagation functions θs only, must vanish identically on account of the vanishing cyclic factor

$$\theta(x_1-x_2)\theta(x_2-x_3)\cdots\theta(x_n-x_1).$$

The self energy of the state $|F\rangle$ is calculated by drawing all "vacuum" to "vacuum" graphs and evaluating them. Quite generally "vacuum" to "vacuum" expectation value of the S-matrix,

$$\langle F|S(t_2\,t_1)|F\rangle=\exp\left[-i/\hbar\cdot\Delta E\ (t_2-t_1)\right]. \tag{30}$$

Actual calculation (Feynman, 1949) shows that

$$\langle F|S(+\infty,-\infty)|F\rangle=\exp\left[-\sigma_1-\frac{\sigma_2}{2}-\frac{\sigma_3}{3}-\cdots\right] \tag{31}$$

where σ_n is the sum of the contributions from proper, connected n-th order graphs, and each σ_n has $\delta^1(o)$ as a common factor. From (30) and (31), ΔE is obtained directly. The "self" energy in state $|F\rangle$ of a real phonon is easily seen to be of the form $\phi(\omega)\pi(\omega)\phi(\omega)\pi(\omega)$ in all orders, where

$$\pi(\omega)=\omega_0^2 f(W,\ \omega,\ K).$$

A part (not dependent on ω) of this expression may be cancelled by an additional term $A\left(\frac{\partial\phi}{\partial t}\right)^2$ in the interaction Lagrangian. This is equivalent to a renormalization of the velocity of sound, or alternatively a renormalization of the coupling constant. These points will not be discussed in detail here.

This work was started in collaboration with Dr. K. S. Singwi, at the Tata Institute for Fundamental Research, Bombay. To him and to Dr. H. J. Bhabha, the author is deeply indebted, for discussions and hospitality.

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Letters to the Editor

Remarks on Lévy's Fourth Order Potential

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April 3, 1953

Using the extended form of Tamm-Dancoff method[1] of the nonadiabatic treatment of two-body problems, Lévy has recently calculated the fourth order nucleon-nucleon interaction via the symmetrical pseudoscalar meson field.[2] In the nonrelativistic region this interaction is expressed in the usual form of potentials and the main part of this was found to be

$$V^4(r)=-3\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\frac{1}{\mu r^2}\left\{\frac{2}{\pi}K_1(2\mu r)+\frac{\mu}{2M}\left[\frac{2}{\pi}K_1(\mu r)\right]^2\right\}. \qquad (1)$$

This comes from the processes which involve the creation of two virtual nucleon pairs in the intermediate states. The first term is the contribution from the processes involving only one pair at a given time (but successively two) while the second is given rise by the processes which involve two pairs at the same time.[3] The first term is in agreement with what results from a scalar pair coupling of pseudoscalar mesons[4] and also with Nakabayasi and Sato's result[5] in the adiabatic limit. Lévy says that the contributions from the processes which involve just one virtual pair, which are also of the order $(G^2/4\pi)^2(\mu/2M)^3$, cancel each other. We found, however, that this statement is not correct and to clarify this point is the aim of the present note.

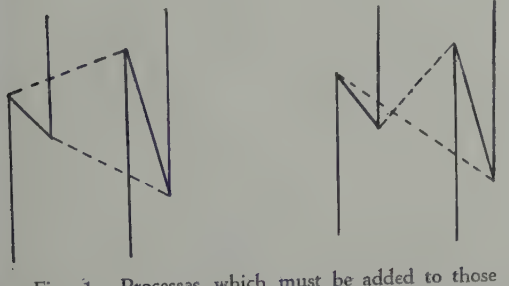

Fig. 1. Processes which must be added to those shown in Fig. 1 of L in order to obtain the second term of (1). See reference 3.

The essential of Lévy's method of calculations will be first briefly illustrated, the notations being the same as those used in L. The two body wave function $\phi(\boldsymbol{r})$ obeys the equation of the form

$$\left(\frac{\Delta}{M}+\varepsilon\right)\phi(\boldsymbol{r})=\int d\boldsymbol{r}'\,U(\boldsymbol{r},\boldsymbol{r}';W)\phi(\boldsymbol{r}') \qquad (2)$$

in the center of mass system. In the analysis of low energy properties W is replaced by $2M$ and

$$U_{2n}(\boldsymbol{r},\boldsymbol{r}')=(2\pi)^{-3}\int d\boldsymbol{p}\,d\boldsymbol{p}'\Delta_{2n}(\boldsymbol{p},\boldsymbol{p}')\,e^{i\boldsymbol{p}\boldsymbol{r}-i\boldsymbol{p}'\boldsymbol{r}'},$$

where $\Delta_{2n}(\boldsymbol{p},\boldsymbol{p}')$ is the interaction term extensively discussed by Lévy previously.[1] For the fourth order processes $(n=2)$ $\Delta_4(\boldsymbol{p},\boldsymbol{p}')$ involves an integration over one arbitrary intermediate momentum. Thus, writing $\Delta_4(\boldsymbol{p},\boldsymbol{p}')=\int d\boldsymbol{k}_1\,\bar{\Delta}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{k}_1)$, we have

$$U_4(\boldsymbol{r},\boldsymbol{r}')=\int d\boldsymbol{k}_1\,d\boldsymbol{k}\,\bar{\Delta}_4(\boldsymbol{k},\boldsymbol{k}_1)\,e^{i\boldsymbol{k}\boldsymbol{r}}\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad (3)$$

if $\bar{\Delta}_4(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{k}_1)=\bar{\Delta}_4(\boldsymbol{p}-\boldsymbol{p}',\boldsymbol{k}_1)$ and $\boldsymbol{p}-\boldsymbol{p}'=\boldsymbol{k}$. From (2) and (3) one can define the fourth order potential by

$$V^4(r)=\int d\boldsymbol{k}_1\,d\boldsymbol{k}\,\bar{\Delta}_4(\boldsymbol{k},\boldsymbol{k}_1)\,e^{i\boldsymbol{k}\boldsymbol{r}}. \qquad (4)$$

Then $V^4(r)$ can readily be calculated by writing down $\bar{\Delta}_4$, which thing is easily done by means of diagram representations of various processes. One-pair processes which give contributions of the order $(G^2/4\pi)^2(\mu/2M)^3$ are shown in Fig. 2 of L.

In writing down Δ's we note the following relations:

$$\begin{aligned}&\langle P|H'|P''\rangle\langle P''|H'|P'\rangle\\&\quad\to u_1^*(\boldsymbol{p})\rho_2 u_1(\boldsymbol{p}'')u_1^*(\boldsymbol{p}'')\rho_2 u_1(\boldsymbol{p}')\\&\quad=\Gamma_{11}(\boldsymbol{p},\boldsymbol{p}'')\Gamma_{11}(\boldsymbol{p}'',\boldsymbol{p}'),\\&\langle P|H'|\bar{P},\bar{P}'',P'\rangle\langle\bar{P},\bar{P}'',P'|H'|P'\rangle\\&\quad\to-\Gamma_{12}(\boldsymbol{p},-\boldsymbol{p}'')\Gamma_{21}(-\boldsymbol{p}'',\boldsymbol{p}'),\end{aligned} \qquad (5)$$

except the meson part of the matrix elements $(2\pi)^{-3/2}(2\omega)^{-1/2}$ and the appropriate isotopic spin factors. $|P,\bar{P}'',P'\rangle$, e.g., corresponds to the states in which two nucleons with their respective momenta $\boldsymbol{p}$, $\boldsymbol{p}'$ and one antinucleon with $\bar{\boldsymbol{p}}''$ exist. u_1 and u_2 are

the positive and negative energy eigenfunctions, respectively. We further note

$$\Gamma_{11}(\boldsymbol{p}, \boldsymbol{p}'') = \frac{}{2M}(\boldsymbol{\sigma}\cdot\boldsymbol{p}'-\boldsymbol{p}'')$$

and $$\Gamma_{12}(\boldsymbol{p}, -\boldsymbol{p}'')\Gamma_{21}(-\boldsymbol{p}'', \boldsymbol{p}') = 1 \qquad (6)$$

in the nonrelativistic region. With (5) and (6) $\bar{\Delta}$'s are readily written down. Once $\bar{\Delta}$'s have been written down the potential can be easily calculated referring eq. (4) above. Thus, diagrams a, b, and c in Fig. 2 of L lead to the expressions (17), (18), and (19) of L, respectively.[6] Hence the contributions from the processes illustrated in Fig. 2 of L do not cancel each other but are given by

$$-6\frac{G^4}{(2\pi)^6}(2M)^{-3}\int d\boldsymbol{k}_1 d\boldsymbol{k}_2 \times\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{\omega_1^2\omega_2^2}e^{-i(\boldsymbol{k}_1+\boldsymbol{k}_2)r}. \qquad (7)$$

Integrations in (7) can be performed using a convention

$$\int d\boldsymbol{k}_1 d\boldsymbol{k}_2 \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{\omega_1^2\omega_2^2}e^{-i(\boldsymbol{k}_1+\boldsymbol{k}_2)r} \to -\lim_{r'\to r}(\nabla\cdot\nabla')\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\frac{1}{\omega_1^2\omega_2^2}e^{-i(\boldsymbol{k}_1 r+\boldsymbol{k}_2 r')}.$$

The potential is repulsive and is given by

$$6\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^3\frac{1}{\mu r^2}\left[\left(1+\frac{1}{\mu r}\right)e^{-\mu r}\right]^2. \qquad (8)$$

This almost cancels out the attraction of the same order given by the second term of eq. (1). One may expect, therefore, that Lévy's numerical values for $G^2/4\pi$ and r_0 should be considerably modified in order to fit the low energy data. It is also interesting to note that the potential (8) is obtained if one calculates the contribution of the processes of the type shown in Fig. 2, the interaction Hamiltonian being of the form

$$\frac{i}{2}\frac{G}{M}\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\frac{\partial\varphi_i}{\partial x_\mu}+\frac{1}{2}\left(\frac{G^2}{M}\right)\bar{\psi}\psi\varphi_i^2, \qquad (9)$$

which is the well-known consequence of Dyson transformation[7] on the pseudo-scalar interaction.

Fig. 2. A typical process which give the potential (8) if the interaction Hamiltonian is given by (9).

Finally it must be remarked that there are six other possible diagrams than those shown in Fig. 2 of L, which give rise to the one-pair terms. These are processes which involve one or two mesons emission (or absorption) by one nucleon while the other nucleon experiences negative energy states. Typical two diagrams are given in Fig. 3. Four diagrams of the type M_1 give contributions of the order $(G^2/4\pi)^2(\mu/2M)^4$ and two of the type M_2 do of the order $(G^2/4\pi)^2(\mu/2M)^5$. These do not affect the potential to the order considered and can be ignored.

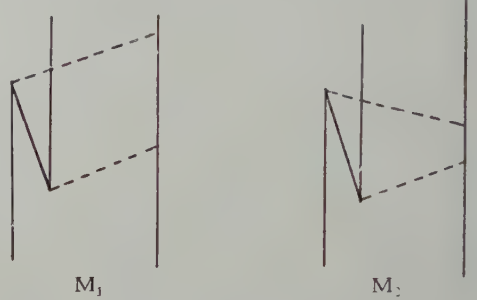

Fig. 3. Two typical processes other than those given in Fig. 2 of L and which give rise to one-pair terms of higher orders.

We are very much indebted to Mr. T. Matsumoto for his earnest discussions in the course of the present analysis One of the writers (T. H.) wishes to express his gratitude to Yomiuri Yukawa Fellowship for financial support.

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6) Our results of calculations coincide formally with expressions (17), (18), and (19) in *L*. We consider, however, that there must be some misprints in these equations since the conclusion that $V_4^{(a')}+V_4^{(b')}+V_4^{(c')}=0$ can never be reached from these expressions.
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A Non-local Wave Equation and its Connection with the Dirac Particle

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April 14, 1953

Since the Schrödinger equation for state vector originally works in the Hilbert space, it must in the ordinary space representation take a form of generally a 'non-local' wave equation, even in the case for a single particle. Really in this case, however, we have the well-known localized wave equations in virtue of the specific simplicity of the Hamiltonian, which is a second or first order *polynomial* in the momentum $\boldsymbol{p}$. Yet even in such a case (e.g. with Hamiltonian $H=\boldsymbol{p}^2/2m+V(\boldsymbol{x})$, if we pass over to the interaction representation through the canonical transformation with the (momentum-dependent) unitary operator, $U=\exp\left(\frac{i}{\hbar}\frac{t}{2m}\boldsymbol{p}^2\right)$, we obtain in the ordinary space representation a 'non-local' wave equation in case of general external potential $V(\boldsymbol{x})$. In this note we shall exhibit and examine a case where the wave equation for a particle is *essentially* 'non local'.

As is well known, the classical relativistic motion of a free particle (of mass m) is derivable from the Hamiltonian

$$H_\pm=\pm(\boldsymbol{p}^2+m^2)^{1/2}, \qquad (1)^*$$

and so we can consider the corresponding quantum-mechanical system.[1] The Schrödinger equation for such system is

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t}\pm(\boldsymbol{p}^2+m^2)^{1/2}\right\}\varphi_\pm(\boldsymbol{p},t)=0, \qquad (2)$$

where $\varphi_\pm(\boldsymbol{p},t)$ is the wave function in the momentum representation. From (2) the wave equation in the coordinate representation is obtained as

$$\frac{1}{i}\frac{\partial\psi_\pm}{\partial t}\pm\int\eta(\boldsymbol{x}-\boldsymbol{x}')\psi_\pm(\boldsymbol{x}',t)\,d\boldsymbol{x}'=0, \qquad (3)$$

where $\hbar\eta(\boldsymbol{x}-\boldsymbol{x}')=(\boldsymbol{x}|H_+|\boldsymbol{x}')$, i.e.,

$$\eta(\boldsymbol{x})=\frac{1}{(2\pi)^3}\int(\boldsymbol{k}^2+\varkappa^2)^{1/2}e^{i\boldsymbol{k}\boldsymbol{x}}\,d\boldsymbol{k}. \quad (\varkappa=m/\hbar) \qquad (4)$$

Its explicit form is

$$\eta(\boldsymbol{x})=-\frac{\varkappa^2}{2\pi^2}\frac{K_2(\varkappa r)}{r^2} \quad (r\equiv|\boldsymbol{x}|)$$

$$\sim\begin{cases}-\dfrac{1}{\pi^2}\dfrac{1}{r^4}+\dfrac{\varkappa^4}{16\pi^2}\log\dfrac{\varkappa r}{2}, & \varkappa r\ll 1\\[2ex] -\dfrac{\varkappa^4}{(2\pi)^{3/2}}\dfrac{e^{-\varkappa r}}{(\varkappa r)^{3/2}}, & \varkappa r\gg 1\end{cases}$$

decreasing exponentially for large r but having appreciable values for $r\lesssim\lambda_0$ (the Compton wave length). Thus the wave equation (3) is surely of non-local type, though $\eta(\boldsymbol{x})$ diverges as r^{-4} for $r\to 0$, with its space integral also diverging.

To see the 'non relativistic' limit, we expand (4) in the descending power of $\varkappa$, and get

$$\eta(\boldsymbol{x})=\varkappa\left(1-\frac{1}{2\varkappa^2}\nabla^2-\frac{1}{8\varkappa^4}\nabla^4+\cdots\right)\delta(\boldsymbol{x}).$$

Here $\eta(\boldsymbol{x})$ is expressed as an *infinite* series in the δ-function and its derivatives, so that it is essentially an integral operator as seen above. Eq. (3) then takes the form

$$\frac{\hbar}{i}\frac{\partial\psi_\pm}{\partial t}\pm\left(m\psi_\pm-\frac{\hbar^2}{2m}\nabla^2\psi_\pm-\frac{1}{8}m\varkappa^4\nabla^4\psi_\pm+\cdots\right)=0,$$

indicating that the usual non-relativistic Schrödinger equation corresponds to the λ_0^2-approximation for our wave equation with positive sign for (1).

The content of our wave equation will more properly be expressed in the form of an initial value problem; that is obtained as

$$\psi_\pm(\boldsymbol{x},t)=\int 2\frac{\partial}{\partial t'}\Delta^\pm(\boldsymbol{x}-\boldsymbol{x}',t-t')\times\psi_\pm(\boldsymbol{x}',t')\,d\boldsymbol{x}', \qquad (5)$$

* We use the units such that $c=1$. In this and the following equations the double sign $\pm$ means that either of $\pm$ is to be taken in the same order, unless otherwise stated.

where the functions $\Delta^{\pm}$ are the positive and negative frequency parts of the invariant Δ-function respectively, and are themselves relativistic invariants:

$$\Delta^{\pm} = \mp \frac{i}{(2\pi)^3} \int_{\substack{k_0>0 \\ \text{or } k_0<0}} \delta(k_\mu^2 + x^2)\, e^{ik_\mu x_\mu}\, d^4k$$

$$= \frac{1}{2}(\Delta \mp i\Delta^{(1)}).$$

The explicit form is

$$\Delta^{+} = -\frac{1}{4\pi}\varepsilon(t)\,\delta(\lambda) + \begin{cases} \dfrac{\varkappa}{8\pi}\dfrac{H_1^{(2)}(\varkappa\lambda^{1/2})}{\lambda^{1/2}}, & t > r > 0 \\ -\dfrac{\varkappa}{8\pi}\dfrac{H_1^{(1)}(\varkappa\lambda^{1/2})}{\lambda^{1/2}}, & t < -r < 0 \\ -\dfrac{i\varkappa}{4\pi^2}\dfrac{K_1(\varkappa(-\lambda)^{1/2})}{(-\lambda)^{1/2}}, & |t| < r \end{cases}$$

with $\Delta^{-} = \Delta^{+}{}^{*}$, where $\lambda = t^2 - r^2$. They have appreciable (imaginary) values also outside the light-cone over the region of the order of λ_c, and moreover diverge in the neighbourhood of the light-cone in both sides as, e.g.,

$$\Delta^{+} \sim D^{+} = \begin{cases} -\dfrac{1}{4\pi}\,\delta_{+}(\lambda), & t > 0 \\ \dfrac{1}{4\pi}\,\delta_{-}(\lambda), & t < 0. \end{cases}$$

Our wave equation selects such solutions of the Klein Gordon equation as to propagate solely with positive (or negative) frequencies, and corresponds to an alternative linearization (in respect to time) of the latter equation other than the Dirac's one. Accordingly, these solutions must of course satisfy the Klein-Gordon equation, as is clear from the expression (5) and the property of $\Delta^{\pm}$: $(\Box - \varkappa^2)\Delta^{\pm} = 0$. The reason why our $\psi_{\pm}$ that propagates extending beyond the light-cone could nevertheless satisfy the propagation equation for Klein-Gordon wave:

$$\psi(\boldsymbol{x}, t) = \int \left\{ -\Delta(\boldsymbol{x}-\boldsymbol{x}',\, t-t')\frac{\partial}{\partial t'}\psi(\boldsymbol{x}'t') + \frac{\partial}{\partial t'}\Delta(\boldsymbol{x}-\boldsymbol{x}',\, t-t')\cdot\psi(\boldsymbol{x}'t') \right\} d\boldsymbol{x}',$$

can be explained by use of (3) and of the relation,

$$\int \Delta(\boldsymbol{x}-\boldsymbol{x}',\, t-t')\,\eta(\boldsymbol{x}'-\boldsymbol{x}'')\,d\boldsymbol{x}' = \frac{\partial}{\partial t}\Delta^{(1)}(\boldsymbol{x}-\boldsymbol{x}'',\, t-t').$$

We started from the 'relativistic' Hamiltonian (1), but this is not sufficient to warrant the relativistic covariance of our wave equation. In fact the non-parametric form of the Hamiltonian formalism we used led to the wave equation (2) or (3), which is clearly not manifestly covariant. It is also to be noted that the right-hand expression of (5) cannot be generalized to an integral over arbitrary space-like surface σ:

$$-2\int_{\sigma} \frac{\partial}{\partial x_\nu'}\Delta^{\pm}(x-x')\cdot\psi_{\pm}(x')\,d\sigma_\nu',$$

and that the action function $I = \int L\,dt$ with

$$L = \int \frac{1}{i}\psi_{\pm}{}^{*}\frac{\partial\psi_{\pm}}{\partial t}\,d\boldsymbol{x} \pm \iint \psi_{\pm}{}^{*}(\boldsymbol{x}\,t)\,\eta(\boldsymbol{x}-\boldsymbol{x}')\,\psi_{\pm}(\boldsymbol{x}'t)\,d\boldsymbol{x}\,d\boldsymbol{x}',$$

which gives (3) via the variational principle, is not Lorentz invariant, at least as far as we assume $\psi_{\pm}$ to be an invariant. Moreover the circumstance that our wave function propagates extending beyond the light-cone is at variance with e.g. the Dirac wave function, and seems to contradict with relativistic requirements.

Since, however, our wave equation is the *Schrödinger* equation that implies in the proper correspondence-theoretic meaning the relativistic relationship between momentum and energy in the classical theory, wave equations for any free relativistic wave fields must have certain connection with this wave equation, insofar as they should involve the particle picture. Thus, for instance, our wave equation can be regarded as the transformed wave equation for a free Dirac particle by suitably arranging $\psi_{\pm}$ into a spinor. More explicitly, we transform the Dirac equation by the canonical transformation due to Foldy and others[2] with the non-covariant unitary operator:

$$U(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\left(1+\frac{m}{f_0}\right)^{1/2} + \frac{1}{\sqrt{2}}\,\frac{\beta\,\boldsymbol{\alpha}\boldsymbol{p}}{\{f_0(f_0+m)\}^{1/2}},$$

with $f_0 = (p^2+m^2)^{1/2}$.

Here, taking the usual representation that makes β diagonal, the transformed Dirac equation is *perfectly* separated into four independent equations for each spinor component, the upper and lower components just satisfying our previous wave equations (2) and so (3), with positive or negative sign respectively. Thus if we regard our $\psi_{\pm}$ not as a scalar but as upper or lower components of a spinor, our wave equation gains the relativistic meaning, though their transformation property for Lorentz group is rather complicated.

The explicit form of the Foldy transformation in its x-representation is

$$\psi(\boldsymbol{x}) \to \psi'(\boldsymbol{x}) = \int U(\boldsymbol{x}-\boldsymbol{x}')\,\psi(\boldsymbol{x}')\,d\boldsymbol{x}',$$

$$U(\boldsymbol{x}) = \frac{1}{2\sqrt{2}\,\pi}\,\frac{\delta(r)}{r^2} + \frac{\boldsymbol{x}^3}{2\sqrt{2}\,\pi^2}\left(A(\boldsymbol{x}r) + i\beta\,\frac{\boldsymbol{\alpha}\boldsymbol{x}}{r}\,B(\boldsymbol{x}r)\right),$$

with

$$A(z) = \sum_{n=1}^{\infty} \frac{1}{2n}\, b_n\, \bar{K}_{(n-1)/2}(z),$$

$$B(z) = \sum_{n=1}^{\infty} b_n\, \bar{K}_{(n-3)/2}(z),$$

$$\bar{K}_\nu(z) \quad z^\nu K_\nu(z),$$

$$b_n = \frac{(-1)^{n-1}\,\Gamma((n-1)/2)}{2^{(n-1)/2}\,\Gamma(n/2)\,\Gamma(n)}.$$

The transformation is such as to make the (Dirac) Hamiltonian diagonal in respect to spinor indices and also make at the same time 'mean position' operator[2] of particle diagonal, which is the combination of the 'true' position operator and the *momentum*-dependent Zitterbewegung subtracted therefrom; so that in the state where true position of particle is localized at $\boldsymbol{x}_0$, the mean position is indefinite over the region of the order of λ_2 about $\boldsymbol{x}_0$, as is exhibited in the above equations. From this reason ψ' must propagate somewhat extending beyond the light-cone, implying the effect of Zitterbewegung.

Finally we briefly remark that the formulation of quantum mechanics in terms of the picture of trajectory ensemble developed in a previous article[3] is inapplicable to the case where the wave equation involves integral operator* as in the present problem.

Fuller account of this and related work will soon be published.

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3) T. Takabayasi, Prog. Theor. Phys. 8 (1952), 143. Also see D. Bohm, Phys. Rev. 84 (1952), 166, 180.

* See Reference 3), § 11 (a).

The Theory of the Structure of Elementary Particles*

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April 26, 1953

Although recent progress in the theory of elementary particles has gained brilliant success on one hand, the essential limitation of the theory itself seems to have become clear especially in these two or three years. It is in connection, for example, with a remarkable regularity between the rest mass of elementary particles as first pointed out by Nambu,[1] with curious nature of V particles which seems hard to be understood within the framework of the current theory, or with divergence difficulties inherent in the quantum field theory from the day of its birth. The most serious weak point of the current theory is that the assignment of spin or rest mass values to elementary particles or the introduction of the mutual interaction between them must be done entirely *ad hoc*.

Under this circumstance, it would be of great interest to investigate the structure of elementary particles, and thus try to elucidate the intrinsic correlation that may lie behind them. In previous papers[4),5)] on Yukawa's theory of non-local field one of the authors suggested to regard the elementary particles as corresponding to various states of the internal motion of a kind of "*Urmaterie*", and pointed out that the non-local field would be nothing but the one that would describe the Urmaterie.** Some development of this idea will be discussed in this note.

Taking the simplest case of scalar, we assume that the Urmaterie is described by a scalar non-local field $U(X_\mu, r_\mu)$. To find constants of internal motion which are responsible to the structure of elementary particles, it is instructive to recall of Fierz's remark

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** Recently Prof. H. Yukawa published a similar attempt.[5)]

that angular momentum of the internal motion just corresponds to spin. Angular momentum being a very inadequate concept from the relativistic view point, however, it is necessary to generalize it so as to fit better the requirement of relativity. For this purpose we introduce an infinitesimal rotational operator of the internal coordinate in four dimensional space;

$$\boldsymbol{R}_{\mu\nu}=\frac{1}{i}\left(r_\mu\frac{\partial}{\partial r_\nu}-r_\nu\frac{\partial}{\partial r_\mu}\right). \qquad (1)$$

Of course the square of $\boldsymbol{R}_{\mu\nu}$ contains spin part, since (1) includes infinitesimal rotation in three dimensional space as its special case. In general the square of $\boldsymbol{R}_{\mu\nu}$ can be separated into spin part $\boldsymbol{S}^2$ and its counter part $\boldsymbol{M}^2$

$$\boldsymbol{R}_{\mu\nu}\boldsymbol{R}^{\mu\nu}=\boldsymbol{S}^2+\boldsymbol{M}^2, \qquad (2)$$

the separation being done in a Lorentz invariant way if some technique is used.

It would be reasonable to expect that $\boldsymbol{M}^2$ is also responsible to the structure of elementary particles just as $\boldsymbol{S}^2$ was. The physical meaning of $\boldsymbol{M}^2$ becomes clear on assuming

$$(r_\mu r^\mu-\lambda^2)U(X_\mu, r_\nu)=0. \qquad (3)$$

Under this condition it can be shown that the eigenvalue of $\boldsymbol{M}$, m is equal to $(\lambda/\hbar c)$ times an expectation value of the energy of the internal motion in the rest system of the center of mass. This fact, together with its invariant character under Lorentz-transformations, enables us to interpret m as the rest mass of elementary particles measured in unit $(\hbar/c\lambda)$.*

Thus, eigenvalue equations for $\boldsymbol{S}^2$ and $\boldsymbol{M}^2$ determine the spin and mass spectrum of elementary particles. Of course the former gives an usual result

$$s(s+1) \quad (s=0, 1, 2, \cdots), \qquad (4)$$

while the latter leads to a hypergeometric equation

$$\left(\frac{d^2}{dx^2}+\frac{3x}{1+x^2}\frac{d}{dx}-\frac{s(s+1)x^2}{(1+x^2)^2}+\frac{m^2}{1+x^2}\right)\varphi(x)=0, \qquad (5)$$

where x is a quantity related in some way to internal coordinates. By solving equation (5) under the boundary condition that $\varphi(x)$ should vanish as $x\to\infty$, eiganvalues of m is given in a relation with s;

$$m^2=(s+1)+4n(s-n),$$
$$n=0, 1, 2, \cdots\cdots\leqq\frac{1}{2}(s-1). \qquad (6)$$

The non-local field provides us, therefore, with an unified description of elementary particles and when decomposed into simultaneous eigenfunctions of $\boldsymbol{S}^2$ and $\boldsymbol{M}^2$, each component describes elementary particles with definite spin and rest mass according to*

$$(\boldsymbol{S}^2-s(s+1))U(X_\mu, r_\mu;\ s, m)=0,$$
$$(\Box-(m/\lambda)^2)U(X_\mu, r_\nu;\ s, m)=0. \qquad (7)$$

Although scalar Urmaterie discussed here gives elementary particles of integer spin only, those with half-integer can easily be obtained if $U(X_\mu r_\mu)$ is taken to be spinor. Thus, our picture stands a possibility of regarding all elementary particles as corresponding to various states of the internal motion of only two kinds of Urmaterie, that is Bose and Fermi.

It would be worthwhile to point out in this connection that new quantum number of our theory is responsible to new selection rules for the transformation of elementary particles. We hope that they play some role in dissolving difficulties on V particles.

The deduction of the spin and mass spectrum in the case of spinor Urmaterie is, together with the investigation of new selection rules suggested above, now in progress, and the result will be reported soon.

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* This means, in other word, to interpret j and j' of the indices of the irreducible Lorentz group $\mathcal{D}_{jj'}$ as spin and rest mass in some rearranged combination.

* Strictly speaking, non-local fields described by equations (7) and (3) are not completely equivalent to corresponding wave functions in local field theory[6] owing to the lack of Yukawa's third equation; $r_\mu(\partial U(X_\mu, r_\mu)/\partial X_\mu)=0$.[4), 7)] This point will be discussed in later investigations.

The Theory of the Interactions with Higher Derivatives

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When we want to develop the theory of the non-local interaction, it is usual to take the method of Yang and Feldman[1] and then construct the S-matrix by connecting the incoming wave with the outgoing wave, because there does not exist the so-called Hamilton formalism. However, it is the essential point of Yang-Feldman's theory that one can define the Hamiltonian in the interaction representation.[2] Even in the case of the non-local interaction, it is shown possible to construct the Hamilton formalism, so long as one does not confuse the canonical variables describing the dynamical system with the independent variables of the variation, and keep in mind the essential differences between the interaction representation and Heisenberg's one.

To clarify these circumstances, we assume that the non-local interactions are the (infinite) sums of the interactions with higher derivatives, and then develop the theory in this case.

We start with the interaction Lagrangian involving with the certain differential operators, $\bar{O}(\partial)$, $O(\partial)$ and $P(\partial)$, such as

$$L_I = g\bar{O}(\partial)\bar{\psi}(x)O(\partial)\psi(x)P(\partial)\psi(x). \quad (1)$$

If we take the wave which satisfies the equation $(\gamma\partial+\varkappa)\psi(x,\sigma)=0$ as the initial condition, we can connect the Heisenberg operators with the interaction operators by a set of equations

$$\begin{aligned}\psi(x) &= \psi(x,\sigma) \\ &- g\int dx'\bar{O}(-\partial)S^{\sigma}(x-x')O(\partial')\psi(x')P(\partial')\psi(x') \\ &+ \int^{\sigma} dx' H_1(x,x'), \quad \text{etc.}\end{aligned} \quad (2)$$

and assuming the existence of the unitary transformation of the operators in the interaction representation between themselves

$$\begin{aligned}\psi(x,\sigma) &= U^{-1}(\sigma,\sigma')\psi(x,\sigma')U(\sigma,\sigma') \\ &= U^{-1}(\sigma,-\infty)\psi_{in}(x)U(\sigma,-\infty),\end{aligned} \quad (3)$$

we can define the Hamiltonian by a set of the following equations:

$$\begin{aligned}&-i[\psi(x,\sigma),\ \mathcal{H}_I(x'|\sigma)] \\ &= g\bar{O}(-\partial)S(x-x')O(\partial')\psi(x')P(\partial')\psi(x') \\ &- H_1(x,x'), \quad \text{etc.}\end{aligned} \quad (4)$$

The additional terms in the equations (3) and (4) are the four-dimensional divergence quantities which satisfy the free equations of motion and are determined so as to exist the interaction Hamiltonian. We can also rewrite the equations (2), such as

$$\begin{aligned}\psi(x) &- \frac{g^2}{2}\int d\lambda'[O(-\partial)(\gamma\partial-\varkappa),\ \varepsilon(x-\lambda')] \\ &\times \Delta(x-\lambda')O(\partial')\psi(x')P(\partial')\psi(\lambda') \\ &- \int^{x} dx' H_1(x,\lambda') \\ &= \psi(x,\sigma) - \frac{i}{2}\int dx'[\varepsilon(x-x') \\ &- \varepsilon(\sigma,x')][\psi(x,\sigma),\ \mathcal{H}_I(x'|\sigma)]\end{aligned} \quad (5)$$

by virtue of the definitions of Hamiltonian (4).

In the cases of the interactions with higher derivatives, the interaction Hamiltonian thus obtained are generally expressed by the infinite series of the coupling constant. But we can confirm that they become to be the time-displacement operators and satisfy the integrability conditions.

While the orthodox method of the Hamilton formalism was developed early by Heisenberg-Pauli.[3] According to their procedures, firstly we define the canonical conjugates $(Q_0,\ P_0)$ in the free field, and then in the interacting system we also define them $(\boldsymbol{Q},\ \boldsymbol{P})$ analogously, and the quantization is treated by either of the following equations:

$$[Q_0(x),\ P_0(x')]_{t=t'} = i\delta(\boldsymbol{r}-\boldsymbol{r}'), \quad (6)$$

$$[\boldsymbol{Q}(x),\ \boldsymbol{P}(x')]_{t=t'} = i\delta(\boldsymbol{r}-\boldsymbol{r}'). \quad (7)$$

If there would exist the unitary transformation between these two sets, the procedures should be consistent with each other and in fact this consistency holds so far. In the cases of the interactions with higher derivatives, however, there does not exist generally such unitary transformation and the above consistency breaks down.

In order to overcome this difficulty, we can define many sets of the canonical conjugates $(Q_0^r,\ P_0^r)$ by modifiing the free field at the outset, so as to connect them $(\boldsymbol{Q}^r,\ \boldsymbol{P}^r)$ of the interacting system by the unitary transformation. This procedure becomes to be equivalent with the theory of the non-localized action.

There is another way to overcome this difficulty. We can define the canonical conjugates which are

connected with the free ones (Q_0, P_0) by the unitary transformation

$$Q(x/\sigma) = U^{-1}(\sigma, -\infty) Q_0(x) U(\sigma, -\infty)_{x/\sigma},$$
$$P(x/\sigma) = U^{-1}(\sigma, -\infty) P_0(x) U(\sigma, -\infty)_{x/\sigma} \quad (8)$$

and can take them as the canonical variables in the interacting system instead of the Heisenberg operators. Then we find from (5)

$$Q = \boldsymbol{\phi}(x) - \frac{g}{2} \int dx' [\bar{O}(-\partial)(\gamma\partial - \varkappa), \varepsilon(x-x')] \times \Delta(x-x') O(\partial') \boldsymbol{\phi}(x') P(\partial') \boldsymbol{\phi}(x') - \int_{\sigma(x)} dx' H_1(x, x') \equiv \dot{\phi}(x/\sigma); \quad (9)$$

$$Q \equiv Q_0 \text{ (numerically equal)}, \quad (10)$$

$$P \equiv i\bar{\psi}(x/\sigma)\gamma_4; \quad P \equiv P_0 \text{ (numerically equal)}$$

and

$$\dot{Q} = \dot{\phi}(x/\sigma) - i[\psi(x/\sigma), \int d\nu' H(x'/\sigma)]_{t=t'} \quad (11)$$

by differentiating (5) with time. And we also confirm that

$$P = \frac{\delta L_{tot}}{\delta \dot{Q}} \quad \text{and} \quad \frac{\delta L_{tot}}{\delta Q^{(r)}} = 0 \quad (r > 1) \quad (12)$$

hold. The Hamiltonian in the interaction representation is given by the usual method

$$\mathcal{H}_I = P\dot{Q} - P_0\dot{Q}_0 - (L_{tot}(Q, \dot{Q}) - L_0(Q_0, \dot{Q}_0)) \quad (13)$$

and this becomes to be nothing else the Hamiltonian obtained directly from (4).

In this theory, if one takes (9) and (10) as the canonical variables of the system, there does not still remain any higher time derivative and the system is reduced to the ordinary system. Also in the case of the non-local interaction, this procedure still holds. The essential non-locality is cancelled out there and the Hamilton formalism does exist in the interaction representation. But the non-locality which comes from the spatial differentials remains and plays the role to remove the so-called divergences.

The difficulty of the negative energy does not appeared in this theory contrary to Pais-Uhlenbeck's theory, but the converging technique becomes slightly complicated and if there does not appear the negative energy, both theory give the same results. Full account will be appeared soon later in this journal.

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On the Interaction of Additive Electrons with Polarization in Ionic Crystals, I

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As a preliminary, we express the energy increase of an ionic crystal due to arbitrary polarization in quadratic forms of two parts of polarization from the stand-point of dielectric continuum. Using this formulation we investigate how to solve the motion of an additive electron in an ionic crystal under the interaction with the polarization which the electron produces around itself, especially in the case when there is a trapping-potential. We show that one is led to the usual method of treating such problems suggested by Mott and Gurney, when one uses the adiabatic approximation (in the sense of Born and Oppenheimer) for the system of an additive and the crystal electrons. On the other hand, if we proceed on the Hartree's approximation for the system, another method of treating these problems is obtained. The latter method is applied to the calculation of various thermal and optical energies of the F- and F'-centers, conduction electrons and polarons, with a simplified model. The essential aspects of this method are stressed in the last section.

1. Introduction

In dielectric substance, an additive electron moves under interaction with polarization which it produces around itself. In case of ionic crystals, this polarization consists of two parts, that is, "electronic polarization" due to the electron redistribution inside each ion, and "displacement polarization" which is nothing but the relative displacement of positive and negative ions. Usually the motion of crystal electrons which is connected with the former polarization is more rapid than that of the additive electron (which may be a conduction electron, or a loosely bound electron at some imperfection); on the other hand, displacement of each ion corresponding to the latter polarization takes place much more slowly than that of the electron. Under these circumstances, *it is the usual*[1)2)3)] *method of treating motion of the electron to consider the electronic polarization as merely giving an electrostatic shielding effect, and the displacement polarization as being determined by the average distribution of the electron.*

This rather intuitive method based upon classical electrostatics, however, must be criticized from quantum-mechanical standpoint; that is, after formulating the motion of the whole system (crystal + additive electron) quantum-mechanically, one must investigate to what sort of approximation the usual method corresponds.

We shall show, in §§ 4 and 5 of the present paper, that the *adiabatic separation* of the motions of crystal electrons and the additive electron leads to the usual method,

and then point out the question of applicability of the approximation and some difficulties associated with this treatment. In § 6 we take the *self-consistent method in Hartree's sense* and develop a new method of treating the motion of the additive electron in ionic crystals. In § 7, then, we apply this method to F- and F'-centers, a conduction electron and a polaron with a simplified model, and calculate various optical and thermal energies, the results being compared with observations. We suggest, in § 8, that this new method gives a possibility of explaining finiteness of the numbers of excited states in various color-centers despite of their Coulomb-like trapping potentials.

§§ 2 and 3 are devoted to devise a method of describing the polarization energy with a quadratic form in two sorts of polarizations, which will prove to be very convenient in the later sections.

§ 2. Energy due to polarization

As regards the dielectric properties, an ionic crystal is characterized by three quantities : static dielectric constant $\varkappa$, dielectric constant at high frequencies $\varkappa_\infty$, and effective charge e^*. We attempt here to devise appropriate expressions for the energy due to arbitrary polarization (an arbitrary function of position, and not necessarily in electrostatic equilibrium).

As stated in the preceding section, the total polarization $\boldsymbol{P}$ consists of two parts, namely electronic polarization $\boldsymbol{q}_1$ and displacement polarization $\boldsymbol{q}_2$:

$$\boldsymbol{P}=\boldsymbol{q}_1+\boldsymbol{q}_2. \tag{2·1}$$

When the ions of the crystal is of valence Z, $\boldsymbol{q}_2$ is expressed as

$$\boldsymbol{q}_2=NZ\boldsymbol{u},$$

where $\boldsymbol{u}$ is the local mean value of the relative displacements of positive and negative ion pairs, and N is the number of pairs of ions in a unit volume. Moreover we use atomic units.

Now we suppose external field $\boldsymbol{E}_0$ to be applied. ($\boldsymbol{E}_0$ is purely due to external sources and does not contain the shielding effect of the dielectric substance now considered, in contrast to electric field intensity $\boldsymbol{E}$). So long as the linearity of dielectric properties holds, and the retardation of potentials can be neglected, the energy U of the dielectric substance due to arbitrary polarization, under the applied field $\boldsymbol{E}_0$, is expressed in the following quadratic form :

$$\begin{aligned} U(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{E}_0) = &\int (-\boldsymbol{E}_0\cdot\boldsymbol{q}_1-\boldsymbol{E}_0\cdot\boldsymbol{q}_2)dv \\ &+\int\Big(\frac{1}{2}a\boldsymbol{q}_1^2+\frac{1}{2}b\boldsymbol{q}_2^2+c\boldsymbol{q}_1\cdot\boldsymbol{q}_2\Big)dv \\ &+\frac{1}{2}\int \boldsymbol{P}\cdot\Gamma\varphi_P\,dv\,. \end{aligned} \tag{2·2}$$

The first integral expresses the interaction of the polarization with the external field, and the third integral corresponds to the long range dipole-dipole interaction, where φ_P, the

electrostatic potential due to the polarization, can be expressed with $\boldsymbol{P}$ as follows:

$$\varphi_P=\int\frac{\rho_P}{r}dv, \qquad \rho_P=-\Delta\cdot\boldsymbol{P}. \tag{2·3}$$

The second integral in (2·2) expresses the internal energy of the dielectric substance due to the polarization, consisting of the short-range part and that part due to Lorentz field; the latter being expressed for cubic crystals as follows:

$$-\frac{1}{2}\frac{4\pi}{3}\int\boldsymbol{P}^2dv=-\frac{1}{2}\frac{4\pi}{3}\int(\boldsymbol{q}_1+\boldsymbol{q}_2)^2dv.$$

a, b, c in (2·2) are constants characteristic of the dielectric substance and can be expressed with three dielectric quantities $\varkappa$, $\varkappa_0$, c^*, as we shall see in the next section. The term containing c is the interaction between two kinds of polarizations.

Since (2·3) leads to the equation

$$\nabla\cdot(4\pi\boldsymbol{P}-\nabla\varphi_P)=0,$$

we can express $\boldsymbol{P}$ as

$$4\pi\boldsymbol{P}=\nabla\varphi_P+\nabla\times\boldsymbol{A} \tag{2·4}$$

with an appropriate vector field $\boldsymbol{A}$. In the same way $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ can be written as

$$\left.\begin{aligned}4\pi\boldsymbol{q}_1&=\nabla\varphi_1+\nabla\times\boldsymbol{A}_1,\\ 4\pi\boldsymbol{q}_2&=\nabla\varphi_2+\nabla\times\boldsymbol{A}_2,\end{aligned}\right\} \tag{2·4′}$$

where

$$\left.\begin{aligned}\varphi_P&=\varphi_1+\varphi_2,\\ \boldsymbol{A}&=\boldsymbol{A}_1+\boldsymbol{A}_2\end{aligned}\right\} \tag{2·5}$$

are decompositions of φ_P and $\boldsymbol{A}$ corresponding to that of $\boldsymbol{P}$.

Now for an arbitrary scalar field F and a vector field $\boldsymbol{V}$ which vanish at infinity, there is the following relation:

$$\int(\nabla F,\nabla\times\boldsymbol{V})dv=0. \tag{2·6}$$

Consequently the third integral of (2·2) can be written

$$\frac{1}{2}\int\boldsymbol{P}\cdot\nabla\varphi_P dv=\frac{1}{8\pi}\int(\nabla\varphi_P)^2dv=\frac{4\pi}{2}\int\boldsymbol{P}^2dv-\frac{1}{2}\int\boldsymbol{P}\cdot\nabla\times\boldsymbol{A}\,dv. \tag{2·7}$$

Defining new constants a', b' and c' by

$$a'=a+4\pi, \qquad b'=b+4\pi, \qquad c'=c+4\pi, \tag{2·8}$$

(2·2) can also be written in the following form:

$$U=\int(-\boldsymbol{E}_0\cdot\boldsymbol{q}_1-\boldsymbol{E}_0\cdot\boldsymbol{q}_2)dv+\int\Big(\frac{1}{2}a'\boldsymbol{q}_1^2+\frac{1}{2}b'\boldsymbol{q}_2^2+c'\boldsymbol{q}_1\cdot\boldsymbol{q}_2\Big)dv$$
$$-\frac{1}{2}\int(\boldsymbol{q}_1+\boldsymbol{q}_2)\cdot\nabla\times\boldsymbol{A}\,dv. \tag{2·2′}$$

We now proceed to calculate the variation of U as a functional of the vector fields $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. By (2·7), (2·4), and (2·6) we have

$$\delta\left\{\frac{1}{2}\int \boldsymbol{P}\cdot\nabla\varphi_P\,dv\right\}=\frac{1}{4\pi}\int\nabla\varphi_P\cdot\delta\nabla\varphi_P\,dv=\int\nabla\varphi_P\cdot\delta\boldsymbol{P}\,dv,$$

so that for the energy U due to the polarization

$$\delta U=\int(-\boldsymbol{E}_0+a\boldsymbol{q}_1+c\boldsymbol{q}_2+\nabla\varphi_P)\cdot\delta\boldsymbol{q}_1\,dv$$
$$+\int(-\boldsymbol{E}_0+c\boldsymbol{q}_1+b\boldsymbol{q}_2+\nabla\varphi_P)\cdot\delta\boldsymbol{q}_2\,dv\,, \qquad (2\cdot9)$$

or by (2·4) and (2·8),

$$\delta U=\int(-\boldsymbol{E}_0+a'\boldsymbol{q}_1+c'\boldsymbol{q}_2-\nabla\times\boldsymbol{A})\cdot\delta\boldsymbol{q}_1\,dv$$
$$+\int(-\boldsymbol{E}_0+c'\boldsymbol{q}_1+b'\boldsymbol{q}_2-\nabla\times\boldsymbol{A})\cdot\delta\boldsymbol{q}_2\,dv\,. \qquad (2\cdot9')$$

§ 3. Relations between a, b, c and $\varkappa_0$, $\varkappa$, e^*

We can now derive the relations between the three coefficients a, b, c and the three dielectric quantities $\varkappa_0$, $\varkappa$, e^* with use of (2·9) as follows.

(i) *Dielectric constant at high frequencies*: $\varkappa_0=1+4\pi\chi_0$

In an electric field oscillating with high frequencies, the motion of $\boldsymbol{q}_2$ is not induced, so we must take

$$\boldsymbol{q}_2=0,\quad \delta\boldsymbol{q}_2=0$$

in (2·9). The value of $\boldsymbol{q}_1$ at any moment of the vibration should be determined by the condition

$$\delta U=0.$$

Thus we have (neglecting the retardation effect as assumed above)

$$a\boldsymbol{q}_1=\boldsymbol{E}_0-\nabla\varphi_P=\boldsymbol{E}\,, \qquad (3\cdot1)$$

from which the total polarization $\boldsymbol{P}$ becomes

$$\chi_0\boldsymbol{E}=\boldsymbol{P}=\boldsymbol{q}_1=\frac{\boldsymbol{E}}{a}.$$

This gives the first relation

$$\chi_0=\frac{1}{a},\qquad \varkappa_0=1+4\pi\chi_0\,. \qquad (3\cdot2)$$

(ii) *Static dielectric constant*: $\varkappa=1+4\pi\chi$

When the dielectric substance is in an electrostatic equlibrium under the external field $\boldsymbol{E}_0$, U takes the minimum value with respect to $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Thus we have, by (2·9) and the relation

$$\boldsymbol{E}_0-\nabla\varphi_P=\boldsymbol{E},$$

the following values for $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$:

$$\left.\begin{aligned}\boldsymbol{q}_1&=\frac{b-c}{ab-c^2}\boldsymbol{E},\\ \boldsymbol{q}_2&=\frac{a-c}{ab-c^2}\boldsymbol{E},\end{aligned}\right\} \tag{3·3}$$

The total polarization becomes

$$\chi\boldsymbol{E}=\boldsymbol{P}=\boldsymbol{q}_1+\boldsymbol{q}_2=\frac{a+b-2c}{ab-c^2}\boldsymbol{E},$$

from which the second relation is obtained:

$$\chi=\frac{a+b-2c}{ab-c^2},\qquad \varkappa=1+4\pi\chi. \tag{3·4}$$

(iii) *Effective charge*: e^*

We now consider a polarization wave due to relative displacements of opposite signs of ions taking place in a homogeneous dielectric sphere whose dimension is much smaller than the wave length now considered, no external field being applied. Then $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ can be taken as spatially constant throughout the sphere, and the depolarizing field is given by

$$-\nabla\varphi_P=-\frac{4\pi}{3}\boldsymbol{P}. \tag{3·5}$$

The variation of $\boldsymbol{q}_2$ is accompanied by that of $\boldsymbol{q}_1$ just in phase, as the characteristic frequencies of the latter are much higher than that of the polarization wave now considered. Therefore, at any moment of the vibration, the energy U must be minimized with regard to $\boldsymbol{q}_1$; this minimum value, as a functional of $\boldsymbol{q}_2$ only, plays in turn a role of potential energy in determining the motion of $\boldsymbol{q}_2$, the kinetic energy being expressed in terms of $\dot{\boldsymbol{q}}_2$.

Substituting (3·5) and $\boldsymbol{E}_0=0$ into (2·9), and equating the coefficient of $\delta\boldsymbol{q}_1$ to zero, we have

$$\left(a+\frac{4\pi}{3}\right)\boldsymbol{q}_1+\left(c+\frac{4\pi}{3}\right)\boldsymbol{q}_2=0, \tag{3·6}$$

at any moment of the vibration.

Following Fröhlich's[4] definition, the effective charge e^* is the ratio of the total polarization $\boldsymbol{P}$ to the displacement polarization (Z being assumed equal to unity) in such a vibration; thus we have the third relation

$$e^*=\frac{|\boldsymbol{q}_1+\boldsymbol{q}_2|}{|\boldsymbol{q}_2|}=1-\frac{c+\frac{4\pi}{3}}{a+\frac{4\pi}{3}}. \tag{3·7}$$

As stated in § 2, $\left(c+\frac{4\pi}{3}\right)$ is the coefficient of the short range interaction between

q_1 and q_2. Consequently, when there is no interaction of short range between the electronic and displacement polarizations, e^* becomes equal to unity. This is the very reason why Fröhlich adopted this definition of e^* as the third quantity characterizing dielectric properties of an ionic crystal.

The three relations (3·2), (3·4), (3·7) give $\varkappa_0$, $\varkappa$, e^* as functions of a, b, c; or by (2·8) as functions of a', b', c'. Here we write down the relations which are derived from them and which we shall make use of in later sections:

$$\left.\begin{aligned}
&\frac{a'+b'-2c'}{a'b'-c'^2}=\frac{1}{4\pi}\left(1-\frac{1}{\varkappa}\right),\\
&\frac{1}{a'}=\frac{1}{4\pi}\left(1-\frac{1}{\varkappa_0}\right),\\
&\frac{a'-c'}{a'b'-c'^2}=\frac{1}{4\pi}\left(1-\frac{\varkappa_0}{\varkappa}\right)\left\{\frac{3}{e^*(\varkappa_0+2)}\right\},\\
&1-\frac{c'}{a'}=\frac{1}{\varkappa_0}\left\{\frac{e^*(\varkappa_0+2)}{3}\right\}.
\end{aligned}\right\} \qquad (3\cdot8)$$

§ 4. The first method for treating the motion of an additive electron

Consider an ionic crystal, with fixed charge distribution Q if any, and an additive electron e which is moving through the crystal or trapped somewhere, say, in the neighbourhood of Q. We shall denote the position of the additive electron by $\boldsymbol{r}$, those of the crystal electrons and nuclei representatively by $\boldsymbol{r}_j$ and $\boldsymbol{X}_m$, respectively. The total energy $H_{tot.}$ of this system consists of various parts as shown in Fig. 1; firstly the energy of the crystal as a whole, of which the kinetic energy of the nuclei is denoted by K_n, the remainder being written H_c; secondly the Coulomb interactions of e and Q with the crystal electrons and nuclei by V_e, V_n, W_e, W_n, respectively; the third part K is the kinetic energy of e; finally V is the interaction energy between e and Q. Thus we can write as follows:

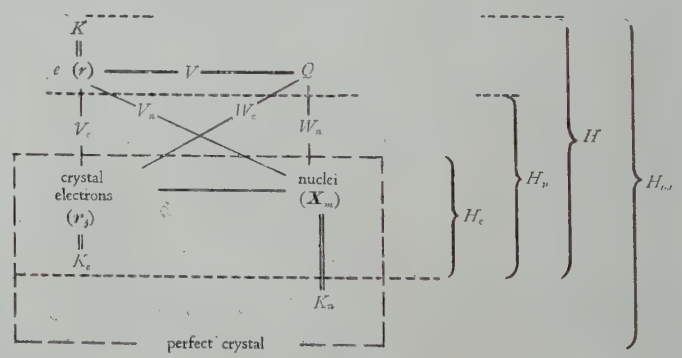

Fig. 1

$$\begin{aligned}
H_{tot.} &= K+V+V_e+V_n+W_e+W_n+H_c+K_n\\
&\quad(\boldsymbol{r},\boldsymbol{r}_j,\boldsymbol{X}_m)\ (\boldsymbol{r})\ (\boldsymbol{r})\ (\boldsymbol{r},\boldsymbol{r}_j)\ (\boldsymbol{r},\boldsymbol{X}_m)\ (\boldsymbol{r}_j)\ (\boldsymbol{X}_m)\ (\boldsymbol{r}_j,\boldsymbol{X}_m)\ (\boldsymbol{X}_m)\\
&=K+V+H_p+K_n\\
&=H+K_n\,, \qquad (4\cdot1)
\end{aligned}$$

where in the first equation the letters in the brackets under each energy operator are the coordinates entailed in the operator. The meanings of H and H_p are clear from (4·1) by themselves.

Now if we take, as usual, the adiabatic separation of the motions of the nuclei and the electrons (both the additive electron and the crystal electrons), the wave function Ψ of the total system can be written as

$$\Psi(\boldsymbol{r}, \boldsymbol{r}_j, \boldsymbol{X}_m) = \Phi(\boldsymbol{r}, \boldsymbol{r}_j; \boldsymbol{X}_m)\chi(\boldsymbol{X}_m), \tag{4·2}$$

where Φ, as a function of $\boldsymbol{r}$ and $\boldsymbol{r}_j$, satisfies the wave equation

$$H(\boldsymbol{r}, \boldsymbol{r}_j; \boldsymbol{X}_m)\Phi(\boldsymbol{r}, \boldsymbol{r}_j; \boldsymbol{X}_m) = E(\ ; \boldsymbol{X}_m)\Phi(\boldsymbol{r}, \boldsymbol{r}_j; \boldsymbol{X}_m). \tag{4·3}$$

In these and the following equations, ; is always used to show that the letters which stand on the right side of it are to be regarded merely as parameters. The eigenvalue $E(\ ; \boldsymbol{X}_m) = E(\boldsymbol{X}_m)$ in (4·3), as a function of $\boldsymbol{X}_m$, is nothing but the adiabatic potential in determining the motion of the nuclei, the wave equation for the nuclei being written

$$\{K_n + E(\boldsymbol{X}_m)\}\chi(\boldsymbol{X}_m) = E_{tot.}\chi(\boldsymbol{X}_m). \tag{4·4}$$

Now the solutions of (4·4) should represent the vibrations of the nuclei near the minimum point of $E(\boldsymbol{X}_m)$. If we neglect the vibrations and consider the nuclei as being fixed in the equilibrium positions (this corresponds to the infinite masses of the nuclei), we have only to solve the equations

$$\frac{\partial E(\boldsymbol{X}_m)}{\partial \boldsymbol{X}_m} = 0 \tag{4·5}$$

instead of solving the equation (4·4). Thus our problems reduces to solving (4·3) and (4·5). The difference between the method of this section and that of § 6 originates in the approximational forms of Φ on solving (4·3).

When we deal with the system consisting of light and heavy particles, the ratio of the masses being far smaller than unity, we can safely use adiabatic approximation (we have already used it in the above), as Born and Oppenheimer have shown,[5] and in such cases the ratio of the velocities is generally much larger than unity. Now coming back to our case, let us call our attention to the fact that in ordinary cases the motion of the additive electron is much slower than those of the crystal electrons. Then one is inclined to use here again the adiabatic approximation in analogy to the case of light and heavey particles, namely the large ratio of their velocities. This procedure is logically not justified because the condition of applicability of adiabatic approximation (mass ratio $\ll 1$) does not hold in our system of the crystal and the additive electrons. In spite of this circumstance, we shall, in this section, assume this method of approximation to be applicable to the system. Then the wave function Φ in (4·3) can be written in the form

$$\Phi(\boldsymbol{r}, \boldsymbol{r}_j; \boldsymbol{X}_m) = \phi(\boldsymbol{r}_j; \boldsymbol{r}, \boldsymbol{X}_m)\psi(\boldsymbol{r}; \boldsymbol{X}_m), \tag{4·6}$$

where the wave function ϕ of the crystal electrons satisfies the equation

$$H_p(\boldsymbol{r}_j; \boldsymbol{r}, \boldsymbol{X}_m)\phi(\boldsymbol{r}_j; \boldsymbol{r}, \boldsymbol{X}_m) = E_p(\ ; \boldsymbol{r}, \boldsymbol{X}_m)\phi(\boldsymbol{r}_j; \boldsymbol{r}, \boldsymbol{X}_m), \tag{4·7}$$

in which $\boldsymbol{r}$, as well as $\boldsymbol{X}_m$, is merely a parameter. The eigenvalue $E_p(\,;\boldsymbol{r},\boldsymbol{X}_m)=E_p(\boldsymbol{r}\,;\boldsymbol{X}_m)$ in the equation (4·7), together with $V(r)$, plays a rôle of adiabatic potential in determining the motion of the additive electron, that is, the wave function ψ of the additive electron satisfies the following equation:

$$\{K+V(\boldsymbol{r})+E_p(\boldsymbol{r}\,;\boldsymbol{X}_m)\}\psi(\boldsymbol{r}\,;X_m)=E(\,;\boldsymbol{X}_m)\psi(\boldsymbol{r}\,;\boldsymbol{X}_m). \tag{4·8}$$

The eigen-equation (4·7) can be replaced by the variational equation

$$\frac{\partial}{\partial\phi}\int\phi H_p\phi\,d\boldsymbol{r}_j=0 \tag{4·9}$$

with a trial function ϕ. Now we recall that (4·7) is the equation which describes the states of the crystal electrons when the additive electron and nuclei are fixed at $\boldsymbol{r}$ and $\boldsymbol{X}_m$, respectively. If the essential aspect of the circumstances now considered is to be described by the two kinds of polarization, we can restrict our trial function ϕ to those class which corresponds to the states (not necessarily stationary) representing some field $\boldsymbol{q}_1$ of the electronic polarization, and in parallel to this, we can, on solving (4·5), also confine the domain of the values of $\boldsymbol{X}_m$'s to that part which corresponds to some field $\boldsymbol{q}_2$ of the displacement polarization.

Then by considering the physical meanings of the operators given in (4·1) and of the energy U defined in § 2, the integral in the equation (4·9) can be written as

$$\int\phi H_p\phi\,d\boldsymbol{r}_j=U(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{E}_0)+\text{const.}, \tag{4·10}$$

where $\boldsymbol{E}_0$ is the electric field due to fixed charge distribution Q and the electron e located at $\boldsymbol{r}$, and is determined by the following formulae:

$$\boldsymbol{E}_0=\boldsymbol{E}_Q+\boldsymbol{E}_e, \tag{4·11}$$

$$\nabla\cdot\boldsymbol{E}_Q=4\pi\rho_Q, \qquad \boldsymbol{E}_Q=\nabla V, \tag{4·12}$$

$$\nabla\cdot\boldsymbol{E}_e(\boldsymbol{r}',\boldsymbol{r})=-4\pi\delta(\boldsymbol{r}'-\boldsymbol{r}), \tag{4·13}$$

in which $\delta(\boldsymbol{r})$ is a three-dimensional δ-function. In other words, e and Q are regarded as the sources of the external field $\boldsymbol{E}_0$ appearing in (2·2) or (2·2′). Thus if we write the integral on the left hand side of (4·10) in the form

$$\int\phi H_p\phi\,d\boldsymbol{r}_j=\int\phi(T_e+T_n+W_e+W_n)\phi\,d\boldsymbol{r}_j+\int\phi H_e\phi\,d\boldsymbol{r}_j$$

with the use of (4·1), we see, at once, that the first term on the right hand side corresponds to the first integral of (2·2), and that the second term corresponds to the second and the third integrals of (2·2) except for a constant term which appears in (4·10). The constant can be taken as zero if we choose the zero point of H_e adequately. Considering the correspondence of ϕ to $\boldsymbol{q}_1$, (4·9) is equivalent to

$$\frac{\delta U(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{E}_0)}{\delta\boldsymbol{q}_1}=0. \tag{4·14}$$

The minimum value of U determined by this condition is nothing but $E_p(\ ;\boldsymbol{r}, \boldsymbol{X}_m)$ on the right hand side of (4·7). By (2·9′), (4·14) leads to

$$a'\boldsymbol{q}_1=\boldsymbol{E}_0-c'\boldsymbol{q}_2+\nabla\times\boldsymbol{A}\,, \tag{4·15}$$

which, combined with (2·4) and (2·1), results in the relation

$$\nabla\times\nabla\times\boldsymbol{A}=4\pi\nabla\times\left(\frac{\boldsymbol{E}_0-c'\boldsymbol{q}_2+\nabla\times\boldsymbol{A}}{a'}+\boldsymbol{q}_2\right).$$

As $\boldsymbol{E}_0$ is the electrostatic field due to c and Q, its curl vanishes identically, and thus we have

$$\nabla\times\nabla\times\boldsymbol{A}=\frac{4\pi(a'-c')}{a'-4\pi}\nabla\times\boldsymbol{q}_2\,.$$

By (2·4′) the above equation can also be written as

$$\nabla\times\left\{\nabla\times\boldsymbol{A}-\left(\frac{a'-c'}{a'-4\pi}\right)\nabla\times\boldsymbol{A}_2\right\}=0,$$

and considering that divergence of the expression in the bracket vanishes identically, we finally get to a simple relation

$$\nabla\times\boldsymbol{A}=\frac{a'-c'}{a'-4\pi}\nabla\times\boldsymbol{A}_2. \tag{4·16}$$

Now that this relation is obtained, $\boldsymbol{q}_1$ can be expressed by the quantities with suffix 2; by substituting (4·16) into (4·15) we have

$$a'\boldsymbol{q}_1=\boldsymbol{E}_0-c'\boldsymbol{q}_2+\left(\frac{a'-c'}{a'-4\pi}\right)\nabla\times\boldsymbol{A}_2\,. \tag{4·15′}$$

Inserting (4·16) and (4·15′) into (2·2′), and applying the theorem (2·6), then decomposing $\boldsymbol{E}_0$ into $\boldsymbol{E}_e$ and $\boldsymbol{E}_Q$, thus rearranging the terms as an integral quadratic form in $\boldsymbol{E}_e$, $\boldsymbol{E}_Q$, $\boldsymbol{q}_2$ and $\nabla\times\boldsymbol{A}_2$, the minimum value (as to $\boldsymbol{q}_1$) of U can be written as

$$\begin{aligned}&U(\boldsymbol{q}_2, \boldsymbol{E}_0)\\&=-\frac{1}{a'}\int\boldsymbol{E}_e\cdot\boldsymbol{E}_Q dv-\left(1-\frac{c'}{a'}\right)\int\boldsymbol{E}_e\cdot\boldsymbol{q}_2 dv-\frac{1}{2a'}\int\boldsymbol{E}_e^2 dv-\frac{1}{2a'}\int\boldsymbol{E}_Q^2 dv\\&-\left(1-\frac{c'}{a'}\right)\int\boldsymbol{E}_Q\cdot\boldsymbol{q}_2 dv+\frac{a'b'-c'^2}{2a'}\int\boldsymbol{q}_2^2 dv-\frac{1}{2}\frac{(a'-c')^2}{a'(a'-4\pi)}\int\boldsymbol{q}_2\cdot\nabla\times\boldsymbol{A}_2 dv.\end{aligned} \tag{4·17}$$

The first integral of (4·17) can be transformed, by (4·12), (4·13) and Green's theorem, as follows:

$$\int\boldsymbol{E}_e\cdot\boldsymbol{E}_Q dv=4\pi\int V(\boldsymbol{r}')\,\delta(\boldsymbol{r}'-\boldsymbol{r})\,d\boldsymbol{r}'=4\pi\,V(\boldsymbol{r}). \tag{4·18}$$

Similarly the second integral can also be written, by (2·4′), (2·6), (4·13) and Green's theorem, as

$$\int \boldsymbol{E}_e \cdot \boldsymbol{q}_2 dv = \frac{1}{4\pi}\int \boldsymbol{E}_e \cdot \boldsymbol{F}\varphi_2 dv = \int \varphi_2(\boldsymbol{r}')\delta(\boldsymbol{r}'-\boldsymbol{r})d\boldsymbol{r}' = \varphi_2(\boldsymbol{r}). \qquad (4\cdot19)$$

The third integral of (4·17) proves to be infinite, as the source of the field $\boldsymbol{E}_e$ is a point charge. This infinity results from the circumstance that we regard here the crystal as a dielectric continuum. On the other hand, its value, though infinite, does not depend on the position $\boldsymbol{r}$ of the electron. Therefore only the first and second integrals in (4·17) depend on $\boldsymbol{r}$, the remaining part we shall denote by $S(\boldsymbol{q}_2)$. Then (4·17) can be written, by (4·18), (4·19) and (3·18), as follows:

$$U(\boldsymbol{q}_2, \boldsymbol{E}_0) = -\left(1-\frac{1}{\varkappa_0}\right)V(\boldsymbol{r}) - \frac{1}{\varkappa_0}\frac{e^*(\varkappa_0+2)}{3}\varphi_2(\boldsymbol{r}) + S(\boldsymbol{q}_2). \qquad (4\cdot20)$$

The value of U given by (4·20) is nothing but the eigenvalue $E_p(\ ;\boldsymbol{r}, X_m)$ in (4·7), and together with $V(\boldsymbol{r})$ forms the adiabatic potential in solving (4·8). As $S(\boldsymbol{q}_2)$ is merely and additive constant in the latter procedure, we have only to solve, instead of (4·8), the following eigenvalue problem:

$$\left[K+\frac{1}{\varkappa_0}V(\boldsymbol{r}) - \frac{1}{\varkappa_0}\left\{\frac{e^*(\varkappa_0+2)}{3}\right\}\varphi_2(\boldsymbol{r})\right]\psi(\boldsymbol{r}) = \varepsilon\psi(\boldsymbol{r}). \qquad (4\cdot21)$$

Of course, solutions $\psi(\boldsymbol{r})$ of (4·21) contains $\boldsymbol{q}_2$ (or $\boldsymbol{X}_m$) as a parameter, as $\varphi_2(\boldsymbol{r})$ is a functional of $\boldsymbol{q}_2$. (4·21) is the wave equation for the electron with the trapping potential $V(\boldsymbol{r})$, when the displacement polarization is frozen to $\boldsymbol{q}_2$ (an arbitrary function of position); we see, from this equation, that the potential $V(\boldsymbol{r})$ due to the fixed charge is shielded by a factor $1/\varkappa_0$, whereas the potential $\varphi_2(\boldsymbol{r})$ due to the polarization is shielded by a ratio

$$\frac{1}{\varkappa_0}\left\{\frac{e^*(\varkappa_0+1)}{3}\right\}.$$

The latter statement is worthy of attention, as a simple consideration might lead to the erroneous result that the shielding factor is $1/\varkappa_0$ in this case, too.

Although the above stated procedure, that is, to solve (4·8) with $E_p(\boldsymbol{r}\,;\boldsymbol{X}_m) = U$ given by (4·20), then to solve (4·4) or (4·5) with thus obtained $E(\ ;\boldsymbol{X}_m)$, is the ordinary one, we may reverse these two steps as follows. Writing $\boldsymbol{q}_2$ instead of the equivalent $\boldsymbol{X}_m$, and defining a new integral

$$\int \psi(K+V+E_p)\psi d\boldsymbol{r} = E(\psi, \boldsymbol{q}_2) \qquad (4\cdot22)$$

with a trial function ψ (the condition of normalization being fulfilled), (4·8) is equivalent to the variational problem:

$$\frac{\delta E(\psi, \boldsymbol{q}_2)}{\delta\psi} = 0 \qquad (4\cdot23)$$

and the minimum value $E(\boldsymbol{q}_2)$ determined by this equation corresponds to $E(\ ;\boldsymbol{X}_m)$ in (4·8); further the condition (4·5) on this $E(\ ;\boldsymbol{X}_m)$ is equivalent to

$$\frac{\delta E(\boldsymbol{q}_2)}{\delta \boldsymbol{q}_2}=0. \tag{4·24}$$

(4·23) and (4·24) is a variation problem of $E(\psi;\boldsymbol{q}_2)$ as to both of ψ and $\boldsymbol{q}_2$; consequently these two steps may be reversed.

Let us first solve the equation

$$\frac{\delta E(\psi,\boldsymbol{q}_2)}{\delta \boldsymbol{q}_2}=0. \tag{4·25}$$

As stated above, the third integral in (4·17) is, though infinite, merely an additive constant in the sense that it depends neither on $\boldsymbol{r}$ nor on $\boldsymbol{q}_2$; thus we shall henceforce drop this term, confining the additive electron always inside the crystal. We can, then, calculate the integral (4·22), with the use of (4·17) and (4·20), as follows:

$$\begin{aligned}E(\psi,\boldsymbol{q}_2)&=\int\psi(\boldsymbol{r})\{K+V(\boldsymbol{r})+U(\boldsymbol{q}_2,\boldsymbol{E}_0)\psi(\boldsymbol{r})d\boldsymbol{r}\\&=\int\psi(\boldsymbol{r})\left\{K+\frac{1}{\varkappa_0}V(\boldsymbol{r})-\left(1-\frac{c'}{a'}\right)\varphi_2(\boldsymbol{r})\right\}\psi(\boldsymbol{r})d\boldsymbol{r}\\&-\frac{1}{2a'}\int\boldsymbol{E}_Q^2dv-\left(1-\frac{c'}{a'}\right)\int\boldsymbol{E}_Q\cdot\boldsymbol{q}_2dv+\frac{a'b'-c'^2}{2a'}\int\boldsymbol{q}_2^2dv\\&-\frac{1}{2}\frac{(a'-c')^2}{a'(a'-4\pi)}\int\boldsymbol{q}_2\cdot\nabla\times\boldsymbol{A}_2dv.\end{aligned} \tag{4·26}$$

Now we define the electric field $\tilde{\boldsymbol{E}}_e$ due to the probability distribution of the electronic charge:

$$\left.\begin{aligned}&\tilde{\rho}_e(\boldsymbol{r})=-\{\psi(\boldsymbol{r})\}^2,\\&\Delta\tilde{\varphi}_e=-4\pi\tilde{\rho}_e,\\&\tilde{\boldsymbol{E}}_e=-\nabla\tilde{\varphi}_e.\end{aligned}\right\} \tag{4·27}$$

With this notation, the term containing φ_2 in (4·26) can be transformed as follows:

$$\begin{aligned}\left(1-\frac{c'}{a'}\right)\int\tilde{\rho}_e(\boldsymbol{r})\varphi_2(\boldsymbol{r})d\boldsymbol{r}&=-\frac{1}{4\pi}\left(1-\frac{c'}{a'}\right)\int\tilde{\boldsymbol{E}}_e\cdot\nabla\varphi_2dv\\&=-\left(1-\frac{c'}{a'}\right)\int\tilde{\boldsymbol{E}}_e\cdot\boldsymbol{q}_2dv.\end{aligned} \tag{4·28}$$

The variation of the last integral can be calculated, by using (2·4′) and (2·6) repeatedly as follows:

$$\begin{aligned}\delta\left\{\frac{1}{2}\int\boldsymbol{q}_2\cdot\nabla\times\boldsymbol{A}_2dv\right\}&=\delta\left\{\frac{1}{2}\frac{1}{4\pi}\int(\nabla\times\boldsymbol{A}_2)^2dv\right\}=\frac{1}{4\pi}\int\nabla\times\boldsymbol{A}_2\cdot\delta\nabla\times\boldsymbol{A}_2dv\\&=\int\nabla\times\boldsymbol{A}_2\cdot\delta\boldsymbol{q}_2dv.\end{aligned} \tag{4·29}$$

By (4·26), (4·28) and (4·29), the condition (4·25) leads to

$$-\left(1-\frac{c'}{a'}\right)(\tilde{\boldsymbol{E}}_e+\boldsymbol{E}_Q)+\frac{a'b'-c'^2}{a'}\boldsymbol{q}_2-\frac{(a'-c')^2}{a'(a'-4\pi)}\nabla\times\boldsymbol{A}_2=0. \tag{4·30}$$

Taking curl of this equation we obtain

$$(a'b'-c'^2)\nabla\times\boldsymbol{q}_2-\frac{(a'-c')^2}{a'-4\pi}\nabla\times\nabla\times\boldsymbol{A}_2=0. \tag{4·31}$$

On the other hand, operation of curl on the second equation of (2·6′) leads to

$$4\pi\nabla\times\boldsymbol{q}_2-\nabla\times\nabla\times\boldsymbol{A}_2=0. \tag{4·32}$$

Unless $ab-c^2$ is equal to zero (which means $\varkappa=\infty$), we obtain, by (4·31) and (4·32),

$$\nabla\times\boldsymbol{q}_2=0, \qquad \nabla\times\nabla\times\boldsymbol{A}_2=0. \tag{4·33}$$

The latter equation, together with the identity

$$\nabla\cdot(\nabla\times\boldsymbol{A}_2)=0,$$

leads to more stringent condition

$$\nabla\times\boldsymbol{A}_2=0. \tag{4·34}$$

Inserting this in (4·30) and with use of (3·8) we obtain

$$\boldsymbol{q}_2=\frac{a'-c'}{a'b'-c'^2}(\tilde{\boldsymbol{E}}_e+\boldsymbol{E}_Q) \tag{4·35}$$

$$=\left\{\frac{3}{\varepsilon^*(\varkappa_0+2)}\right\}\frac{1}{4\pi}\left(1-\frac{\varkappa_0}{\varkappa}\right)(\tilde{\boldsymbol{E}}_e+\boldsymbol{E}_Q). \tag{4·35′}$$

Comparing (4·35) with (4·15′), we see that the electronic polarization $\boldsymbol{q}_1$ is controlled by the instantaneous field $\boldsymbol{E}_e$ due to the additive electron at $\boldsymbol{r}$, while the displacement polarization $\boldsymbol{q}_2$ is determined by the average field $\tilde{\boldsymbol{E}}_e$ of the electron. This is nothing but the stand-point from which the usual method started, as has been stated in § 1. We must notice, in (4·35′), that the factor which relates $\boldsymbol{q}_2$ with $\tilde{\boldsymbol{E}}_e+\boldsymbol{E}_Q$ is not $\frac{1}{4\pi}\left(1-\frac{\varkappa_0}{\varkappa}\right)$ but of the more complicated form as is seen.

Inserting (4·34), (4·35) and (4·28) in (4·26), we obtain, by (3·8), the following expression for the energy:

$$E(\psi,\boldsymbol{q}_2(\psi))=E(\psi)$$

$$=\int\psi(\boldsymbol{r})\left(K+\frac{1}{\varkappa_0}V(\boldsymbol{r})\right)\psi(\boldsymbol{r})d\boldsymbol{r}$$

$$-\frac{1}{8\pi}\left(\frac{1}{\varkappa_0}-\frac{1}{\varkappa}\right)\int(\tilde{\boldsymbol{E}}_e^2+2\tilde{\boldsymbol{E}}_e\cdot\boldsymbol{E}_Q)dv-\frac{1}{8\pi}\left(1-\frac{1}{\varkappa}\right)\int\boldsymbol{E}_Q^2dv.$$

The last term of this expression is the energy due to the polarization when there is only Q with no additive electron in the system. Taking this state as zero point of the energy, we have, instead of the above, the following expression:

$$E(\psi)=\int\psi(\boldsymbol{r})\Big(K+\frac{1}{\varkappa_0}V(\boldsymbol{r})\Big)\psi(\boldsymbol{r})d\boldsymbol{r}-\frac{1}{8\pi}\Big(\frac{1}{\varkappa_0}-\frac{1}{\varkappa}\Big)\int(\tilde{\boldsymbol{E}}_e^2+2\tilde{\boldsymbol{E}}_e\cdot\boldsymbol{E}_Q)d\boldsymbol{r}. \tag{4·36}$$

Further we can apply Green's theorem to the integral containing $\boldsymbol{E}_e\cdot\boldsymbol{E}_Q$ by (4·21) and (4·27) as follows:

$$\frac{1}{4\pi}\int\tilde{\boldsymbol{E}}_e\cdot\boldsymbol{E}_Q d\boldsymbol{r}=-\int\tilde{\varphi}_e\,I'd\boldsymbol{r}=\int\psi(\boldsymbol{r})I'(\boldsymbol{r})\psi(\boldsymbol{r})d\boldsymbol{r}.$$

Inserting this in (4·36), we obtain an alternative form:

$$E(\psi)=\int\psi(\boldsymbol{r})\Big(K+\frac{1}{\varkappa}V(\boldsymbol{r})\Big)\psi(\boldsymbol{r})d\boldsymbol{r}-\frac{1}{8\pi}\Big(\frac{1}{\varkappa_0}-\frac{1}{\varkappa}\Big)\int\tilde{\boldsymbol{E}}_e^2 d\boldsymbol{r}. \tag{4·36′}$$

If we solve the variational equation

$$\frac{\delta E(\psi)}{\delta\psi}=0 \qquad \Big(\int\psi^2 d\boldsymbol{r}=1\Big) \tag{4·37}$$

with $E(\psi)$ given by (4·36) or (4·36′), the problem we are considering is completely solved. By (4·27) and Green's theorem, (4·37) leads to the eigen-value problem

$$\Big\{K+\frac{1}{\varkappa}V(\boldsymbol{r})+\Big(\frac{1}{\varkappa_0}-\frac{1}{\varkappa}\Big)\tilde{\varphi}_e(\boldsymbol{r})\Big\}\psi(\boldsymbol{r})=\lambda\psi(\boldsymbol{r}), \tag{4·38}$$

where $\tilde{\varphi}_e(\boldsymbol{r})$ is to be determined by

$$\Delta\tilde{\varphi}_e(\boldsymbol{r})=4\pi\{\psi(\boldsymbol{r})\}^2. \tag{4·39}$$

(4·38) is not a usual one-body eigen-value problem in quantum mechanics; it is, above all, not a linear equation. To solve (4·37) directly by Ritz's procedure, that is, to solve (4·37) with appropriate trial functions inserted in (4·36) or (4·36′), is the method which Kubo[2] has obtained on a rather intuitive consideration based on classical electrostatics; while solving (4·38) and (4·39) simultaneously self-consistently is that which was suggested by Mott and Gurney[1] and has recently been applied to certain simple trapping-centers by Simpson[3]. Both methods are equivalent.

All that has been stated is as regards the states in which the positions of the nuclei $\boldsymbol{X}_m$ (consequently the displacement polarization $\boldsymbol{q}_2$) are in equilibrium. If the additive electron makes an optical transition by excitation, $\boldsymbol{X}_m$ will stay, immediately after the transition, in their former values by Franck and Condon's principle (this approximational consideration is quite in parallel with using (4·5) in place of (4·4)). We shall now consider such a state B', and distinguish the quantities corresponding to this state from those of the state A before transition by attaching primes as shown in Fig. 2.

A	optical transition	B'
ψ		ψ'
$\tilde{\boldsymbol{E}}_e, \tilde{\varphi}_e$		$\tilde{\boldsymbol{E}}_e', \tilde{\varphi}_e'$
$\boldsymbol{q}_2$		$\boldsymbol{q}_2'=\boldsymbol{q}_2$

Fig. 2

As $\boldsymbol{q}'_2=\boldsymbol{q}_2$ by Franck-Condon's principle, we see, from (4·34) and (4·35), that

$$\nabla \times \boldsymbol{A}_2' = \nabla \times \boldsymbol{A}_2 = 0,$$

$$\boldsymbol{q}_2' = \boldsymbol{q}_2 = \frac{a'-c'}{a'b'-c'^2}(\tilde{\boldsymbol{E}}_e + \boldsymbol{E}_Q),$$

$$\varphi_2' = \varphi_2 .$$

Inserting these values in the formula which is obtained from (4·26) by attaching primes to ψ, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{A}_2$, then transforming certain integrals by Green's theorem, and finally taking the same zero-point of energy as before, we obtain the following expression for E' of the state B' corresponding to (4·36) and (4·36′) of the state A:

$$\begin{aligned} E'(\psi', \boldsymbol{q}_2(\psi')) &= \int \psi'(\boldsymbol{r})\left(K + \frac{1}{\chi_0} V(\boldsymbol{r})\right)\psi'(\boldsymbol{r})\,d\boldsymbol{r} \\ &\quad - \frac{1}{8\pi}\left(\frac{1}{\chi_0} - \frac{1}{\chi}\right)\int (2\tilde{\boldsymbol{E}}_e' \cdot \tilde{\boldsymbol{E}}_e - \tilde{\boldsymbol{E}}_e^2 + 2\tilde{\boldsymbol{E}}_e' \cdot \boldsymbol{E}_Q)\,dv \qquad (4\cdot40) \\ &= \int \psi'(\boldsymbol{r})\left(K + \frac{1}{\chi} V(\boldsymbol{r})\right)\psi'(\boldsymbol{r})\,d\boldsymbol{r} \\ &\quad - \frac{1}{8\pi}\left(\frac{1}{\chi_0} - \frac{1}{\chi}\right)\int (2\tilde{\boldsymbol{E}}_e' \cdot \tilde{\boldsymbol{E}}_e - \tilde{\boldsymbol{E}}_e^2)\,dv . \qquad (4\cdot40') \end{aligned}$$

Another form

$$\begin{aligned} E'(\psi', \boldsymbol{q}_2(\psi')) &= \int \psi'(\boldsymbol{r})\left(K + \frac{1}{\chi} V(\boldsymbol{r})\right)\psi'(\boldsymbol{r})\,d\boldsymbol{r} \\ &\quad - \frac{1}{8\pi}\left(\frac{1}{\chi_0} - \frac{1}{\chi}\right)\int \tilde{\boldsymbol{E}}_e'^2\,dv + \frac{1}{8\pi}\left(\frac{1}{\chi_0} - \frac{1}{\chi}\right)\int (\tilde{\boldsymbol{E}}_e' - \tilde{\boldsymbol{E}}_e)^2\,dv \qquad (4\cdot40'') \end{aligned}$$

will also be useful in the following discussion.

To obtain the wave function ψ' and the energy E' of the state B', we have only to solve the variational equation

$$\frac{\delta E'(\psi', \boldsymbol{q}_2(\psi'))}{\delta \psi'} = 0 \qquad (\int \psi'^2\,d\boldsymbol{r} = 1) \qquad (4\cdot41)$$

with $E'(\psi', \boldsymbol{q}_2(\psi'))$ given by (4·40), in which we already know the wave function ψ of the state A. Comparing (4·40″) with (4·36), we see that, besides the primes to ψ and $\tilde{\boldsymbol{E}}_e$, there is added the third term in the former which was not in the latter. When the system has relaxed from non-equilibrium state B' to equilibrium state B (belonging to the same electronic state as B'), we have only to minimize the sum of the first and the second terms in (4·40″) to know the energy and wave function for the latter state (while we must also include the third term in order to treat the non-equilibrium state B'). As the third term is of a positive definite form, the minimum value of the sum of the first two terms is smaller than that of the total sum. Thus we see that the state B has smaller energy than the state B', which is a natural result.

From (4·41) with E' given by (4·40′), we obtain the equation

$$\left\{K+\frac{1}{\chi}V(\boldsymbol{r})+\left(\frac{1}{\chi_0}-\frac{1}{\chi}\right)\tilde{\varphi}_e(\boldsymbol{r})\right\}\psi'(\boldsymbol{r})=\lambda'\psi'(\boldsymbol{r}), \tag{4·42}$$

where λ' is a parameter. Comparing this with the corresponding equation of state A, we find that here we have only to solve a usual Schrödinger equation, in contrast to (4·38) which must be solved simultaneously with (4·39)*.

Thus, defining an operator

$$\Lambda=K+\frac{1}{\chi}V(+\boldsymbol{r})\left(\frac{1}{\chi_0}-\frac{1}{\chi}\right)\tilde{\varphi}_e(\boldsymbol{r}) \tag{4·43}$$

with $\tilde{\varphi}_e(\boldsymbol{r})$ determined in the state A, we can write (4·38) and (4·42) as follows:

$$\Lambda\psi(\boldsymbol{r})=\lambda\psi(\boldsymbol{r}), \tag{4·38′}$$

$$\Lambda\psi'(\boldsymbol{r})=\lambda'\psi'(\boldsymbol{r}). \tag{4·42′}$$

Further, (4·36′) and (4·40′) can be written as

$$E=\int\psi\Lambda\psi\,d\boldsymbol{r}+G(\psi)=\lambda+G(\psi), \tag{4·44}$$

$$E'=\int\psi'\Lambda\psi'\,d\boldsymbol{r}+G(\psi)=\lambda'+G(\psi), \tag{4·45}$$

where

$$G(\psi)=\frac{1}{8\pi}\left(\frac{1}{\chi_0}-\frac{1}{\chi}\right)\int\tilde{\boldsymbol{E}}_e^2\,dv. \tag{4·46}$$

Thus we may regard Λ as a hamiltonian in all optical processes starting from the state A, and the total energies $E, E', \cdots$ as sums of the eigenvalues $\lambda, \lambda', \cdots$ and the lattice energy $G(\psi)$, the latter being determined by the initial state A.

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* Note that in (4·42) $\varphi_e(\boldsymbol{r})$ depends not on ψ' but on ψ.

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On the Inelastic Collision between Molecules, III

—— *Rotational Transitions in H_2-gas* ——

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The same method of approximation as in the previous paper has been used to calculate the probabilities of rotational transitions of para-hydrogen at the room temperature. Three types of the process are considered in which the rotational quantum numbers change as

$$\begin{cases} 2 \to 0 \text{ (the first molecule)} \\ X \to X \text{ (the second molecule)} \end{cases}$$

$$\begin{cases} 4 \to 2 \\ X \to X \end{cases}$$

and

$$\begin{cases} 4 \to 2 \\ 0 \to 2. \end{cases}$$

Relative probabilities of these three processes at T 298.4°K explain the shape of the experimental curve obtained by Rhodes fairly well. The absolute values of these probabilities, however, are a few times larger than the required values. The discrepancy of this order is inevitable to our extremely simplified method of calculation.

The effect of the statistics for the identical molecules is discussed in the Appendix.

§ 1. Introduction

In the inelastic collisions between molecules, the transition probabilities are small for the processes in which the large energy exchanges between the internal and the translational degrees of freedom are involved. It is true at least in the coupling of the translation with the vibration, as investigated previously.[1] It may be considered, therefore, that the rotational transition $2 \to 0$ of the para-hydrogen molecule has a much larger probability than the transition $4 \to 2$ for the same initial kinetic energy, since the ratio of the changes of translational energy for these two processes is 6 : 14. If it could be true, the dispersion of the sound velocity should occur in two separated steps at the room temperatures. But the recent measurement by Rhodes[2] showed that the dispersion occurred apparently in one step. Besides, he said that the transition $4 \to 2$ caused the dispersion at higher frequencies

than the transition $2 \rightarrow 0$ did. This means that the transition probability of the process $4 \rightarrow 2$ is larger than that of $2 \rightarrow 0$, in contradiction to the above anticipation.

In the present paper, the direct calculations of the transition probabilities are carried out which are compared with the experimental values.

In the above anticipation, we have tacitly compared the process

$$\begin{cases} 2 \rightarrow 0 \text{ (the first molecule)} \\ X \rightarrow X \text{ (the second molecule)} \end{cases} \tag{1}$$

with the process

$$\begin{cases} 4 \rightarrow 2 \\ X \rightarrow X\,. \end{cases} \tag{2}$$

But, we must consider, in addition, the third process

$$\begin{cases} 4 \rightarrow 2 \\ 0 \rightarrow 2\,, \end{cases} \tag{3}$$

for the ratio of the changes of the translational energy for (1) and (3) is 6 : 8, being close to unity. In the case of the vibration-translation coupling, the processes of type (3) have larger probabilities than the processes of type (2).[1] Of course, the state of affairs is altered by passing from the vibration-translation coupling to the rotation-translation coupling because of the small level spacing in the latter case, and yet the probability of (3) may be comparable with that of (2). In the next section, we shall calculate the transition probabilities for these three processes.

§ 2, Calculation of the transition probabilities

In the distorted wave approximation, the cross section for the transition $L_0 \rightarrow L$ is given by[3]

$$Q^{L_0 \rightarrow L} = \frac{1}{(2l_{10}+1)(2l_{20}+1)} \sum_{M, M_0} Q^{L_0 \rightarrow L}_{M_0 \rightarrow M}, \tag{4}$$

$$Q^{L_0 \rightarrow L}_{M_0 \rightarrow M} = \frac{1}{4\pi} \frac{k_f}{k_0} \sum_{j,m} (2j+1) \times \left| 2\mu \int_0^\infty G^{jm}_{LM}(R) \sum_{j'} \left(\frac{2j'+1}{2j+1} \right)^{1/2} (LMjm|V|L_0 M_0 j' 0) G_0^{j'm}(R) dR \right|^2, \tag{4a}$$

where L, L_0, M, M_0 stand for the set of values (l_1, l_2), (l_{10}, l_{20}), (m_1, m_2), (m_{10}, m_{20}) respectively and l_i, m_i are the rotational quantum numbers for the i-th molecule. Suffices "o" and "f" are employed for the initial and the final states, respectively. As for the other notations, see paper II.[3]*) Atomic units are used throughout this paper

*) The matrix elements $(L \cdots | V | L_0 \cdots)$ are different from those used in the paper II by a factor $(2j'+1/2j+1)^{1/2}$.

unless otherwise are specified.

In order to calculate the cross section $Q^{L_0 \to L}$, we shall adopt the following approximate procedure. Firstly, the spheres with suitable diameters are imagined in place of the molecules. For simplicity equal diameter is taken for all rotational states of hydrogen molecule. In the collisions of these representative spheres with each other, it is assumed that the transition probability is a function of the component $v_\parallel$ of relative velocity along the line joining the centers at the instant of contact. Now, if the temperature of gas is $T°K$, the average value of $\epsilon = \mu v_\parallel^2/2$ is, as is well known, $\bar{\epsilon} = \varkappa T$, $\varkappa$ being the Boltzmann constant. Thus, by putting the initial wave number as

$$k_0 = \mu \sqrt{\overline{v_\parallel^2}} = \sqrt{2\mu\varkappa T}, \tag{5}$$

we may carry out the integration over the Maxwellian distribution and the summation $\sum_j'(2j+1)$ without essential error. (See Appendix I.) Of course the suitable averaging over j must be made for the remaining factors.

In the next place, let us assume the potential of the intermolecular force to be

$$V = v \cdot \exp(-2R), \tag{6}$$

where the first factor v is independent of the intermolecular distance R. Then we may write

$$Q_{M_0 \to M}^{L_0 \to L} = C \cdot k_f \sum_m' w \cdot A_v \left| \sum_{j'} \left(\frac{2j'+1}{2j+1} \right)^{1/2} (LMjm \mid v \mid L_0 M_0 j' 0) \right|^2 \cdot \left| \int_0^\infty G^j(k_f) e^{-2R} G^{j'}(k_0)\, dR \right|^2, \tag{7}$$

C is the common factor to the processes (1), (2) and (3). A_v means the suitable averaging over j and we may roughly replace it by $\lim_{j \to \infty}$ as in the previous paper. w is the statistical factor which is $1/2$ when the molecules are identical in the final state (because of the reduction of the number of permissible final states), otherwise it is simply unity. $G(k)$ in the final factor of the above equation is the S-wave radial function multiplied by the distance R for the wave number k. For this function, we use the distorted wave under the potential

$$V_0 = 31 \exp(-2R), \tag{8}$$

as in the previous paper. Then the integration over R is carried out easily employing the Jackson-Mott formula.[3]

Finally, we must determine the explicit form of the v-factor of (6) and then its matrix elements. For this sake, we assume that the potential V is expressed by the sum of the interatomic potentials and write it in the form of Maclaurin series. Then the main terms which cause the processes (1) and (2) come from (See Appendix II, and Fig. 3)

$$2\left[(x^2+x'^2)\varphi''(R)+(y^2+y'^2+z^2+z'^2)\,\Phi_1(R)\right]$$
$$=2A[\xi_1^2(1+1/2R)\cos^2\theta_1+\xi_2^2(1+1/2R)\cos^2\theta_2-(\xi_1^2+\xi_2^2)/2R]\exp(-2R)\,.$$

Here R in the bracket may be replaced by the suitable mean value in the immediate neighbourhood of the classical turning point which is the region from which the greatest contribution to the cross section comes. In the thermal collisions, this is about 5 A. U. Thus we may replace $(1+1/2R)$ by 1.1. Then the effective potential for the process (1) and process (2) is

$$V_1=V_2=1.47\,\xi_1^2AP_2(\cos\theta_1)\exp(-2R). \tag{9}$$

On the other hand, the effective potential for the process (3) comes from

$$x^2x'^2\varphi^{(IV)}(R)+(x^2y'^2+x^2z'^2+x'^2y^2+x'^2z^2)\,\Phi_2(R).$$

Thus we have

$$V_3=0.55\xi_1^2\xi_2^2AP_2(\cos\theta_1)P_2(\cos\theta_2)\exp(-2R). \tag{10}$$

The matrix elements of $P_2(\cos\theta_1)$ and $P_2(\cos\theta_1)\times P_2(\cos\theta_2)$ are calculated easily as in the Appendix III.

Using these formulae, we can obtain the ratio of the cross sections for the processes (1), (2) and (3), with "X"="0" for (1) and (2). Especially, for $T=298.4°K$, in which Rhodes' measurement was made, we have

$$Q^{(1)}:Q^{(2)}:Q^{(3)}=1:0.46:0.19. \quad (\text{for "}X\text{"="0"}) \tag{11}$$

As an example for the absolute value of the transition probability, we have

$$P^{(1)}=1.0\times10^{-2} \quad \text{per collision,} \tag{12}$$

for the process (1). Here we have used the following values: $\xi_1=\xi_2=1.44$; $A=5.0$; $2\mu=3675.0$.

It is seen from the values (11) that the processes (1), (2) and (3) have same order probabilities, and yet the probability of (1) is the largest of them, in contradiction with the interpretation by Rhodes. It seems, however, that he has considered from the beginning that the process (2) is more probable than (1). We must, therefore, analyze his experimental data in a more general way taking into account of the process (3).

§ 3. Comparison with the Rhodes' experimental data

Para-hydrogen at room temperature is essentially a three level gas with most of its molecules in rotational levels with $l=0$, 2 and 4, and we may safely neglect the molecules in the higher levels. (For example, the number of molecules in rotational level $l=6$ is of the order of 10^{-4} compared to that of $l=0$, and small enough to be neglected for the present discussions.) Of the many inelastic collisions between such kinds of molecule, we shall consider only the following seven processes which seem to occur more frequently compared with the other processes.

$$(0)+(0) \underset{k_1}{\overset{k_1'}{\rightleftarrows}} (2)+(0) \tag{13a}$$

$$(0)+(2) \underset{k_2}{\overset{k_2'}{\rightleftarrows}} (2)+(2) \tag{13b}$$

$$(0)+(4) \underset{k_3}{\overset{k_3'}{\rightleftarrows}} (2)+(4) \tag{13c}$$

$$(2)+(2) \underset{K}{\overset{K'}{\rightleftarrows}} (0)+(4) \tag{13d}$$

$$(2)+(0) \underset{\varkappa_1}{\overset{\varkappa_1'}{\rightleftarrows}} (4)+(0) \tag{13e}$$

$$(2)+(2) \underset{\varkappa_2}{\overset{\varkappa_2'}{\rightleftarrows}} (4)+(2) \tag{13f}$$

$$(2)+(4) \underset{\varkappa_3}{\overset{\varkappa_3'}{\rightleftarrows}} (4)+(4), \tag{13g}$$

where (l) stands for H_2-molecule in the rotational level l and $k, k', K, K', \varkappa, \varkappa'$ are the constants of reaction rates. (See the equation (15) which will appear later on.) For simplicity, we shall assume the following relations

$$\begin{aligned} &2k_1=10k_2/9=k_3\,;\ \varkappa_1=10\varkappa_2/9=18\varkappa_3/17\,;\\ &2k_1'=10k_2'/9=k_3'\,;\ \varkappa_3'=10\varkappa_2'/9=18\varkappa_3'/17. \end{aligned} \tag{14}$$

The fractional coefficients are due to the effect of statistics. To illustrate this effect, we shall compare the two processes

$$(4)+(0) \overset{\varkappa_1}{\rightarrow} (2)+(0) \tag{13e'}$$

and

$$(4)+(2) \overset{\varkappa_2}{\rightarrow} (2)+(2). \tag{13f}$$

In the final state of the latter process, two hydrogen molecules have the same angular momentum, $l_1=l_2=2$. But their z-components m_1 and m_2 are not the same in general. They have a same value only once in five times. If $m_1=m_2$, these two hydrogen molecules must be treated as identical and then the transition probability is reduced by half owing to the statistical factor w in the equation (7). Therefore, we have

$$\varkappa_2=\left\{\left(\frac{1}{2}\right)\times\frac{1}{5}+(1)\times\left(1-\frac{1}{5}\right)\right\}\varkappa_1=\frac{9}{10}\varkappa_1.$$

The same ratio is obtained by considering the inverse processes

$$(2)+(0) \overset{\varkappa_1'}{\rightarrow} (4)+(0) \tag{13e''}$$

$$(2)+(2) \overset{\varkappa_2'}{\rightarrow} (4)+(2). \tag{13f''}$$

If the two molecules in the initial state are identical, the number of such collisions per unit time and unit volume is reduced by half. On the other hand, the cross section is not altered. (See Appendix IV.) Thus the reaction rate $\varkappa_2'$ is reduced by half, once in five, as in the inverse process (13f').

Of course, the intermolecular potentials in these processes (13a)–(13g) are different, but we shall neglect this effect since, as we suppose, it does not seriously affect our final result.

Suppose that we take some volume of gas which contains molecules distributed in three levels, the numbers being n_0, n_2 and n_4 respectively ($\sum_l n_l = n$), then we obtain the following set of differential equations.

$$\begin{aligned}
\frac{dn_4}{dt} &= -K\{n_0 n_4 - n_2^2 f_1(T)\} - \varkappa_1\left(n_0 + \frac{9}{10}n_2 + \frac{17}{18}n_4\right)\{n_4 - n_2 f_3(T)\} \\
\frac{dn_2}{dt} &= -\frac{dn_0}{dt} - \frac{dn_4}{dt} \\
\frac{dn_0}{dt} &= -K\{n_0 n_4 - n_2^2 f_1(T)\} + k_1\left(n_0 + \frac{9}{5}n_2 + 2n_4\right)\{n_2 - n_0 f_2(T)\}.
\end{aligned} \tag{15}$$

The principle of detailed balance allows us to assume that

$$\begin{aligned}
f_1(T) &= (n_0 n_4 / n_2^2)_{\text{equil.}} = (9/25)\exp[-(E_4 - 2E_2)/\varkappa T] \\
f_2(T) &= (n_2/n_0)_{\text{equil.}} = 5\exp(-E_2/\varkappa T) \\
f_3(T) &= (n_4/n_2)_{\text{equil.}} = (9/5)\exp[-(E_4 - E_2)/\varkappa T].
\end{aligned} \tag{16}$$

Suppose T varies periodically, so that

$$\begin{aligned}
T &= T^0 + \Delta T \cdot e^{i\omega t}, \\
n_l &= n^0_l + \Delta n_l \cdot e^{i\omega t}.
\end{aligned} \tag{17}$$

These suppositions in (15) lead to*), after dropping second order terms,

$$\begin{aligned}
i\omega \Delta n_4 = &-K\{n_0 \cdot \Delta n_4 + n_4 \cdot \Delta n_0 - 2n_2 \Delta n_2 f_1(T) - n^2{}_2 \Delta T\, df_1(T)/dT\} \\
&-\varkappa_1(n - n_2/10 - n_4/18)\{\Delta n_4 - \Delta n_2 \cdot f(T) - n_2 \Delta T df_3(T)/dT\} \\
\Delta n_2 = &-\Delta n_0 - \Delta n_4
\end{aligned} \tag{18}$$

*) For simplicity, we shall omit the suffix "$_0$" from T^0 and $n^0{}_l$ in the following.

$$i\omega \Delta n_0 = -K\{n_0 \Delta n_4 + n_4 \Delta n_0 - 2n_2 \Delta n_2 f_1(T) - n_2^2 \Delta T df_1(T)/dT\}$$
$$+ k_1(n + 4n_2/5 + n_4)\{\Delta n_2 - \Delta n_0 f_2(T) - n_0 \Delta T df_2(T)/dT\}.$$

If K, k_1 and $\varkappa_1$ can be determined by any means, we obtain $\Delta n_0/\Delta T$, $\Delta n_2/\Delta T$ and $\Delta n_4/\Delta T$, which may generally be complex quantities, for various values of ω, T and n. These quantities permit exact calculation of effective specific heat of rotational degrees of freedom given by

$$C' = \{E_4 \Delta n_4 + E_2 \Delta n_2\}/\Delta T. \tag{19}$$

(The specific heat per mol is calculated putting $n = 6.03 \times 10^{24}$.) Then the total specific heat of hydrogen gas per mol is

$$C = 3R/2 + C', \tag{20}$$

where R is the universal gas constant.

Now the value of C has been determined experimentally by the measurements of ultrasonic dispersion and anomalous absorption. As was shown by Rhodes[2], the sound velocity, omitting second order terms, is given by

$$v^2 = \frac{RT}{M}\left(1 + 2pB\frac{T_0^2}{T^2}\right) + \frac{R^2T}{M}\left(1 + 2p\frac{T_0^2}{T}\frac{dB}{dT}\right)Re\begin{pmatrix}1\\ C\end{pmatrix}, \tag{21}$$

where $T_0 = 273°\mathrm{K}$; p means the pressure; B the second virial coefficient; $Re(1/C)$ the real part of $1/C$ and M the molecular weight. Thus the experimental values of the specific heat $1/(Re(1/C))$ as a function of frequency may be compared with the theoretical value given by eq. (20). Now we shall use the data by Rhodes at 298.4°K and compare them with the theoretical result which is calculated employing the values in the preceding section. The available values (11) of the cross sections in hydrogen allows us to determine the exact shape of dispersion curve of $1/(Re(1/C))$ as shown by curve B in Fig. 1. The absolute value of k_1 is obtained by the following equation,

$$k_1 = 2R_0^2\left(\frac{2\pi\varkappa T}{\mu}\right)^{1/2}\frac{1}{V_0}\frac{p}{p_0}\frac{T_0}{T}P, \tag{22}$$

R_0 is the collision diameter (taken as 5 A. U.), p the pressure, p_0 1 atm. and V_0 volume of gas per mol for 1 atm. and 273°K; P is given by (12). From this formula we obtain

$$k_1/p = 2.25 \times 10^{-16}\ \mathrm{sec}^{-1}\ \mathrm{atm.}^{-1}\ \mathrm{mol.}$$

The curve B in Fig. 1 was calculated using this value of k_1/p. If the theoretical value k_1 is multiplied by a factor 0.73, the shape of curve becomes more close to the experimental one. (Curve C) In order to make the curve C coincide with the experimental curve A, in regard to the absolute scale, all the values of k_1, $\varkappa_1$, K corresponding to the curve C must be further multiplied by a factor 0.4. This difference between the theoretical and the experimental values of reaction rates is smaller than expected in view of our extremely simplified calculations.

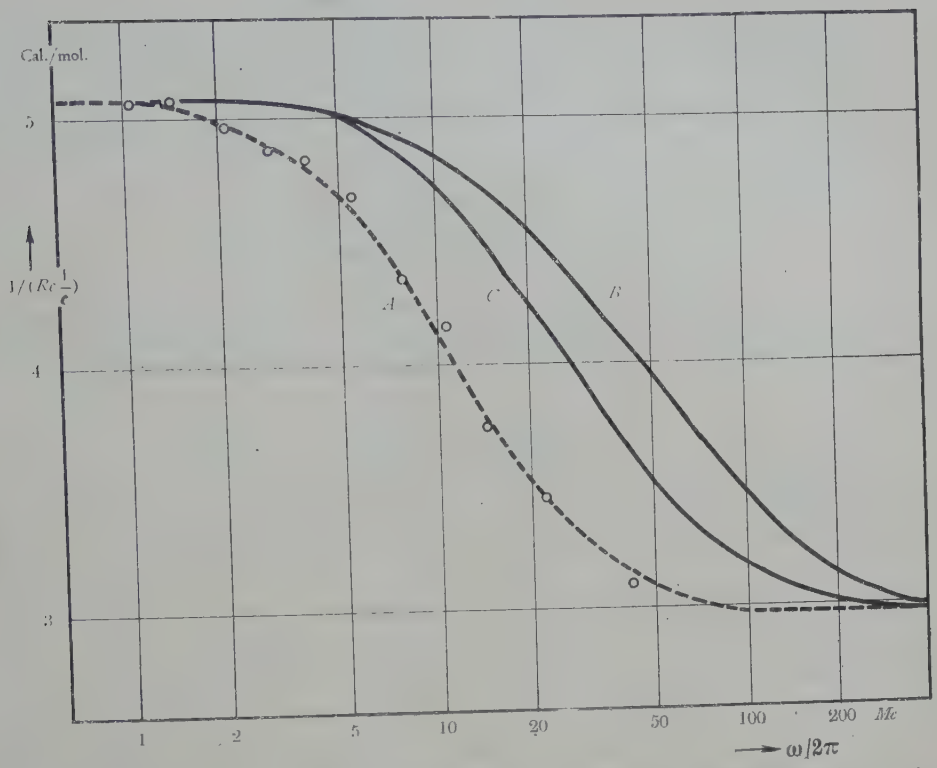

Fig. 1. The dotted curve (A) shows the specific heat $1\ (Re(1/C))$ calculated from the experimental data. (The °'s are experimental points.) The solid curve B is the theoretical curve. The curve C is obtained if the theoretical value of k_1 is multiplied by a factor 0.73..

Rhodes concluded that the relaxation frequencies for the rotational transitions (0—2) and (2—4) are 7.2 mc and 20 mc per atm. respectively from his analysis. Presumably, however, the probability of the rotational transition (2—4) is less than that of (0—2), therefore the relation frequency is lower for the transition (2—4) than for (0–2), in accordance with our theoretical viewpoint.

Finally, it must be noted the substitution of other values (a few times larger or smaller) for $Q^{(2)}/Q^{(1)}$ and $Q^{(3)}/Q^{(1)}$ than calculated above changes the shape of the theoretical curve B considerably in direction to fall apart from the experimental curve and gives rise to a step curve, so that our theoretical calculation is fairly reasonable and gives promising means for the analysis of the phenomena of gas collisions and especially for the interpretation of ultrasonic dispersion and absorption in gases.

§ 4. Conclusion

It has been found that the approximation method, called "classical approximation" tentatively, gives reasonable values of the relative cross sections of various processes and

reasonable order of magnitude of their absolute values, at least for hydrogen. However, the hydrogen is a special case. In other substances, the rotational level spacings are much smaller than those of hydrogen molecules which result in large transition probabilities. Moreover, we must take account of very large number of rotational levels at room temperature. Our next plan is to study such complex cases.

Appendix I. Classical approximation

If we consider that the molecules are the spheres of radius $R_0/2$, the total cross section for the collision between such molecules is πR_0^2, and the partial cross section is given by

$$\frac{2\pi}{k^2} j\,dj \qquad (j \leq kR_0),$$

j being the angular momentum of the relative motion. Thus the cross section of an inelastic collision has the form

$$\sigma = \int_0^{kR_0} \frac{2\pi}{k^2} F(k,j)\, j\,dj,$$

corresponding to the quantummechanical formula

$$\sigma = \frac{\pi}{k^2} \sum_j (2j+1) F(k,j).$$

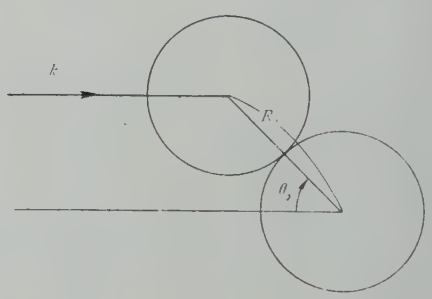

Fig. 2.

Here we shall assume that the transition probability $F(k,j)$ is a function of $k^2 - j^2/R_0^2 (= k^2 \cos^2\theta_j)$ only, which corresponds to the replacement of $k^2 - j(j+1)/R^2$ in the usual wave equation

$$\{d^2/dR^2 + k^2 - j(j+1)/R^2 - 2\mu V(R)\}\psi(R) = 0$$

by a constant $k^2 - j(j+1)/R_0^2$, providing that the potential V is very steep for $R \lesssim R_0$ and negligibly small for larger R. Then we have the cross section, averaging over the Maxwellian distribution $K(k)\,dk$, as

$$\bar{\sigma} = \pi R_0^2 \int_0^\infty K(k)\,dk \int_0^{kR_0} F(k^2 - j^2/R_0^2)\, 2j\,dj/k^2 R_0^2$$

$$= \pi R_0^2 \int_0^\infty F(\tilde{k}^2)\, \tilde{K}(\tilde{k})\, d\tilde{k},$$

where $\tilde{k}^2 = k^2 - j^2/R_0^2$ and

$$\tilde{K}(\tilde{k})\,d\tilde{k} = (1/\mu\varkappa T) \exp\{-\tilde{k}^2/2\mu\varkappa T\}\, \tilde{k}\, d\tilde{k}.$$

Of course $F(\tilde{k}^2)$ must be calculated wave mechanically. But, only the calculation for S-wave is required in our approximation.

If F has a form of $F_1(j)F_2(\tilde{k}^2)$, we must modify the above formulae as follows:

$$\sigma=(2\pi/k^2)\overline{F_1(j)}\int_0^\infty F_2(k^2-j^2/R_0^2)jdj,$$

$$\bar{\sigma}=\pi R_0^2\overline{F_1(j)}\int_0^\infty \cdot F_2(\tilde{k}^2)\tilde{K}(\tilde{k})d\tilde{k}$$

$$\fallingdotseq \pi R_0^2\cdot\overline{F_1(j)}\cdot F_2(2\mu\alpha T).$$

Appendix II. The intermolecular force potential

We shall assume that the intermolecular potential V is given by the sum of the central force potentials between atoms belonging to the different molecules. If the intermolecular distance R is large enough compared with the internuclear distances $\tilde{\xi}_1, \tilde{\xi}_2$ of these molecules, we may write this potential V in the form of Maclaurin series.[4] Especially, if the both molecules are homonuclear diatomic molecules we have

$$V=\varphi(r_{13})+\varphi(r_{14})+\varphi(r_{23})+\varphi(r_{24})$$

$$=4\sum_{\substack{l+m+n=\text{even}\\ i+j+k=\text{even}}}\frac{x^l y^m z^n}{l!\,m!\,n!}\cdot\frac{x'^i y'^j z'^k}{i!\,j!\,k!}\left[\frac{\partial^{l+m+n+i+j+k}\varphi(r)}{\partial X^l\partial Y^m\partial Z^n\partial X'^i\partial Y'^j\partial Z'^k}\right]_{\substack{X=Y=Z=0\\ X'=Y'=Z'=0}},$$

where $r^2=(X'-X-R)^2+(Y'-Y)^2+(Z'-Z)^2$. Carrying out the differentiations, we get

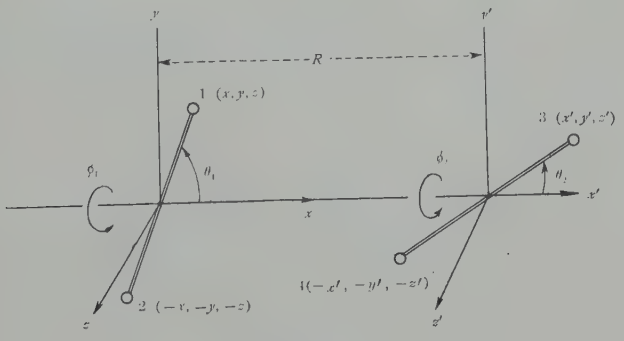

Fig. 3.

$$V=4\varphi(R)+2\left[(x^2+x'^2)\varphi''(R)+(y^2+y'^2+z^2+z'^2)\Phi_1(R)\right]$$
$$+(1/6)\left[(x^4+x'^4)\varphi^{(IV)}(R)+3(y^4+y'^4+z^4+z'^4)\Phi_2(R)\right]$$
$$+\left[x^2x'^2\varphi^{(IV)}(R)+(x^2y^2+x^2z^2+x'^2y'^2+x'^2z'^2+x^2y'^2+x^2z'^2+x'^2y^2+x'^2z^2)\Phi_2(R)\right.$$
$$\left.+(y^2z^2+y'^2z'^2+y^2z'^2+y'^2z^2)\Phi_2(R)+3(y^2y'^2+z^2z'^2)\Phi_2(R)\right]+4yzy'z'\Phi_2(R)$$
$$+4xx'(yy'+zz')\Phi_3(R)+\text{higher order terms},$$

where

$$\Phi_1(R)=\varphi'(R)/R, \quad \Phi_2(R)=\Phi_1'(R)/R=[\varphi''(R)-\varphi'(R)/R]/R^2$$
$$\Phi_3(R)=\Phi_1''(R)=[\varphi'''(R)-2\varphi''(R)/R+2\varphi'(R)/R^2]/R,$$
$$\varphi'(R)=d\varphi(R)/dR, \quad \varphi''(R)=d^2\varphi(R)/dR^2, \text{ etc.}$$

For the hydrogen molecules, it is fairly good approximation to take the function $\varphi(r)$ as

$$\varphi(R)=A\cdot\exp(-2R).$$

Then $A=5.0$ is the suitable value of this parameter.

Of course

$$(2x)^2+(2y)^2+(2z)^2=\xi_1^2$$
$$(2x')^2+(2y')^2+(2z')^2=\xi_2^2$$

and $\xi_1=\xi_2=1.44$ in the case of hydrogen.

Appendix III. Matrix elements of the potential

As stated in § 2, the matrix elements of $P_2(\cos\theta_1)$ and $P_2(\cos\theta_1)\times P_2(\cos\theta_2)$ are required for our purpose. The former have already been evaluated in the previous paper[3], i.e.,

$$(l_1m_1l_2m_2jm \mid P_2(\cos\theta_1)\mid l_1'm_1'l_2'm_2'j'm')$$
$$=C^2(jm;j'm')\,C^2(l_1'm_1';l_1m_1)\,\delta(m_1+m_2,m_1'+m')\,\delta(l_2,l_2')\,\delta(m_2,m_2').$$

On the other hand, noting that

$$P_2(\cos\theta_1)\,P_2(\cos\theta_2)$$
$$=(4\pi/5)^2\sum_{\mu,\mu'=-2}^{2}Y_{2\mu'}^*(\theta_1\varphi_1)\,Y_{2\mu'}^*(\theta_2\varphi_2)\,Y_{2\mu}(\Theta\Phi)\,Y_{2\mu'}(\Theta\Psi)$$
$$=(4\pi/5)^{3/2}\sum_{\mu,\mu',l}C^2(l,\mu+\mu';2,\mu)\,Y_{2\mu}^*(\theta_1\varphi_1)\,Y_{2\mu'}^*(\theta_2\varphi_2)\,Y_{l,\mu+\mu'}(\Theta\Phi)$$

(Θ, Φ are the angular components of the vector $\boldsymbol{R}$), we obtain

$$(l_1m_1l_2m_2jm \mid P_2(\cos\theta_1)P_2(\cos\theta_2)\mid l_1'm_1'l_2m_2'j'm')$$
$$=\sum(2l+1/5)^{1/2}C^2(l,m_1'+m_2'-m_1-m_2;2,m_1'-m_1)\,C^l(jm;j'm')$$
$$\times C^2(l_1'm_1';l_1m_1)\,C^2(l_2'm_2';l_2m_2)\,\delta(m_1+m_2+m,m_1'+m_2'+m').$$

Since our calculation are made for $l_1=l_2=2$, $l_1'=4$, $l_2'=0$, $m'=0$, the numerical values of $C^l(j,m;j',0)$ for large j are required. Among them, $C^2(j,m;j',0)$ was calculated in the previous paper[3] and

$$C^0(j,m;j',0)=\delta(j,j')\,\delta(m,0).$$

The remaining coefficients $C^1(j, m; j', 0)$ have been calculated straight-forwardly employing the general formula for $C^l(j, m; j', m')$. The results are

$$C^1(j, 0; j, 0) = 9(j+2)(j+1)j(j-1)/I$$

$$C^1(j, 1; j, 0) = 9(j+2)(j-1)\sqrt{5j(j+1)}/I$$

$$C^1(j, 2; j, 0) = -3(j^2+j-9)\sqrt{10(j+2)(j+1)j(j-1)}/I$$

$$C^1(j, 3; j, 0) = -9\sqrt{35(j+3)(j+2)(j+1)j(j-1)(j-2)}/I$$

$$C^1(j, 4; j, 0) = 3\sqrt{70(j+4)(j+3)(j+2)(j+1)j(j-1)(j-2)(j-3)}/2I$$

$$I = 4(2j+5)(2j+3)(2j-1)(2j-3)$$

$$C^1(j, 0; j+2, 0) = 10(j+3)(j+2)(j+1)j/J$$

$$C^1(j, 1; j+2, 0) = -(j+3)(j+2)(2j-7)\sqrt{5j(j+1)}/J$$

$$C^1(j, 2; j+2, 0) = -2(j+7)(j+3)\sqrt{10(j+2)(j+1)j(j-1)}/J$$

$$C^1(j, 3; j+2, 0) = (2j+9)\sqrt{35(j+3)(j+2)(j+1)j(j-1)(j-2)}/J$$

$$C^1(j, 4; j+2, 0) = -\sqrt{70(j+4)(j+3)(j+2)(j+1)j(j-1)(j-2)(j-3)}/J$$

$$J = 4(2j+7)(2j+3)(2j-1)\sqrt{(2j+5)(2j+1)}$$

$$C^1(j, 0; j+4, 0) = 35(j+4)(j+3)(j+2)(j+1)/K$$

$$C^1(j, 1; j+4, 0) = -14(j+4)(j+3)(j+2)\sqrt{5j(j+1)}/K$$

$$C^1(j, 2; j+4, 0) = 7(j+4)(j+3)\sqrt{10(j+2)(j+1)j(j-1)}/K$$

$$C^1(j, 3; j+4, 0) = -2(j+4)\sqrt{35(j+3)(j+2)(j+1)j(j-1)(j-2)}/K$$

$$C^1(j, 4; j+4, 0) = \sqrt{70(j+4)(j+3)(j+2)(j+1)j(j-1)(j-2)(j-3)}/2K$$

$$K = 8(2j+7)(2j+5)(2j+3)\sqrt{(2j+9)(2j+1)}$$

$$C^1(j, 0; j-2, 0) = 10(j+1)j(j-1)(j-2)/L$$

$$C^1(j, 1; j-2, 0) = (j-1)(j-2)(2j+9)\sqrt{5j(j+1)}/L$$

$$C^1(j, 2; j-2, 0) = -2(j-2)(j-6)\sqrt{10(j+2)(j+1)j(j-1)}/L$$

$$C^1(j, 3; j-2, 0) = -(2j-7)\sqrt{35(j+3)(j+2)(j+1)j(j-1)(j-2)}/L$$

$$C^1(j, 4; j-2, 0) = -\sqrt{70(j+4)(j+3)(j+2)(j+1)j(j-1)(j-2)(j-3)}/L$$

$$L = 4(2j+3)(2j-1)(2j-5)\sqrt{(2j+1)(2j-3)}$$

$$C^4(j,0;j-4,0)=35j(j-1)(j-2)(j-3)/M$$

$$C^4(j,1;j-4,0)=14(j-1)(j-2)(j-3)\sqrt{5j(j+1)}/M$$

$$C^4(j,2;j-4,0)=7(j-2)(j-3)\sqrt{10(j+2)(j+1)j(j-1)}/M$$

$$C^4(j,3;j-4,0)=2(j-3)\sqrt{35(j+3)(j+2)(j+1)j(j-1)(j-2)}/M$$

$$C^4(j,4;j-4,0)=\sqrt{70(j+4)(j+3)(j+2)(j+1)j(j-1)(j-2)(j-3)}/2M$$

$$M=8(2j-1)(2j-3)(2j-5)\sqrt{(2j+1)(2j-7)}.$$

Appendix IV. Effect of the statistics of the identical molecules in the inelastic molecular collision*)

Two hydrogen molecules are treated usually as a dissimilar pair of particles when these are in the different rotational states. (The rotational state is defined by the two quantum numbers corresponding to the absolute value and its z-component of the rotational angular momentum.) It must be noted, however, that these are not completely dissimilar, because the total wave function, including the rotational wave functions, must be symmetric with respect to the exchange of two molecules.

For example, the unsymmetrized wave function**)

$$Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)\,\varphi(\boldsymbol{R})$$

can be rewritten as

$$(1/2)[Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)\,\varphi(\boldsymbol{R})+Y_{lm}(\theta_2\varphi_2)\,Y_{l'm'}(\theta_1\varphi_1)\,\varphi(-\boldsymbol{R})]$$
$$+(1/2)[Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)\,\varphi(\boldsymbol{R})-Y_{lm}(\theta_2\varphi_2)\,Y_{l'm'}(\theta_1\varphi_1)\,\varphi(-\boldsymbol{R})],$$

and such wave function is excluded because it contains the antisymmetric part. Thus we must always employ the symmetrized wave function of the form

$$(1/\sqrt{2})[Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)\,\varphi(\boldsymbol{R})+Y_{lm}(\theta_2\varphi_2)\,Y_{l'm'}(\theta_1\varphi_1)\,\varphi(-\boldsymbol{R})].$$

This is rewritten as

$$(1/\sqrt{2})[Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)+Y_{lm}(\theta_2\varphi_2)\,Y_{l'm'}(\theta_1\varphi_1)](1/2)[\varphi(\boldsymbol{R})+\varphi(-\boldsymbol{R})]$$
$$+(1/\sqrt{2})[Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)-Y_{lm}(\theta_2\varphi_2)\,Y_{l'm'}(\theta_1\varphi_1)](1/2)[\varphi(\boldsymbol{R})-\varphi(-\boldsymbol{R})]$$
$$\equiv \mathcal{Y}^{ll'}_{mm'}(1,2)^{+}\varphi(\boldsymbol{R})^{+}+\mathcal{Y}^{ll'}_{mm'}(1,2)^{-}\varphi(\boldsymbol{R})^{-},$$

---

*) The authors express their appreciation to Prof. M. Kotani (University of Tokyo) for discussions on this problem.

**) θ_i, φ_i give the direction of the i-th molecule in the fixed coordinate system in space. $\boldsymbol{R}$ is the vector joining the centers of the two hydrogen molecules.

where $\mathcal{Y}^{ll'}_{mm'}(1,2)^{\pm}$ is the normalized and symmetrized rotational wave functions. Especially, if $(l,m)=(l',m')$, it is simply

$$(1/\sqrt{2})[\varphi(\boldsymbol{R})+\varphi(-\boldsymbol{R})]Y_{lm}(\theta_1\varphi_1)\,Y_{lm}(\theta_2\varphi_2).$$

In this latter case, the relative motion of the two molecules has only the even values of quantum number j. Accordingly, there is the explicit appearance of the effect of indistinguishability of two hydrogen molecules. Clearly, this effect is remarkable in the low energy collisions where the S-wave predominates.

The total wave function of the two-molecular system is of the form

$$\Psi=\sum_{ll'mm'}\mathcal{Y}^{ll'}_{mm'}(1,2)^{+}\varphi^{ll'}_{mm'}(\boldsymbol{R})^{+}+\sum_{ll'mm'}\mathcal{Y}^{ll'}_{mm'}(1,2)^{-}\varphi^{ll'}_{mm'}(\boldsymbol{R})^{-}.$$

Substituting this in the wave equation $H\Psi=E\Psi$, we have

$$(\Delta^2+k^2_{ll'})\varphi^{ll'}_{mm'}(\boldsymbol{R})^{+}=2\mu\sum_{pqp'q'}(\mathcal{Y}^{ll'}_{mm'}{}^{+}\mid V\mid\mathcal{Y}^{pp'}_{qq'}{}^{+})\varphi^{pp'}_{qq'}(\boldsymbol{R})^{+},$$

$$(\Delta^2+k^2_{ll'})\varphi^{ll'}_{mm'}(\boldsymbol{R})^{-}=2\mu\sum_{pqp'q'}(\mathcal{Y}^{ll'}_{mm'}{}^{-}\mid V\mid\mathcal{Y}^{pp'}_{qq'}{}^{-})\varphi^{pp'}_{qq'}(\boldsymbol{R})^{-},$$

since the matrix elements of the type $(+\mid V\mid -)$vanish in the present case. In these equations

$$k^2_{ll'}/2\mu=E-l(l+1)/2M\xi^2-l'(l'+1)/2M\xi^2,$$

and ξ is the internuclear distance in a molecule and M is half the mass of hydrogen atom.

$\varphi^{\pm}$ can be further expanded as usual.

$$\varphi^{ll'}_{mm'}(\boldsymbol{R})^{+}=\sum_{j=\text{even}}\sum_{\nu}(2j+1)^{1/2}\{G^{j\nu}_{ll'mm'}(R)/R\}\,Y_{j\nu}(\theta\Phi),$$

$$\varphi^{ll'}_{mm'}(\boldsymbol{R})^{-}=\sum_{j=\text{odd}}\sum_{\nu}(2j+1)^{1/2}\{G^{j\nu}_{ll'mm'}(R)/R\}\,Y_{j\nu}(\theta\Phi).$$

Using these G-functions the cross sections of the inelastic collision processes are given by (4), (4a) in the distorted wave approximation. We shall consider the process of the type

$$(l,m)+(l,m)\rightarrow(l',m')+(l,m). \qquad (*)$$

If the two molecules are distinguishable with each other, these can be numbered. Thus the excitation of the first molecule is separated from the excitation of the second molecule. Since two final state wave functions $(Y_{l'm'}(\theta_1\varphi_1)\,Y_{lm}(\theta_2\varphi_2)\varphi(\boldsymbol{R})$ and $Y_{lm}(\theta_1\varphi_1)\,Y_{l'm'}(\theta_2\varphi_2)\varphi(\boldsymbol{R}))$ are orthogonal with each other, the cross section of the process (*) is twice the cross section of the process

$$(l'',m'')+(l,m)\rightarrow(l'',m'')+(l',m') \qquad (**)$$

when we neglect the small difference of the intermolecular forces for these two processes (*) and (**). It is not true, however, in the case of the indistinguishable molecules. In this case only the final wave functions of the form

$$\mathcal{Y}^{w}_{m,m'}(1,2)^{+}\varphi(\boldsymbol{R})^{+}$$

are permissible, because the $\mathcal{Y}$-function in the initial state is of the "+"-character (symmetrical) and the matrix elements $(-|V|+)$ vanish for our present case. Therefore the cross sections of the process (*) and (**) are the same with each other. For hetero-nuclear cases, however, non-vanishing matrix elements of the form $(\pm|V|\mp)$ may exist. But in this case $(\pm|V|\pm)$ and $(\pm|V|\mp)$ come from the different terms of the function I^{ν} (symmetric and antisymmetric part resp.) so that the cross section does not double.

Additional notes to paper I[1] and II[3]

(1) In the paper I, the molecular collisions were considered for the mean thermal energy as in the present paper. Recently, independent of mine, Schwartz, Slawsky and Herzfeld[5] investigated the same problem in which they carried out the summation over the Maxwellian distribution.

(2) In the paper II, we proposed "the method of modified wave number" (§ 4, (iv)), in which the wave number k was simply replaced by a constant

$$\tilde{k}=\{k^2-j(j+1)/R_0^2\}^{1/2}.$$

It will be better, however, to formulate with a little modifications as follows:

The radial wave equation to be solved is

$$\{d^2/dR^2+k^2-j(j+1)/R^2-V(R)\}\varphi_j(k^2;R)=0.$$

First, we solve the wave equation

$$\{d^2/dR^2+\tilde{k}^2-V(R)\}\varphi(\tilde{k}^2;R)=0.$$

regarding $\tilde{k}$ as a constant. Denote the solution of this equation which satisfies the boundary conditions (e.g., $\varphi(0)=0$ and $\varphi\sim\sin(\tilde{k}R+\delta)$ asymptotically) by

$$\boldsymbol{\Phi}(\tilde{k}^2,R).$$

Then the approximate wave function of the original equation will be given by

$$\varphi_j(k^2,R)=\boldsymbol{\Phi}(k^2-j(j+1)/R^2,R)$$

provided that the potential $V(R)$ is steep compared with $j(j+1)/R^2$.

(3) "Formulae for $C^2(j,m;j,0)$ with even j" (paper II, Appendix II) can be applied to the case with odd j as well.

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On the Convergence of the Perturbation Method in the Quantum Field Theory

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The convergence of the Green-functions of one-nucleon and one-meson problems is investigated. In order to get the finite expressions for these Green-functions, we define the vertex function Γ by (1·9). By virtue of this definition the number of graphs of the proper vertex-type of the n-th order in g increases so slowly that the majorant series of the Green-function can be made to converge.

Introduction

In the preceding paper[1] of the present author and T. Imamura they showed that the S-matrix represented as a power series in the coupling constant diverges in case of a scalar field with non-linear self-interaction though the relativistic cut-off (non-local interaction) was carried out. This divergence results from the fact that the number N_n of graphs corresponding to the n-th term in the series stated above increases rapidly with the increase in n. This rapid increase of the number of graphs is due to the structure of the functional equation satisfied by Γ (vertex-function).

In the present paper we shall investigate how to modify the functional equation of Γ in order to get the convergent results in the perturbation method, especially to obtain a finite expression of the Green-functions for the one-body problem.

§ 1. Fundamental equations

In what follows we put $\hbar=c=1$, and $\mu=1$ where μ is the rest mass of a meson field, thus all the quantities appearing in this paper are dimensionless. Let us consider a system of nucleon- and neutral scalar meson-fields with the Lagrangian density

$$\begin{aligned} L(x) = &-\bar{\psi}(x)\{r_\mu(-i\partial_\mu)+M\}\psi(x)-\frac{1}{2}\{\partial_\mu\phi\partial_\mu\phi+\phi^2\} \\ &+g\int U_{\alpha\beta}(x', x'' : x)\, T\{\phi(x), \bar{\psi}_\alpha(x')\psi_\beta(x'')\}\, dx'dx'' \\ &+J(x)\phi(x)+\bar{\eta}_\alpha(x)\psi_\alpha(x)+\bar{\psi}_\alpha(x)\eta_\alpha(x), \end{aligned} \tag{1·1}$$

where $\bar{\psi}_\alpha$, ψ_α's represent the nucleon field, and ϕ the neutral scalar meson field. $J(x)$ is a c-number source function and η_α, $\bar{\eta}_\alpha$'s are c-number source spinors being anti-commutative with each other and with $\bar{\psi}$ and ψ.

Following the Schwinger's method let us define one-nucleon Green-function $G_{\alpha\beta}(x, x')$ and one-meson Green-function $\mathfrak{G}(x, x')$ which satisfy the following system of equations:

$$\{\gamma^\mu(-i\partial_\mu)+M\}_{\alpha\rho}G_{\rho\beta}(x, x')-g\int\langle\phi(y)\rangle U_{\alpha\rho}(xx'':y)G_{\rho\beta}(x''x')dx''dy$$

$$+\int\sum{}^*_{\alpha\rho}(xy)G_{\rho\beta}(yx')dy=\delta_{\alpha\beta}\delta(x-x'), \tag{1·2}$$

$$(\Box-1)\mathfrak{G}(x, x')-\int P(xy)\mathfrak{G}(y, x')dy=-\delta(x-x'), \tag{1·3}$$

$$(\Box-1)\langle\phi(x)\rangle+ig\int U_{\alpha\beta}(x'x'':x)G_{\beta\alpha}(x''x')dx'dx''=-J(x). \tag{1·4}$$

Here the two kinds of self-energy operator $\sum^*$ and P corresponding to nucleon and meson fields are defined as

$$\sum{}^*_{\alpha\beta}(x, x')=ig^2\int U_{\alpha\rho}(xz:y)G_{\rho\sigma}(z, z')\Gamma_{\sigma\beta}(z'x':y')\mathfrak{G}(y'y)dydy'dzdz', \tag{1·5}$$

$$P(x, x')=-ig^2\int U_{\alpha\beta}(y'y'':x)G_{\beta\rho}(y''z'')\Gamma_{\rho\sigma}(z''z':x')G_{\sigma\alpha}(z'y')dy'dy''dz'dz''. \tag{1·6}$$

The vertex-operator Γ appearing in (1·5) (1·6) is defined by

$$\Gamma_{\alpha\beta}(x, x':y)=-\frac{1}{g}\frac{\delta G^{-1}_{\alpha\beta}(x, x')}{\delta\langle\phi(y)\rangle}=U_{\alpha\beta}(x, x':y)-\frac{1}{g}\frac{\delta\sum{}^*_{\alpha\beta}(x, x')}{\delta\langle\phi(y)\rangle}. \tag{1·7}$$

By substituting with (1·5) into (1·7), Γ is written down as

$$\Gamma_{\alpha\beta}(xx':y)=U_{\alpha\beta}(xx':y)$$

$$-ig^2U(x1:2)G(13)\Gamma(34:y)G(45)\Gamma(5x':6)\mathfrak{G}(62)$$

$$-igU(x1:2)G(13)\frac{\delta}{\delta\langle\phi(y)\rangle}\{\Gamma(3x':4)\mathfrak{G}(42)\}. \tag{1·8}$$

The second and third terms in the right-hand side of (1·8) are graphically represented by Fig. 1 and Fig. 2a, b, c, respectively.

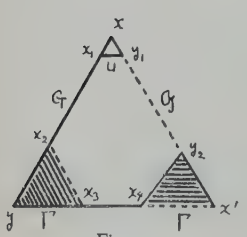

Fig. 1

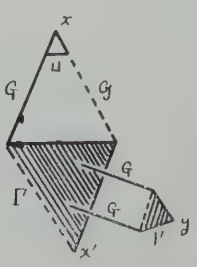

Fig. 2a

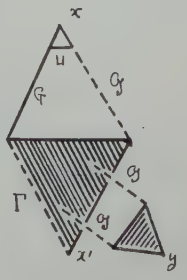

Fig. 2b

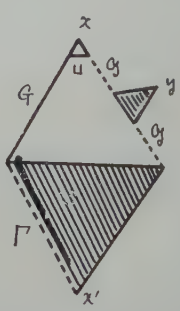

Fig. 2c

Now let us propose that Γ is defined by

$$\Gamma_{\alpha\beta}(x, x' : y) = U_{\alpha\beta}(x, x' : y)$$
$$-ig^2\int U_{\alpha\rho'}(xx_1 : y_1) G_{\rho\rho'}(x_1x_2)\Gamma_{\rho'\sigma'}(x_2x_3 : y) G_{\sigma'\sigma}(x_3x_4)\Gamma_{\sigma\beta}(x_4x' : y_2)\mathfrak{G}(y_2y_1)$$
$$dx_1dx_2dx_3dx_4dy_1dy_2. \qquad (1\cdot9)$$

in place of $(1\cdot8)$. That is to say, we propose to omit the third term in the right-hand side of $(1\cdot8)$, whereas we retain the definition of $\sum^*$ and P. From this new definition of Γ, it is easily seen that Γ consists of the sum of contributions arising from all the graphs which are obtained by inserting the graph itself shown in Fig. 1 into the two vertices appearing in Fig. 1. These graphs will be called Γ-graphs, the examples of which being shown in Fig. 3. The important feature of the Γ-graph is that it has only one open-polygon of G-lines and never has closed-loops of G-lines, and this open-polygon necessarily passes over all the U-vertices of the Γ-graph.

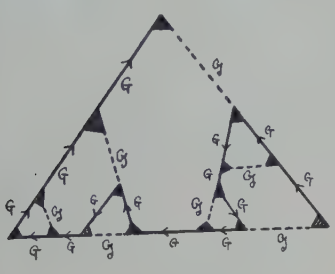

Fig. 3

In case of $J=0$, $\eta=\bar{\eta}=0$, all the quantities considered above are represented in the energy-momentum space in the following way,

$$G_{\alpha\beta}(x) = (2\pi)^{-4}\int G_{\alpha\beta}(k)e^{ikx}dk,$$

$$\mathfrak{G}(x) = (2\pi)^{-4}\int \mathfrak{G}(k)e^{ikx}dk,$$

$$iS^F_{\alpha\beta}(x) = (2\pi)^{-4}\int \overset{\circ}{G}_{\alpha\beta}(k)e^{ikx}dk = (2\pi)^{-4}\int \frac{(-\gamma k+M)_{\alpha\beta}}{k^2+M^2}e^{ikx}dk,$$

$$i\Delta^F(x) = (2\pi)^{-4}\int \overset{\circ}{\mathfrak{G}}(k)e^{ikx}dk = (2\pi)^{-4}\int \frac{e^{ikx}}{k^2+1}dk.$$

$$\langle\phi(x)\rangle = (2\pi)^{-2}\varphi_0 = \text{constant}, \qquad (1\cdot10)$$

$$\sum^*_{\alpha\beta}(x) = (2\pi)^{-4}\int \sum^*_{\alpha\beta}(k)e^{ikx}dk,$$

$$P(x) = (2\pi)^{-4}\int P(k)e^{ikx}dk,$$

$$U_{\alpha\beta}(xy : z) = (2\pi)^{-6}\int U_{\alpha\beta}(k\cdot l)e^{ik(x-z)-il(y-z)}dk\,dl,$$

$$\Gamma_{\alpha\beta}(xy : z) = (2\pi)^{-6}\int \Gamma_{\alpha\beta}(k\cdot l)e^{ik(x-z)-il(y-z)}dk\,dl.$$

By taking account of $(1\cdot10)$, $(1\cdot2)$ can be written in the momentum representation as

$$G_{\alpha\beta}(k)=\overset{\circ}{G}_{\alpha\beta}(k)+g\varphi_0\overset{\circ}{G}_{\alpha\sigma}(k)U_{\sigma\rho}(k\cdot k)G_{\rho\beta}(k)-\overset{\circ}{G}_{\alpha\nu}(k)\sum\nolimits^{*}_{\rho\sigma}(k)G_{\sigma\beta}(k). \quad (1\cdot11)$$

Similarly (1·4) turns out into

$$\varphi_0=ig\overset{\circ}{\mathfrak{G}}(0)\int U_{\alpha\beta}(l\cdot l)G_{\beta\alpha}(l)dl, \quad (1\cdot12)$$

and (1·3) is transformed into

$$\mathfrak{G}(k)=\overset{\circ}{\mathfrak{G}}(k)-\overset{\circ}{\mathfrak{G}}(k)P(k)\mathfrak{G}(k). \quad (1\cdot13)$$

§ 2. Proof for existence of solution

As in case of the preceding paper,[1] let us suppose that $k^2+1>0$, $k^2+M^2>0$ in order to avoid the difficulties concerning the so-called displaced poles arising when energy variables are transformed into pure imaginary ones. Then in course of obtaining the solution of (1·11)$\sim$(1·13) the integrations with regard to virtual energy-momentum variables are allowed to be carried out over the four dimensional Euclidian space.

By using the iteration-method the solution of (1·9) can be formally obtained with regard to Γ as follows;

$$\Gamma_{\alpha\beta}(x,\,x':y)=\sum_{n=0}^{\infty}c_n\,g^{2n}\sum_{\Gamma\text{-graph}}$$
$$\times\int\prod_{j=1}^{2n+1}U_{\alpha_j\beta_j}(x_j\cdot x_j':y_j)\prod_{j=1}^{2n}G_{\rho_j\sigma_j}(\xi_j'-\xi_j)\prod_{k=1}^{n}\mathfrak{G}(\eta_k-\eta_k')\prod_{j\neq1}^{2n+1}dx_j\prod_{j\neq2}^{2n+1}dx_j'\prod_{j\neq3}^{2n+1}dy_j, \quad (2\cdot1)$$

where

$$c_n=\frac{(-i)^n}{(n-1)!}\quad n\geq1,\quad c_0=1,$$

and $\qquad x_1=x,\quad x_2'=x',\quad y_3=y.$

In (2·1), the sets of variables $(\xi_1\cdots\xi_n)$, $(\xi_1'\cdots\xi_n')$ and $(\eta_1\cdots\eta_n:\eta_1'\cdots\eta_n')$ should be chosen out of the sets $(x_2\cdots x_{2n+1})$, $(x_1'x_3'\cdots x_{2n+1}')$ and $(y_1y_2y_4\cdots y_{2n+1})$ respectively in such a way as to get those integrands in (2·1) which correspond to all the possible Γ-graphs with $2n+1$ U-vertices. $\sum_{\Gamma\text{-graph}}$ in (2·1) means to sum up integrals over all these Γ-graphs. (2·1) can be written in the momentum representation as

$$\Gamma_{\alpha\beta}(k\cdot l)=U_{\alpha\beta}(k\cdot l)+\sum_{n=1}^{\infty}c_n\,g^{2n}\sum_{\Gamma\text{-graph}}$$
$$\times\int\prod_{j=1}^{2n+1}U_{\alpha_j\beta_j}(k_j,\,l_j)\prod_{j=1}^{2n}G_{\rho_j\sigma_j}(p_j)\prod_{k=1}^{n}\mathfrak{G}(p_k')\prod_{k=1}^{n}dq_k \quad (2\cdot2)$$

with $k_1=k$, $l_2=l$. Here k_j, l_j, and $h_j=(l_j-k_j)$ are the energy-momentum variables carried by two G-lines and one $\mathfrak{G}$-line being incident on the j-th U-vertex, and $(q_1\cdots q_n)$ are the n linearly independent variables chosen out of $3n$ variables $(p_1\cdots p_{2n}:p_1'\cdots p_n')$. From the very character of Γ-graphs we can put

$$q_k=p_k'\qquad(k=1\cdots n).$$

Suppose now that the form factor $U_{\alpha\beta}(k, l)$ is bounded in the four dimensional Euclidian space E, and further both integrals

$$\int_E |U(k+q, q)\mathring{G}_{\alpha\beta}(q)|dq$$

and

$$\int_E |U(k, k+q)\mathring{\mathfrak{G}}(q)|dq$$

are convergent and bounded functions of k in E. That is to say, we assume that there is a finite constant W defined by

$$W=\underset{k,l\in E}{\mathrm{Max}}\{|U(k,l)|,\ |\mathring{G}|,\ |\mathring{\mathfrak{G}}|, \int |U(k+q,q)\mathring{G}(q)|dq,\ \int |U(k,k+q)\mathring{\mathfrak{G}}(q)|dq\}. \tag{2·3}$$

Furthermore let it be assumed that there exists a finite constant M defined as

$$M=\underset{k\in E}{\mathrm{Max}}\{|\mathfrak{G}(k)|,\ |G(k)|, \int_E |U(k+q,q)G_{\alpha\beta}(q)|dq, \int |U(k,k+q)\mathfrak{G}(q)|dq\}. \tag{2·4}$$

Then the following inequality holds for any one of integrals with fixed n in (2·2);

$$\begin{aligned}&\left|\int_E \prod_{j=1}^{2n+1} U_{\alpha_j\beta_j}(k_j l_j) \prod_{j=1}^{2n} G_{\rho_j\sigma_j}(p_j) \prod_{k=1}^{n} \mathfrak{G}(q_k) \prod_{k=1}^{n} dq_k\right| \\ &<(\mathrm{Max}|G|)^{2n}(\mathrm{Max}|U|)^{n+1}\cdot 4^{2n-1}\cdot\{\mathrm{Max}\int|U(k,k+q)\mathfrak{G}(q)|dq\}^n \\ &<4^{2n-1}W^{n+1}M^n.\end{aligned} \tag{2·5}$$

Here the trivial inequality

$$|\{G(1)G(2)\cdots G(n)\}_{\alpha\beta}|<4^{n-1}(\mathrm{Max}|G|)^n$$

has been taken into account. Thus we get

$$|\Gamma_{\alpha\beta}(k,l)|<\{W+\sum_{n=1}^{\infty}|c_n|N_n g^{2n}4^{2n-1}W^{n+1}M^{3n}\}, \tag{2·6}$$

where N_n is the number of Γ-graphs with fixed n and is given by

$$|c_n|\cdot N_n=\frac{(2n)!}{n!(n+1)!}. \tag{2·7}$$

The calculation of N_n will be given in the appendix.

Now the infinite series in the right-hand side of (2·6) is absolutely convergent if

$$4^3g^2WM^3\leq\frac{1}{4}$$

is satisfied. For the sake of simplicity, putting

$$f(x)=\sum_{n=1}^{\infty}|c_n|\cdot N_n g^{2n}4^{2n-1}W^{n+1}x^{3n},$$

we get

$$|\Gamma_{\alpha\beta}| < W+f(M). \tag{2·6}'$$

Since (1·5) can be written down in the momentum representation as

$$\sum{}^{*}_{\alpha\beta}(k)=ig^2\int U(k,\,k-l)\,G_{\alpha\rho}(k-l)\,\Gamma_{\rho\beta}(k-l,\,k)\mathfrak{G}(l)\,dl, \tag{1·5}'$$

the inequality

$$|\sum{}^{*}_{\alpha\beta}(k)| < 4g^2\,\mathrm{Max}|G|\cdot\mathrm{Max}|\Gamma|\cdot\mathrm{Max}\int|U(k,\,k-l)\mathfrak{G}(l)|dl < 4g^2M^2(W+f(M)) \tag{2·9}$$

holds. Similarly the representative of (1·6) in the momentum space is

$$P(k)=-ig^2\int U(l-k,\,l)G_{\alpha\beta}(l-k)\,\Gamma_{\beta\rho}(l-k,\,l)\,G_{\rho\alpha}(l)\,dl, \tag{1·6}'$$

and the upper bound of $|P(k)|$ is given by

$$\begin{aligned}|P(k)| &< 4^3g^2\mathrm{Max}|G(l-k)|\cdot\mathrm{Max}|\Gamma|\cdot\mathrm{Max}\int|U(l-k,\,l)\,G(l)|dl\\ &< 4^3g^2M^2(W+f(M)).\end{aligned} \tag{2·10}$$

We are at the state of solving the equations (1·11)$\sim$(1·13) by using the method of successive approximation. By substituting with (1·12) into (1·11), the latter equation turns out into

$$\begin{aligned}G_{\alpha\beta}(k)=\overset{\circ}{G}_{\alpha\beta}(k)&+ig^2\overset{\circ}{\mathfrak{G}}(0)\int U(l,\,l)\,G_{\mu\mu}(l)dl\cdot U(k,\,k)\,\overset{\circ}{G}_{\alpha\sigma}(k)\,G_{\sigma\beta}(k)\\ &-\overset{\circ}{G}_{\alpha\rho}(k)\sum{}^{*}_{\rho\sigma}(k)\,G_{\sigma\beta}(k),\end{aligned} \tag{2·11}$$

where we put

$$U_{\alpha\beta}(k,\,l)=\delta_{\alpha\beta}U(k,\,l).$$

In the first place we put the solution in the m-th approximation to be

$$\overset{(m)}{G}_{\alpha\beta}(k),\quad \overset{(m)}{\mathfrak{G}}(k),$$

and suppose that there is a finite constant M_m given by

$$M_m=\underset{k\in K}{\mathrm{Max}}\{|\overset{(m)}{G}_{\alpha\beta}(k)|,\ |\overset{(m)}{\mathfrak{G}}(k)|,\int|U(k+q,\,q)\overset{(m)}{G}_{\alpha\beta}(q)|dq,\ \int|U(k,\,k+q)\overset{(m)}{\mathfrak{G}}(q)|dq\}. \tag{2·12}$$

$\overset{(m)}{G}$ and $\overset{(m)}{\mathfrak{G}}$ are defined in the following way:

$$\begin{aligned}&\overset{(0)}{G}_{\alpha\beta}(k)=\overset{\circ}{\mathfrak{G}}_{\alpha\beta}(k)\\ &\overset{(m)}{G}_{\alpha\beta}(k)=\overset{\circ}{G}_{\alpha\beta}(k)+ig^2\overset{\circ}{\mathfrak{G}}(0)\int U(l,\,l)\overset{(m-1)}{G}_{\mu\mu}(l)dl\cdot U(k,\,k)\,\overset{\circ}{G}_{\alpha\sigma}(k)\overset{(m-1)}{G}_{\sigma\beta}(k)\\ &\qquad-\overset{\circ}{G}_{\alpha\rho}(k)\overset{(m-1)}{\sum{}^{*}_{\rho\sigma}}(k)\overset{(m-1)}{G}_{\sigma\beta}(k),\qquad m\geqq 1,\end{aligned} \tag{2·13}$$

and

$$\overset{(0)}{\mathfrak{S}}(k)=\overset{\circ}{\mathfrak{S}}(k)$$

$$\overset{(m)}{\mathfrak{S}}(k)=\overset{\circ}{\mathfrak{S}}(k)-\overset{\circ}{\mathfrak{S}}(k)\overset{(m-1)}{P}(k)\overset{(m-1)}{\mathfrak{S}}(k), \qquad m\geq 1, \tag{2·14}$$

where $\overset{(m)}{P}$ and $\overset{(m)}{\sum}{}^{*}$ are the expressions given by replacing G and $\mathfrak{S}$ in the right-hand sides of $(1\cdot6)'$ and $(1\cdot5)'$ with $\overset{(m)}{G}$ and $\overset{(m)}{\mathfrak{S}}$ respectively.

From (2·13) we have the relation

$$\begin{aligned}|\overset{(m)}{G}_{\alpha\beta}(k)| &< W+g^2\mathrm{Max}|\{\overset{\circ}{\mathfrak{S}}(0)U(k,k)\overset{\circ}{G}(k)\}|\cdot 4^2\mathrm{Max}|\overset{(m-1)}{G}| \\ &\quad\times\mathrm{Max}\int|U(l,l)\overset{(m-1)}{G}_{\mu\nu}(l)|dl+4^2W\cdot\mathrm{Max}|\overset{(m-1)}{\sum}{}^{*}|\cdot\mathrm{Max}|\overset{(m-1)}{G}| \\ &\leq W+4^2g^2W^3M_{m-1}^2+4^3g^2M_{m-1}^3\{W+f(M_{m-1})\}W \\ &=W+F(M_{m-1}),\end{aligned} \tag{2·15}$$

where we have put

$$F(x)=g^2W[4^2W^2x^2+4^3x^3\{W+f(x)\}].$$

Similarly from (2.14) we get

$$|\overset{m}{\mathfrak{S}}(k)|<W+F(M_{m-1}). \tag{2·16}$$

Further the following inequalities can be easily derived by taking into consideration (2·9), (2·10), (2·12);

$$\begin{aligned}&\int|U(k+q,q)\overset{(m)}{G}_{\alpha\beta}(q)|dq\leq\int|U(k+q,q)\overset{\circ}{G}_{\alpha\beta}(q)|dq \\ &+4^2g^2\ \mathrm{Max}|(\overset{\circ}{\mathfrak{S}}(0)U(l,l)|)\cdot\int|U(k+q,q)\overset{\circ}{G}(q)|dq\ \mathrm{Max}|\overset{(m-1)}{G}|\cdot\int|U(l,l)\overset{(m-1)}{G}(l)|dl \\ &+4^2\ \mathrm{Max}|\overset{(m-1)}{(\sum}{}^{*}\overset{(m-1)}{G})|\cdot\int|U(k+q,q)\overset{\circ}{G}(q)|dq<W+F(M_{m-1}),\end{aligned} \tag{2·17}$$

and

$$\int|U(k,k+q)\overset{(m)}{\mathfrak{S}}(q)|dq<W+F(M_{m-1}). \tag{2·18}$$

Thus we arrive at the result

$$M_m<W+F(M_{m-1}). \tag{2·19}$$

Now the infinite series

$$F(x)=Wg^2[4^2W^2x^2+4^3x^3\{W+f(x)\}]$$

is absolutely convergent if

$$|x|\leq\lambda \qquad 4^2g^2W\lambda^3<\frac{1}{4} \tag{2·20}$$

is satisfied. In other words, $F(x)$ is a regular function of x for $|x| \leq \lambda$. By means of the mean value theorem, we can put $F(x)$ into the form

$$F(x) = \frac{x^2}{2} \left. \frac{d^2F(\xi)}{d\xi^2} \right|_{\xi=\theta x} \qquad 0 < \theta < 1, \quad |x| \leq \lambda,$$

since the series F begins at the term with x^2. By virtue of the character of regular functions, there exists such a finite constant A as

$$\left| \frac{d^2F(x)}{dx^2} \right| \leq g^2 WA, \quad |x| \leq \lambda. \tag{2·21}$$

Thus it holds

$$|F(x)| < \frac{g^2 AW}{2} x^2, \quad |x| \leq \lambda.$$

Let it be assumed that

$$M_{m-1} \leq \lambda, \tag{2·20}'$$

then (2·19) can be rewritten as

$$M_m \leq W\left[1 + \frac{g^2 A}{2} M_{m-1}^2\right]. \tag{2·22}$$

In the second place let us consider the equation

$$x = W\left[1 + \frac{g^2 A}{2} x^2\right]. \tag{2·23}$$

Suppose now that g satisfies the condition

$$D = 1 - 2g^2 AW^2 > 0,$$

then (2·23) has two positive real roots

$$\lambda_1 = \frac{1 + \sqrt{D}}{g^2 AW} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{D}}{g^2 AW} \equiv \tau.$$

Putting

$$y(x) = \frac{g^2 AW}{2} x^2 - x + W,$$

we have the relation

$$\left. \frac{dy}{dx} \right|_{x=\tau} = g^2 AW\tau - 1 < 0$$

i.e.

$$a \equiv g^2 AW\tau < 1. \tag{2·24}$$

Further we propose that by choosing g to be sufficiently small the inequality

$$4^2 g^2 W\tau^3 < \frac{1}{4} \tag{2·25}$$

can be satisfied. Under these assumptions we can prove the existence of a solution of (1·11)~(1·13) in the following way.

From (2·23) it is easily seen that

$$M_0 \equiv W < \tau,$$

i.e.

$$\mathrm{Max}\{|\overset{\circ}{G}|, |\overset{\circ}{\mathfrak{G}}|, \int |U\overset{\circ}{G}|dq, \int |U\overset{\circ}{\mathfrak{G}}|dq\} < \tau .$$

Assume now that

$$M_{m-1} < \tau$$

is satisfied. Then by virtue of (2·22), it holds that

$$M_m < W\left[1 + \frac{g^2 A}{2}\tau^2\right] = \tau.$$

Thus we get

$$M_m < \tau \quad \text{for every} \quad m(=0, 1, 2, \cdots). \tag{2·26}$$

From this result it is seen that all the assumptions so far used are justified except the assumption of the existence of τ and (2·25).

Next let us put

$$\Delta M_m = \mathrm{Max}\{\overset{(m)}{\Delta G}_m, \overset{(m)}{\Delta \mathfrak{G}}, \Delta(\int |UG|dq)_m, \Delta(\int |U\mathfrak{G}|dq)_m\}, \tag{2·27}$$

where the following abbreviations have been used;

$$\overset{(m)}{\Delta G} = \mathrm{Max}|\overset{(m+1)}{G} - \overset{(m)}{G}|$$

$$\overset{(m)}{\Delta \mathfrak{G}} = \mathrm{Max}|\overset{(m+1)}{\mathfrak{G}} - \overset{(m)}{\mathfrak{G}}|$$

$$\Delta(\int |UG|dq)_m = \mathrm{Max}\int |U(\overset{(m+1)}{G} - \overset{(m)}{G})|dq$$

$$\Delta(\int |U\mathfrak{G}|dq)_m = \mathrm{Max}\int |U(\overset{(m+1)}{\mathfrak{G}} - \overset{(m)}{\mathfrak{G}})|dq.$$

Remembering the trivial relation

$$|\overset{(m)}{A}\overset{(m)}{B} - \overset{(m-1)}{A}\overset{(m-1)}{B}| \leq |\overset{(m)}{A}| \cdot |\overset{(m)}{B} - \overset{(m-1)}{B}| + |\overset{(m)}{A} - \overset{(m-1)}{A}| \cdot |\overset{(m-1)}{B}|,$$

we can estimate the upper bound of

$$\overset{(m-1)}{\Delta \Gamma} = \mathrm{Max}|\overset{(m)}{\Gamma} - \overset{(m-1)}{\Gamma}|.$$

The result is given as follows;

$$\overset{(m-1)}{\Delta \Gamma} \leq \sum_{n=1}^{\infty} |c_n| \cdot g^{2n} \sum_{\Gamma\text{-graph}}$$

$$\times\int\prod_{j=1}^{2n+1}|U_{\alpha_j\beta_j}|\cdot\Big|\prod_{j=1}^{2n}\prod_{k=1}^{n}\overset{(m)}{G}(p_j)\overset{(m)}{\mathfrak{G}}(q_k)-\prod_{j=1}^{2n}\prod_{k=1}^{n}\overset{(m-1)}{G}(p_j)\overset{(m-1)}{\mathfrak{G}}(q_k)\Big|\prod_{k=1}^{n}dq_k$$

$$\leq\sum_{n=1}^{\infty}|c_n|\cdot g^{2n}N_n\cdot 4^{2n-1}\cdot(\mathrm{Max}|U|)^{n+1}\cdot\big[2n(\mathrm{Max}|G|)^{2n-1}(\mathrm{Max}\int|U\mathfrak{G}|dq)^{n}\cdot\overset{(m-1)}{\Delta G}$$

$$+n(\mathrm{Max}|G|)^{2n}\cdot(\mathrm{Max}\int|U\mathfrak{G}|dq)^{n-1}\Delta(\int|U\mathfrak{G}|dq)_{m-1}\big].$$

Using (2·26), (2·21), we can further transform this result into

$$\overset{(m-1)}{\Delta\Gamma}<\sum_{n=1}^{\infty}|c_n|g^{2n}N_n4^{2n-1}\cdot W^{n+1}\cdot 3n\tau^{3n-1}\cdot\Delta M_{m-1},$$

i.e.,

$$\overset{(m-1)}{\Delta\Gamma}<\frac{df(\tau)}{d\tau}\Delta M_{m-1}. \tag{2·28}$$

Hence the upper bound of $|\overset{(m)}{\sum}{}^{*}-\overset{(m-1)}{\sum}{}^{*}|$ can be estimated in the following way;

$$\overset{(m-1)}{\Delta\sum}{}^{*}=\mathrm{Max}|\overset{(m)}{\sum}{}^{*}_{\alpha\beta}(k)-\overset{(m-1)}{\sum}{}^{*}_{\alpha\beta}(k)|$$

$$\leq 4g^2\mathrm{Max}\big(\int|U\mathfrak{G}|dq\big)\cdot\mathrm{Max}|\Gamma|\cdot\Delta G_{m-1}$$

$$+4g^2\mathrm{Max}|G|\cdot\mathrm{Max}|\Gamma|\cdot\Delta\big(\int|U\mathfrak{G}|dq\big)_{m-1}$$

$$+4g^2\mathrm{Max}|G|\cdot\mathrm{Max}\big(\int|U\mathfrak{G}|dq\big)\cdot\overset{(m-1)}{\Delta\Gamma}$$

$$\leq 4g^2\frac{d}{d\tau}\{W\tau^2+\tau^2f(\tau)\}\Delta M_{m-1}. \tag{2·29}$$

In quite the similar way we get

$$\overset{(m-1)}{\Delta P}=\mathrm{Max}|\overset{(m)}{P}(k)-\overset{(m-1)}{P}(k)|\leq 4^3g^2\frac{d}{d\tau}\{W\tau^2+\tau^2f(\tau)\}\Delta M_{m-1}. \tag{2·30}$$

Accordingly from (2·13) we have

$$|\overset{(m+1)}{G}_{\alpha\beta}(k)-\overset{(m)}{G}_{\alpha\beta}(k)|\leq[4^2g^2W^3\cdot 2\tau+4^2W\,\mathrm{Max}|\textstyle\sum^{*}|$$

$$+4^3W\tau\,g^2\frac{d}{d\tau}\{\tau^2(W+f(\tau))\}]\Delta M_{m-1}$$

$$=\frac{d}{d\tau}F(\tau)\cdot\Delta M_{m-1}.$$

Since it holds that

$$\frac{d}{d\tau}F(\tau)=\tau\frac{d^2F(x)}{dx^2}\Big|_{x=\theta\tau}<\tau g^2AW\equiv\alpha,\quad(0<\theta<1),$$

we arrive at the result

$$|\overset{(m+1)}{G} - \overset{(m)}{G}| \leq a\Delta M_{m-1}. \tag{2·31}$$

Similarly we can obtain the following relations:

$$\Delta\overset{(m)}{\mathfrak{G}} \leq a\cdot\Delta M_{m-1},$$

$$\Delta\left(\int |UG|dq\right)_m \leq a\cdot\Delta M_{m-1}$$

$$\Delta\left(\int |U\mathfrak{G}|dq\right)_m \leq a\cdot\Delta M_{m-1}.$$

Therefore the relation

$$\Delta M_m \leq a\cdot\Delta M_{m-1} \qquad (m=1, 2\cdots) \tag{2·32}$$

has been verified. From (2·32), and (2·24)

$$\lim_{m\to\infty} \Delta M_m = 0.$$

Thus we get the solutions

$$\begin{aligned} &\lim_{m\to\infty} \overset{(m)}{G}_{\alpha\beta} = \mathfrak{G}_{\alpha\beta} \\ &\lim_{m\to\infty} \overset{(m)}{\mathfrak{G}} = \mathfrak{G} \end{aligned} \qquad \text{(absolutely convergent)}.$$

Of course these solutions satisfy (1·11)$\sim$(1·13) and are regular functions of g if g satisfies the condition (2·25). Therefore we can expand these functions into power series in g which are just the expressions usually obtained by the perturbation method.

Now our last task is to prove the uniqueness of the solution. Let $(G_1, \mathfrak{G}_1)$ and $(G_2, \mathfrak{G}_2)$ be two sets of solutions of (1·11)$\sim$(1·13). And further we assume that

$$\text{Max}\{|G_i|, |\mathfrak{G}_i|, \int |UG_i|dq, \int |U\mathfrak{G}_i|dq\} < \tau \quad (i=1, 2) \tag{2·33}$$

is satisfied. Putting

$$\Delta = \text{Max}\{|G_1-G_2|, |\mathfrak{G}_1-\mathfrak{G}_2|, \int |U(G_1-G_2)|dq, \int |U(\mathfrak{G}_1-\mathfrak{G}_2)|dq\},$$

we can prove the relation

$$\Delta \leq a\Delta \tag{2·34}$$

in a way quite similar to that used in deriving (2·32). From (2·34), it is easily seen that

$$\Delta = 0$$

by virtue of (2·24). That is to say, we obtain the conclusion that there is only one solution of (1·11)$\sim$(1·13) which satisfies the condition (2·33).

§ 3. Discussion

Our arguments so far developed are essentially based on the following four assumptions;

i) that the vertex function Γ' is defined by (1·9) instead of (1·8),

ii) the proposal concerning the form factor U, i.e., the existence of a finite constant W defined by (2·3),

iii) that the equation (2·23) is supposed to have two positive real roots $\lambda_1 > \lambda_2 (=\tau)$ and

iv) that $\lambda_2 (=\tau)$ satisfies the condition (2·25).

On account of the assumption (i), we have omitted the contributions to Γ' arising from

$$\frac{\partial \Gamma_{\alpha\beta}(xx' : y)}{\partial \langle \phi(y') \rangle} \quad \text{and} \quad \frac{\partial \mathfrak{G}(x, x')}{\partial \langle \phi(y) \rangle},$$

which are graphically represented in Fig. 4 and 5.

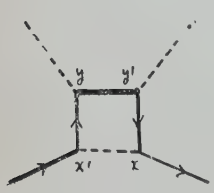

Fig. 4

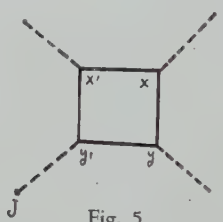

Fig. 5

Fig. 4 represents Compton-scattering of a meson by a nucleon, whereas Fig. 5 the meson-meson scattering. From the physical point of view it seems at a glance a fatal defect to neglect these types of graphs. However we do not take a pessimistic view of this problem. From our standpoint we can give the Green-function for the process of Compton-scattering in the following way. Let us introduce a quantity

$$\Gamma'_{\alpha\beta}(xx' : y) = -\frac{1}{g} \frac{\delta G^{-1}_{\alpha\beta}(x \cdot x')}{\partial \langle \phi(y) \rangle}. \tag{3·1}$$

This can be expressed in the following way by using the symbolic notations.

$$\Gamma'(xx' : y) = U(xx' : y) - ig\Big\{ gU(x1 : 2)G(1, 3)\Gamma'(34 : y)G(45)\Gamma'(5x' : 6)\mathfrak{G}(6 \cdot 2)$$

$$+ U(x1 : 2)G(13)\frac{\partial \Gamma'(3x' : 4)}{\partial \langle \phi(y) \rangle}\mathfrak{G}(42) + U(x1 : 2)G(13)\Gamma'(3x' : 4)\frac{\partial \mathfrak{G}(42)}{\partial \langle \phi(y) \rangle}\Big\} \tag{3·2}$$

where (1·2) and (1·5) have been used. Since it is easily seen from (1·9), (1·3) and (1·6) that the third and fourth terms in the right-hand side of (3·2) are power series in g beginning at the terms with g^3, Γ' can be approximately expressed by

$$\Gamma'(xx' : y) = U(xx' : y) - ig^2 U(x1 : 2)G(13)U(34 : y)G(45)\Gamma'(5x' : 6)\mathfrak{G}(6 \cdot 2), \tag{3·3}$$

the second term of which corresponds to the graph shown in Fig. 6. Now in order to derive the Green-function for the Compton-scattering defined by

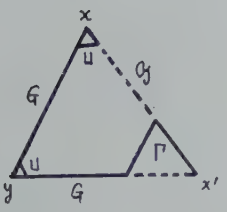

Fig. 6

$$C_{\alpha\beta}(xx' : yy') = \frac{\delta^2 G_{\alpha\beta}(x, x')}{\delta\langle\phi(y)\rangle\delta\langle\phi(y')\rangle},$$

it is further necessary to introduce a new quantity

$$\Pi_{\alpha\beta}(xx' : yy') = \frac{1}{g^3}\frac{\delta\Gamma_{\alpha\beta}(xx' : y)}{\delta\langle\phi(y')\rangle}. \qquad (3\cdot4)$$

Substituting with (3·2) or (3·3) into (3·4), we can obtain an approximate expression of $\Pi_{\alpha\beta}$ given by

$$\Pi_{\alpha\beta}(xx'\,yy') \cong -i\{U(x1:2)G(13)U(34:y')G(45)U(56:y)G(67)\Gamma(7x':8)\mathfrak{G}(82)$$
$$+U(x1:2)G(13)U(34;y)G(45)U(56:y')G(67)\Gamma(7x':8)\mathfrak{G}(82)\}.$$

This is graphically shown in Fig. 7.

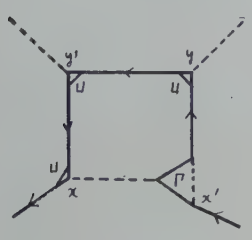

Fig. 7a

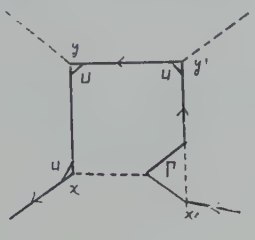

Fig. 7b

Thus our omission of contributions to Γ from the graphs Fig. 4 and 5 does not mean that the effects arising from those types of graphs have been completely excluded from the present theory. In fact we have seen from Fig. 7 that these effects are correctly included in the Green-function $C_{\alpha\beta}$.

The remaining task on which we should investigate is that A) in what a way the definition (1·9) modifies the S-matrix, B) whether (1·9) is sufficient or not to ensure the convergence of all the elements of S-matrix, and finally C) whether the experimental data agree or not with the results obtained by using the present formalism.

The assumption ii) is satisfied if we choose $U(k, l)$ in the following way;

$$U(k, l) = [(1+r_0^2k^2)(1+r_0^2l^2)\{1+r_0^2(k-l)^2\}]^{-1} \quad \left(r_0 \ll \frac{1}{M}\right).$$

In fact in this case both integrals

$$\int_E |U(k, k+q)\overset{\circ}{\mathfrak{G}}(q)|dq \quad \text{and} \quad \int |U(k+q, q)\overset{\circ}{G}_{\alpha\beta}(q)|dq$$

are convergent and bounded for any k in E.

Next let us investigate the assumptions iii) and iv). (2·25) is approximately re-written as

$$4^3g^2W(\tau^3 - W^3) + 4^3g^2W^4 \approx 4^3g^2W^4\left\{1+\frac{3}{3}g^2AW^2\right\} < 1,$$

if g is assumed to be sufficiently small. Therefore g is restricted by

$$g^2 < \frac{1}{4^3W^4}.$$

At the same time g should satisfy

$$g^2 < \frac{1}{2AW^2}$$

in order to realize the assumption iii). Hence we must choose g as follows;

$$0 \leq g^2 < \mathrm{Min}\left\{\frac{1}{4^3 W^4}, \frac{1}{2AW^2}\right\}. \tag{3·1}$$

Thus we arrive at the conclusion that if we choose g in such a way as to satisfy the condition (3·1), then the perturbation method is permissible for obtaining the solution of (1·11)~(1·13).

Appendix

Calculation of N_n

Following the line of reasoning used in the preceding paper,[1] we can easily calculate N_n by regarding Γ, U, G and $\mathfrak{G}$ as c-numbers. By putting

$$\Gamma = Uy(x), \qquad x = -ig^2U^2G^2\mathfrak{G},$$

(1·9) is written down as

$$y = 1 + xy^2.$$

The solution of this equation is

$$y = \frac{1 - \sqrt{1-4x}}{2x}$$

or

$$y = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n \quad \text{for} \quad |4x| < 1,$$

i.e.,

$$\Gamma = \sum_{n=0}^{\infty} \frac{(2n)!(-i)^n}{n!(n+1)!} U^{2n+1} g^{2n} G^{2n} \mathfrak{G}^n.$$

Therefore we get

$$|c_n| \cdot N_n = \frac{(2n)!}{n!(n+1)!}.$$

References

1) R. Utiyama and T. Imamura, Prog. Theor. Phys. 9 (1953), 431.
2) cf. (1).

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Scattering Problem in the Intermediate-coupling Theory, I

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The intermediate coupling theory is generalized so as to include dynamical problems such as meson-nucleon collisions. The underlying physical picture, as well as the results obtained, resembles in many respects the dispersion theoretical treatment of nuclear processes. Relations of the method with other approximation methods are investigated.

§ 1. Introduction

The intermediate coupling theory has been hitherto applied only to static problems such as a calculation of the self-energies of a nucleon and its isobars.[1] The recent development of the experiments using artificial mesons[2] makes it desirable to develop a method applicable to the intermediate coupling region including also dynamical problems such as scattering of a meson by a nucleon. The experiments seem to indicate a broad resonance of the colliding meson with an isobar state which is the characteristic of the intermediate coupling. The extending field of application of the intermediate-coupling theory, e.g., an application to the theory of solid state,[2] may also welcome a generalization in a similar direction.

The purpose of the present paper is to suggest how a method suitable for the treatment of such problems can be developed. The contents of the present paper is only of methodological interest, because the model taken up in this paper is too much simplified to allow any comparison of derived results with actual experimental data. There are different methods of approach, e.g., the Tamm-Dancoff approximation[4] which was successfully applied by Cornell group to the meson-nucleon collison problem.[5] The method developed in the present paper, however, may be of some interest because it provides us with a very clear physical picture of the process which resembles in many respects the familiar dispersion-theoretical treatment of nuclear collision processes, and the formula obtained by our method may be used as a basis of a phenomenological description of more complicated collision processes. The usefulness of such phenomenological description has been in fact shown by several authors.[6]

The essential point of our method consists in the distinction of meson configurations between bound and unbound ones. They may be also called inner and outer configurations. The bound, inner configuration is the configuration which belongs to the self-field of the nucleon; mesons in this configuration are bound to the nucleon and make, so to speak, the "clothes" of the nucleon. The outer, unbound configurations on the other hand, are

configurations in which mesons are unbound and move in the other space without belonging to the "clothes", thus the incident meson is to be considered to belong to one of these configurations. In the static problems hitherto treated the application of the intermediate-coupling theory concerns only with the inner configuration, where various states of the nucleon clothes correspond to various isobar states of the nucleon. In the dynamical problems, however, the unbound configurations and their coupling with the bound configuration play a rôle; the meson-nucleon scattering process, for instance, is pictured as an absorption and re-emission of the incident meson into and out of the bound configuration from and to an unbound configuration.

The approximation in our theory first consists in a limitation of the number of mesons in the outer configurations. We neglect, namely, the possibility of the presence of more than one meson in the outer, unbound configurations. In the inner configuration, however, we take into account the possibility of any number of mesons. Thus the feature of our approximation may be illustrated most clearly by comparing it with the Tamm-Dancoff approximation. Also in the latter, one limits the highest number of mesons, but here the highest number is two,* where one does not distinguish between inner and outer configurations. So we may say that in the Tamm-Dancoff approximation all meson configurations are treated on an equal footing while in ours one lays more weight on the inner than the outer configurations; the inner configuration is treated more accurately, and the outer configurations are treated less accurately than the Tamm-Dancoff approximation.

One defect of our method is that it is not possible to include the effect of the recoil of the nucleon, much less relativistic effects. However we may hope that it will not be impossible to improve the method so as to include, at least partly, these effects, after the suggestions by Matthews and Salam[7]. The method using a canonical transformation first introduced by Jost[8] and then applied to the intermediate coupling theory by Lee et al.[9] will be also useful.

Another defect of our theory is that one must introduce the cut-off procedure in order to avoid the divergence characteristic to the quantized field theory; it is impossible to develop the theory on the basis of the more satisfactory renormalization technics.

The simplified model taken up in this paper is the same one as was discussed in the earlier papers, I and I′. It consists of an infinitely heavy nucleon interacting with the charged longitudinal meson field. This model is taken up for the sake of simplicity. We are quite aware of the necessity of developing the theory on more realistic models; but in doing it a considerable complication will appear. We believe, however, that most parts of the general features of the results will remain valid in such more general considerations.

§ 2. Derivation of an approximate equation

As was given in I or I′ the Hamiltonian of our system is

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* Thus the word "Tamm-Dancoff approximation" is used throughout this paper in the sense somewhat different from its original one.

$$H=\int K[a^*(\mathfrak{k})a(\mathfrak{k})+b^*(\mathfrak{k})b(\mathfrak{k})]\,d\mathfrak{k}$$
$$-\frac{f}{\varkappa}\int\frac{k}{\sqrt{K}}\{[a(\mathfrak{k})+b^*(\mathfrak{k})]\,Q+[a^*(\mathfrak{k})+b(\mathfrak{k})]\,Q^*\}\,d\mathfrak{k}\,, \tag{2·1}$$

where the meaning of notations is given in the cited papers. In these papers we then introduced the Fock representation to solve the eigenvalue problem

$$(H-E)\,\Psi=0, \tag{2·2}$$

but it is more simple to use an alternative method given in II. In this method, which was discussed also by Watson and Hart,[10] we first expand the meson field by a properly chosen complete orthonormal set of functions $\varphi_0(\mathfrak{k})$ and $\varphi_{\mathfrak{s}}(\mathfrak{k})$:

$$\begin{aligned} a(\mathfrak{k})&=A_0\varphi(\mathfrak{k})+\int a_{\mathfrak{s}}\varphi_{\mathfrak{s}}(\mathfrak{k})\,d\mathfrak{s}\\ b(\mathfrak{k})&=B\varphi_0(\mathfrak{k})+\int b_{\mathfrak{s}}\varphi_{\mathfrak{s}}(\mathfrak{k})\,d\mathfrak{s}\,, \end{aligned} \tag{2·3}$$

where the first member of the orthogonal set is

$$\varphi_0(\mathfrak{k})=\frac{1}{\boldsymbol{K}_2}\frac{k}{K\sqrt{K}} \tag{2·4}$$

with

$$\boldsymbol{K}_2^2=\int\frac{k^2}{K^3}d\mathfrak{k}\,. \tag{2·4}$$

Then in terms of A, B, $a_{\mathfrak{s}}$, $b_{\mathfrak{s}}$ and their Hermitic conjugates the Hamiltonian (2·1) is expressed as

$$\begin{aligned} H=&K_{00}\{A^*A+B^*B-V[(A+B^*)Q+(A^*+B)Q^*]\}\\ &+\int K_{0\mathfrak{s}}[(A^*-VQ)a_{\mathfrak{s}}+(B^*-VQ^*)b_{\mathfrak{s}}]\,d\mathfrak{s}\\ &+\int K_{\mathfrak{s}0}[a_{\mathfrak{s}}^*(A-VQ)+b_{\mathfrak{s}}^*(B-VQ)]\,d\mathfrak{s}\\ &+\iint K_{\mathfrak{s}\mathfrak{s}'}(a_{\mathfrak{s}}^*a_{\mathfrak{s}'}+b_{\mathfrak{s}}^*b_{\mathfrak{s}'})\,d\mathfrak{s}\,d\mathfrak{s}' \end{aligned} \tag{2·5}$$

with

$$\begin{aligned} K_{00}&=\int\varphi_0^*(\mathfrak{k})K\varphi_0(\mathfrak{k})\,d\mathfrak{k},\\ K_{0\mathfrak{s}}&=\int\varphi_0^*(k)K\varphi_{\mathfrak{s}}(\mathfrak{k})\,d\mathfrak{k},\\ K_{\mathfrak{s}0}&=\int\varphi_{\mathfrak{s}}^*(\mathfrak{k})K\varphi_0(\mathfrak{k})\,d\mathfrak{k},\\ K_{\mathfrak{s}\mathfrak{s}'}&=\int\varphi_{\mathfrak{s}}^*(\mathfrak{k})K\varphi_{\mathfrak{s}'}(\mathfrak{k})\,d\mathfrak{k} \end{aligned} \tag{2·5′}$$

and

$$V=\frac{f}{\varkappa}\boldsymbol{K}_2 . \tag{2·5''}$$

The intermediate coupling approximation consists in the neglect of the second and the third terms of (2·5), where we already make the error as small as possible by our choice (2·4) of the first member of the orthogonal set.

As was discussed by Watson and Hart, the error is not very large so far as one deals with the states without unbound mesons; one can thus successfully determine energies of the nucleon and its isobars. However, as will be shown later these terms play an essential rôle when the unbound mesons are to be included in the problem. For instance, in the case of scattering of an unbound meson colliding with a nucleon, these terms cause an absorption and re-emission of the incident meson giving rise to a scattering, so it is these terms that are essential in such processes.

The choice of the functions $\varphi_{\mathfrak{s}}(\mathfrak{f})$'s is quite arbitrary; the final results ought to be independent of this choice, and, if one will, one could proceed without using such functions at all. But it will be found in § 4 that a particular choice of $\varphi_{\mathfrak{s}}(\mathfrak{f})$'s has a direct physical meaning, so that it is convenient to use these $\varphi_{\mathfrak{s}}(\mathfrak{f})$'s from the beginning. It will be shown in Appendix I that in this choice $\varphi_{\mathfrak{s}}(\mathfrak{f})$'s make a continuous set, the Mächtigkeit of which is equal to that of the values of $\mathfrak{f}$. Thus the suffix $\mathfrak{s}$ is to be considered as a vector whose domain coincides with the domain of $\mathfrak{f}$. In this choice, moreover, $K_{\mathfrak{s}\mathfrak{s}'}$'s make a diagonal matrix of the form

$$K_{\mathfrak{s}\mathfrak{s}'}=S\delta(\mathfrak{s}-\mathfrak{s}'). \tag{2·6}$$

We shall now introduce the following abbreviations:

$$\begin{aligned} &A^*A+B^*B-V[(A+B^*)Q+(A^*+B)Q^*]=\mathcal{Q}, \\ &A-VQ^*=\boldsymbol{A}, \qquad A^*-VQ=\boldsymbol{A}^*, \\ &B-VQ=\boldsymbol{B}, \qquad B^*-VQ^*=\boldsymbol{B}^*. \end{aligned} \tag{2·7}$$

Then (2·5) becomes

$$\begin{aligned} H=K_{00}\mathcal{Q}&+\int K_{0\mathfrak{s}}(\boldsymbol{A}^*a_{\mathfrak{s}}+\boldsymbol{B}^*b_{\mathfrak{s}})\,d\mathfrak{s}+\int K_{\mathfrak{s}0}(a_{\mathfrak{s}}^*\boldsymbol{A}+b_{\mathfrak{s}}^*\boldsymbol{B})\,d\mathfrak{s} \\ &+\int S(a_{\mathfrak{s}}^*a_{\mathfrak{s}}+b_{\mathfrak{s}}^*b_{\mathfrak{s}})\,d\mathfrak{s}\,. \end{aligned} \tag{2·8}$$

Our problem is now to solve the equation (2·2) with this H.

The variables A, B, $a_{\mathfrak{s}}$, $b_{\mathfrak{s}}$ and their Hermitic conjugates satisfy the commutation relations

$$\begin{aligned} &[A, A^*]=[B, B^*]=1, \\ &[a_{\mathfrak{s}}, a_{\mathfrak{s}'}^*]=[b_{\mathfrak{s}}, b_{\mathfrak{s}'}^*]=\delta(\mathfrak{s}-\mathfrak{s}'), \end{aligned} \tag{2·9}$$

the remaining commutators being zero. If we define n^+, n^-, $n^{+\mathfrak{s}}$ and $n^{-\mathfrak{s}}$ by means of

$$A^*A=n^+, \qquad B^*B=n^-,$$
$$a_{\hat{s}}{}^*a_{\hat{s}}=n^{+\hat{s}}, \qquad b_{\hat{s}}{}^*b_{\hat{s}}=n^{-\hat{s}}, \tag{2·10}$$

it follows from (2·9) that n^-, n^-, $n^{+\hat{s}}d\hat{s}$ and $n^{-\hat{s}}d\hat{s}$ are integers. Their physical meaning is obvious: n^+ and n^- are respectively the numbers of positive and negative mesons in the configuration $\varphi_0(\mathfrak{f})$; $n^{+\hat{s}}$ and $n^{-\hat{s}}$ are the corresponding numbers of mesons in the configuration $\varphi_{\hat{s}}(\mathfrak{f})$.

The function $\varphi_0(\mathfrak{f})$, when we go back to the r-representation from the f-representation, has its value only in the neighbourhood of the origin, so that mesons in this configuration are bound to the nucleon. The mesons in this configuration form the "clothes" of the nucleon. As we shall see in § 4, the functions $\varphi_{\hat{s}}(\mathfrak{f})$, on the other hand, are not bound to the nucleon and have non-vanishing values in the external space: each of them consists of a plane wave and a scattered wave. Thus the mesons in this configuration do not belong to the "clothes" but move in the external space. In this way we make a distinction between bound and unbound mesons as was mentioned in the introduction.

So far we have made no approximations. An essential simplification is now obtained by introducing the following approximation:

Approximation I. We neglect the possibility of the presence of more than one meson in the unbound configurations. (Number of mesons in the bound configuration may be arbitrary).

Then the Schrödinger functional consists of only three parts $\Phi^0(n^+n^-)$, $\Phi^{+\hat{s}}(n^+n^-)$ and $\Phi^{-\hat{s}}(n^+n^-)$:

$$\Phi=\begin{Bmatrix}\Phi^0(n^+n^-)\\ \Phi^{+\hat{s}}(n^+n^-)\\ \Phi^{-\hat{s}}(n^+n^-)\end{Bmatrix} \tag{2·11}$$

where $\Phi^0(n^+n^-)$ is the probability amplitude for that there are n^+ positive and n^- negative mesons in the bound configuration and no mesons at all in the unbound configurations, $\Phi^{+\hat{s}}(n^+n^-)$ is the corresponding probability amplitude when there is one positive meson in the configuration $\varphi_{\hat{s}}(\mathfrak{f})$, and $\Phi^{-\hat{s}}(n^+n^-)$ the probability amplitude when there is one negative meson in $\varphi_{\hat{s}}(\mathfrak{f})$.

By requirement that Φ be a solution of (2·1) with H of (2·8), it follows that Φ^0, $\Phi^{+\hat{s}}$ and $\Phi^{-\hat{s}}$ satisfy

$$\begin{cases}(K_{00}\Omega-E)\Phi^0+\boldsymbol{A}^*\int K_{0\hat{s}}\Phi^{+\hat{s}}+\boldsymbol{B}^*\int K_{0\hat{s}}\Phi^{-\hat{s}}d\hat{s}=0,\\ (K_{00}\Omega+S-E)\Phi^{+\hat{s}}+\boldsymbol{A}K_{\hat{s}0}\Phi^0=0,\\ (K_{00}\Omega+S-E)\Phi^{-\hat{s}}+\boldsymbol{B}K_{\hat{s}0}\Phi^0=0.\end{cases} \tag{2·12}$$

Our neglect introduced above is essential in the intermediate coupling approximation. As we mentioned above we have neglected the second and the third term of (2·5) in the problem hitherto treated. But we can easily infer that this is the same approximation as the neglect of the possible presence of mesons in the unbound configurations. In the scattering problem, however, such a total neglect of outer mesons is not allowed because

of the presence of the colliding (and scattered) meson, but the result will be sufficiently good if we take one step forward and introduce our approximation I. This point will be discussed quantitatively in § 7.

§ 3. Case of $\pi^+ + P \to \pi^+ + P$ reaction

For the sake of definitness we take up a scattering of a positive meson by a proton. This problem is of particular interest because in this case a resonance of the isobar state comes into question.

In this problem it is consistent to neglect the possible presence of negative mesons in the unbound configurations (so long as the coupling is not too strong), because, as we shall see in § 7, the error caused by this neglect is of the same order of magnitude as the error possessed primarily by the intermediate coupling approximation arising from the above mentioned neglect. We thus introduce the following approximation :

Approximation II. We neglect the possibility of the presence of negative mesons in the unbound configurations.

This means that we put $\Phi^{-\hat{s}}=0$ in (2·11). Then we obtain

$$\begin{cases} (K_{00}\Omega - E)\Phi^0 + \boldsymbol{A}^* \int K_{0\hat{s}}\Phi^{\hat{s}} d\hat{s} = 0, \\ (K_{00}\Omega + S - E)\Phi^{\hat{s}} + \boldsymbol{A} K_{\hat{s}0}\Phi^0 = 0. \end{cases} \tag{3·1}$$

In order to solve (3·1), we first solve the eigenvalue problem

$$(\Omega - \Omega_\nu)\chi_\nu = 0 \tag{3·2}$$

where Ω_ν is the ν-th eigenvalue and χ_ν the corresponding eigenfunction. This is the problem treated in earlier papers of the intermediate couping theory ; their objective was to find various states of the nucleon and its " clothes ". The lowest state χ_0 corresponds to the ordinary proton (or neutron) surrounded by bound mesons, and the next higher state χ_1 corresponds to the proton isobar with charge 2 (or neutron isobar with charge -1). Discussions about various nucleon states will be given in § 7.

Having found the eigenvalues Ω_ν and eigenfunctions χ_ν, we can write down the equation (3·1) in the representation in which Ω is diagonal. Writing namely

$$\Phi^0 = \sum_\nu \Phi_\nu{}^0 \chi_\nu , \qquad \Phi^{\hat{s}} = \sum_\nu \Phi_\nu^{\hat{s}} \chi_\nu , \tag{3·3}$$

the equation (3·1) becomes

$$\begin{cases} (K_{00}\Omega_\nu - E)\Phi_\nu{}^0 + \sum_{\nu'} (\nu|\boldsymbol{A}^*|\nu') \int K_{0\hat{s}}\Phi_{\nu'}{}^{\hat{s}} d\hat{s} = 0, \\ (K_{00}\Omega_\nu + S - E)\Phi_\nu^{\hat{s}} + \sum (\nu|\boldsymbol{A}|\nu') K_{\hat{s}0}\Phi_{\nu'}{}^0 = 0. \end{cases} \tag{3·4}$$

$\Phi_\nu{}^0$ and $\Phi_\nu^{\hat{s}}$ occurring in (3·4) have the following physical meaning : $\Phi_\nu{}^0$ is the probability amplitude for that the nucleon is in the state χ_ν while there are no mesons in the outer space ; $\Phi_\nu^{\hat{s}}$ is, on the other hand, the probability amplitude for that the nucleon is in the ν-th state and there is one outer meson in the configuration $\varphi_{\hat{s}}(\mathfrak{k})$. The equation (3·4) shows how these amplitudes couple with each other, and how the state of nucleon

changes when an outer meson is absorbed into or emitted out of the inner configuration: it is apparent in the equation that such emission and absorption are described by the matrix elements $K_{0\mathfrak{k}}$ and $K_{\mathfrak{k}0}$, and the accompanying changes of the nucleon states are described by the matrix elements $(\nu|\boldsymbol{A}|\nu')$ and $(\nu|\boldsymbol{A}^*|\nu')$.

We now proceed on solving (3·4). The first equation of (3·4) gives

$$\Phi_\nu{}^0 = -\sum_{\nu'} \frac{(\nu|\boldsymbol{A}^*|\nu')}{K_{00}\Omega_\nu - E} \int K_{0\mathfrak{k}} \Phi_\nu^{\mathfrak{k}} \, d\mathfrak{k} \,,$$

and the substitution of this into the second gives

$$(K_{00}\Omega_\nu + S - E)\Phi_\nu^{\mathfrak{k}} - \sum_{\nu'} \left[\sum_{\nu''} \frac{(\nu|\boldsymbol{A}|\nu'')(\nu''|\boldsymbol{A}^*|\nu')}{K_{00}\Omega_{\nu''} - E} \right] K_{\mathfrak{k}0} \int K_{0\mathfrak{k}} \Phi_{\nu'}^{\mathfrak{k}} \, d\mathfrak{k} = 0 \,. \qquad (3\cdot5)$$

Now it is convenient to go back to the f-representation from our $\mathfrak{k}$-representation. Putting namely

$$\Phi_\nu(\mathfrak{f}) = \int \Phi_{\nu'}^{\mathfrak{k}} \, \varphi_{\mathfrak{k}}(\mathfrak{f}) \, d\mathfrak{k} \qquad (3\cdot6)$$

we defind that $\Phi_\nu(\mathfrak{f})$ satisfies

$$(K_{00}\Omega_\nu + K - E)\Phi_\nu(\mathfrak{f}) - \varphi_0(\mathfrak{f}) \int \varphi(\mathfrak{f}) K \Phi_\nu(\mathfrak{f}) \, d\mathfrak{f}$$

$$- \sum_{\nu'} \left[\sum_{\nu''} \frac{(\nu|\boldsymbol{A}|\nu'')(\nu''|\boldsymbol{A}^*|\nu')}{K_{00}\Omega_{\nu''} - E} \right] (K - K_{00}) \varphi_0(\mathfrak{f}) \int \varphi_0(\mathfrak{f}) K \Phi_{\nu'}(\mathfrak{f}) \, d\mathfrak{f} = 0 \,. \qquad (3\cdot7)$$

One can easily see that our result (3·7) is independent of the choice of $\varphi_{\mathfrak{k}}(\mathfrak{f})$, and, accordingly the final result does depend only on $\varphi_0(\mathfrak{f})$. The solution of (4·7), when solved under the boundary condition that at infinity there is only an outgoing spherical wave beside an incident plane wave, gives all what we want to know about the scattering.

The equation (3·7) shows how $\Phi_\nu(\mathfrak{f})$'s with different ν couple with each other. It can be shown, however, that, when the coupling constant f (or V) is not too large, such a coupling of different ν can be neglected. In an extreme case, for instance, of vanishing V, this is exactly the case, because in this case non-vanishing matrix elements of $\boldsymbol{A}$ and $\boldsymbol{A}^*$ are only $(\nu-1|\boldsymbol{A}|\nu)$ and $(\nu+1|\boldsymbol{A}^*|\nu)$, so that $[\sum_{\nu''}]$ terms of (3·7) is of the form

$$\left[\sum_{\nu''} \frac{(\nu|\boldsymbol{A}|\nu'')(\nu''|\boldsymbol{A}^*|\nu')}{K_{00}\Omega_{\nu''} - E} \right] = \frac{(\nu|\boldsymbol{A}|\nu+1)(\nu+1|\boldsymbol{A}^*|\nu)}{K_{00}\Omega_{\nu+1} - E} \, \delta_{\nu\nu'} \,. \qquad (3\cdot8)$$

The appearance of $\delta_{\nu\nu'}$ means that different Φ_ν do not couple with each other. In the case of small, but non-vanishing V, (3·8) is valid only approximately, but we shall find in § 7 that this approximation is rather good also in the intermediate coupling region.

The use of the approximate expression (3·8) in (3·7) corresponds to the neglect of other excitation of nucleon levels except $\nu \to \nu+1$ in the course of the collision process. In our problem $\pi^+ + P \to \pi^+ + P$, this means that other nucleon levels except the first isobar state are neglected. Thus we introduce here the following approximation:

Approximation III. Only one nucleon level is excited in the course of the collision process. This approximation may be called a "one-level approximation".

As the result of this approximation, it suffices to solve the simpler equation

$$(K_{00}\Omega_0+K-E)\Phi_0(\mathfrak{k})-\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})K\Phi_0(\mathfrak{k})d\mathfrak{k}$$
$$-\frac{(0|\boldsymbol{A}|1)(1|\boldsymbol{A}^*|0)}{K_{00}\Omega_1-E}(K-K_{00})\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})K\Phi_0(\mathfrak{k})d\mathfrak{k}=0. \tag{3·9}$$

Introducing the abbreviations

$$(0|\boldsymbol{A}|1)=\alpha, \qquad (1|\boldsymbol{A}^*|0)=\alpha^* \tag{3·10}$$

and

$$\begin{aligned} E-K_{00}\Omega_0&=E_0,\\ \Omega_1-\Omega_0&=\Omega_{10}, \end{aligned} \tag{3·10′}$$

(3·9) becomes

$$(K-E_0)\Phi_0(\mathfrak{k})$$
$$=\frac{K_{00}(\Omega_{10}-|\alpha|^2)+(|\alpha|^2K-E_0)}{K_{00}\Omega_{10}-E_0}\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})K\Phi_0(\mathfrak{k})d\mathfrak{k}. \tag{3·11}$$

It is quite easy to solve (3·11). The solution satisfying the required boundary condition is

$$\Phi_0(\mathfrak{k})=\delta(\mathfrak{k}-\mathfrak{p}_0)-\frac{F(E_0)\varphi_0(\mathfrak{p}_0)}{1+F(E_0)\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2d\mathfrak{k}}\delta_+(K-E_0)\varphi_0(\mathfrak{k})$$
$$-\frac{\varphi_0(\mathfrak{p}_0)\varphi_0(\mathfrak{k})}{1+F(E_0)\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2d\mathfrak{k}}, \tag{3·12}$$

with

$$F(E_0)=\frac{1}{|\alpha|^2}[K_{00}(\Omega_{10}-|\alpha|^2)+E_0(|\alpha|^2-1)], \tag{3·12′}$$

$$\delta_+(K-E_0)=\frac{1}{K-E_0}+i\pi\delta(K-E_0) \tag{3·12″}$$

and $\mathfrak{p}_0$, a vector satisfying

$$\sqrt{\mathfrak{p}_0^2+\varkappa^2}=E_0. \tag{3·12‴}$$

As is well known, the first term of the right-hand side of (3·12) represents the incident plane wave, $\mathfrak{p}_0$ being its propagating vector, while the second term represents the scattered spherical wave; the third term vanishes asymptotically so that it has nothing to do with the scattering.

It is instructive to study the solution when the coupling constant V vanishes. In this case no scattering should take place. In our theory, however, this fact does not follow in direct manner, because, also for $V=0$, neither the matrix elements $K_{0\tilde{\mathfrak{s}}}$, $K_{\tilde{\mathfrak{s}}0}$ nor $(\nu|\boldsymbol{A}|\nu')$, $(\nu|\boldsymbol{A}^*|\nu')$ vanish. This fact means that an absorption and re-emission of the incident

meson is still taking place which might give rise to a scattering. We can check, however, that actually there is no scattering at all, the argument being as follows:

When $V=0$, we have

$$\begin{aligned} &\Omega=A^*A+B^*B, \\ &\boldsymbol{A}=A,\ \boldsymbol{B}=B,\ \boldsymbol{A}^*=A^*,\ \boldsymbol{B}^*=B^*. \end{aligned} \tag{3·13}$$

Accordingly, the eigenvalues of Ω are $0, 1, 2, \cdots$ and the matrix elements of $\boldsymbol{A}$ and $\boldsymbol{A}^*$ do not vanish only when ν and ν' differ by one. In particular, for $\nu, \nu'=0$ or 1 we have

$$\begin{cases} a=(0|\boldsymbol{A}|1)=1, \\ a^*=(1|\boldsymbol{A}^*|0)=1, \\ \Omega_{10}=\Omega_1-\Omega_0=1, \end{cases} \tag{3·14}$$

which, substituted in $(3·12')$, give rise to

$$F(E_0)=0 \tag{3·15}$$

and then we get from $(3·12)$

$$\Phi_0(\mathfrak{k})=\delta(\mathfrak{k}-\mathfrak{p}_0)-\varphi_0(\mathfrak{p}_0)\varphi_0(\mathfrak{k}). \tag{3·16}$$

The result $(3·16)$ shows that there is no scattered wave in this case; the non-plane wave part of $(3·16)$ vanishes asymptotically so that it has nothing to do with the scattering. By the way, $(3·11)$ is reduced into

$$(K-E_0)\Phi_0(\mathfrak{k})=\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})K\Phi_0(\mathfrak{k})\,d\mathfrak{k} \tag{3·17}$$

in the case of the vanishing coupling constant.

Before entering into a quantitative discussion about our result $(3·12)$, we now proceed to some discussions which will illustrate a physical aspect of our method.

§ 4. "Potential" scattering

In the preceding paragraph we have seen that, when V vanishes, no scattering takes place in spite of the fact that there is still an absorption and re-emission of the incident meson. In this and the next paragraphs we shall investigate how this comes out and what physical picture can be given to our scattering process.

First we notice that the configuration $\varphi_s(\mathfrak{k})$ itself has a scattered wave as a result of the requirement that it should be orthogonal to $\varphi_0(\mathfrak{k})$. Now the total scattered wave will be a superposition of the scattered wave which belongs to $\varphi_s(\mathfrak{k})$ itself and a scattered wave caused by the absorption and re-emission of the incident meson. Then the fact that no scattering occurs when V vanishes is to be understood as it is due to an interference of these two kinds of scattered waves.

Thus we are led to consider that the scattered wave is generally composed of two parts: one belongs to $\varphi_s(\mathfrak{k})$ itself, and the other is due to the absorption and re-emission of the incident wave. In the case of vanishing V these two waves completely cancel each other. On the other hand when V does not vanish they do not cancel each other but

the remaining scattering results in the non-vanishing total scattering. How this fact mathematically follows will be shown in the next paragraph.

This interpretation of the meson-nucleon scattering resembles in many respects the dispersion-theoretical interpretation of nuclear collision processes. Also in the case of nuclear collision the total scattering was considered to be composed of two parts: the potential and the resonance scatterings. The former is the scattering due to a repulsive potential at the surface of the nucleus, which is to be assumed in order that the incident particle may be in a state not belonging to the compound nucleus. The latter scattering, on the other hand, is due to an absorption and re-emission of the incident particle into and out of the compound nucleus. In our theory, the scattering possessed by $\varphi_{\xi}(\mathfrak{f})$ itself is not a result of a repulsive potential but a result of the orthogonality requirement of $\varphi_{\xi}(\mathfrak{f})$ to $\varphi_0(\mathfrak{f})$. But this requirement is to be satisfied in order that the incident meson may be in a state not belonging to the "clothes" of the nucleon. The other part of the scattering, on the other hand, is caused by the absorption and re-emission of the incident meson into and out of the "clothes". Thus the analogy between our theory and the dispersion theory of nuclear processes is obvious.

By this analogy we shall call the first part of the meson scattering the "potential" scattering, and in this paragraph we will investigate this part of the scattering. Here the quotation marks are necessary in order to emphasize that this is not the result of a repulsive potential in its original sence. The second part of the scattering, on the other hand, will be called the "resonance" scattering.

If we will give such an interpretation to our process, it becomes now necessary to specify our $\varphi_{\xi}(\mathfrak{f})$'s more precisely than the mere requirement of the orthogonality to $\varphi_0(\mathfrak{f})$. This can be done in the following way.

By the reasoning given above, the "potential" scattering would be the total scattering itself, if there were no absorption and re-emission of the incident mesons. This means that the scattered wave possessed by $\varphi_{\xi}(\mathfrak{f})$ itself must be that scattered wave which is possessed by $\Phi_0(\mathfrak{f})$ of (3·12) when a and a^*, the constants decisive for the absorption and re-emission of the incident meson, are put to be zero. Then it follows that $\varphi_{\xi}(\mathfrak{f})$ is a solution of the equation of the form of (3·11) with vanishing a and a^*. In this way it has become clear that $\varphi_{\xi}(\mathfrak{f})$ satisfies

$$(K-S)\varphi_{\xi}(\mathfrak{f})=\varphi_0(\mathfrak{f})\int\varphi_0(\mathfrak{f})K\varphi_{\xi}(\mathfrak{f})d\mathfrak{f} \qquad (4\cdot1)$$

S being a constant. The orthogonality of $\varphi_{\xi}(\mathfrak{f})$ to $\varphi_0(\mathfrak{f})$ and the completeness of the set $\varphi_0(\mathfrak{f})$ and $\varphi_{\xi}(\mathfrak{f})$, are then secured if we add the term $\varphi_0(\mathfrak{f})K\int\varphi_0(\mathfrak{f})\varphi_{\xi}(\mathfrak{f})d\mathfrak{f}$ to the right-hand side of (4·1) and claim that our $\varphi_0(\mathfrak{f})$ and $\varphi_{\xi}(\mathfrak{f})$ be the eigenfunctions of the equation

$$(K-S)\varphi(\mathfrak{f})=\varphi_0(\mathfrak{f})K\int\varphi_0(\mathfrak{f})\varphi(\mathfrak{f})d\mathfrak{f}+\varphi_0(\mathfrak{f})\int\varphi_0(\mathfrak{f})K\varphi(\mathfrak{f})d\mathfrak{f}. \qquad (4\cdot2)$$

It can be easily seen firstly that $\varphi_0(\mathfrak{f})$ itself is an eigenfunction of (4·2) (with the eigenvalue $S=-K_{00}$). Then other eigenfunctions are all orthogonal to $\varphi_0(\mathfrak{f})$ and satisfy (4·1). Moreover, from the orthonormal relation

$$\int \varphi_{\mathfrak{s}}^{*}(\mathfrak{f})\varphi_{\mathfrak{s}'}(\mathfrak{f})d\mathfrak{f}=\delta(\mathfrak{s}-\mathfrak{s}'), \tag{4·3}$$

it follows that

$$\int \varphi_{\mathfrak{s}}^{*}(\mathfrak{f})K\varphi_{\mathfrak{s}'}(\mathfrak{f})d\mathfrak{f}=S\delta(\mathfrak{s}-\mathfrak{s}') \tag{4·4}$$

because, when multiplied by $\varphi_{\mathfrak{s}'}^{*}(\mathfrak{f})$ and integrated over $\mathfrak{f}$, the right-hand side of (4·1) vanishes. Thus our set of functions $\varphi_0(\mathfrak{f})$ and $\varphi_{\mathfrak{s}}(\mathfrak{f})$ has the required property and moreover $\varphi_{\mathfrak{s}}(\mathfrak{f})$ satisfy the relation (1·6).

In (4·3) we have assumed without proof that the eigenfunctions $\varphi_{\mathfrak{s}}(\mathfrak{f})$ belong to a continuous spectrum. This fact, as well as the fact, that the number of the eigenfunctions is of the Mächtigkeit which is equal to that of $\mathfrak{f}$ will be proven in Appendix I.

The eigenfunctions of (4·1) can be found now quite easily. They are

$$\begin{aligned}\varphi_{\mathfrak{s}}(\mathfrak{f}) &=\delta(\mathfrak{f}-\mathfrak{s})+\int\varphi_0(\mathfrak{f})K\varphi_{\mathfrak{s}}(\mathfrak{f})d\mathfrak{f}\cdot\delta_{+}(K-S)\varphi_0(\mathfrak{f})\\ &=\delta(\mathfrak{f}-\mathfrak{s})+K_{0\mathfrak{s}}\delta_{+}(K-S)\varphi_0(\mathfrak{f})\end{aligned} \tag{4·5}$$

where $K_{0\mathfrak{s}}$ can be obtained immediately:

$$K_{0\mathfrak{s}}=-\frac{\varphi_0(\mathfrak{s})}{\int\delta_{+}(K-S)\varphi_0(\mathfrak{f})^2d\mathfrak{f}}, \tag{4·6}$$

so that the required solution is

$$\varphi_{\mathfrak{s}}(\mathfrak{f})=\delta(\mathfrak{f}-\mathfrak{s})-\frac{\varphi_0(\mathfrak{s})}{\int\delta_{+}(K-S)\varphi_0(\mathfrak{f})^2d\mathfrak{f}}\delta_{+}(K-S)\varphi_0(\mathfrak{f}). \tag{4·7}$$

In (4·5), (4·6) and (4·7) $\mathfrak{s}$ is a vector whose length is related to the eigenvalue S by

$$\sqrt{\mathfrak{s}^2+\varkappa^2}=S. \tag{4·8}$$

That (4·7) is correctly normalized can be shown easily.

We see that $\varphi_{\mathfrak{s}}(\mathfrak{f})$ consists actually of a plane wave and a scattered spherical wave, the "potential" scattering in our normenclature, the vector $\mathfrak{s}$ being the propagation vector of the plane wave and S the energy of that wave.

We can further verify that, the scattered wave in (3·12) is actually reduced to the scattered wave in (4·7) when we put $\alpha=\alpha^{*}=0$, that is, to put $F(E_0)=\infty$ in (3·12).

§ 5. "Resonance" scattering

Having found the "potential" scattering, we now investigate the rest of the scattering. First we must prove that the total scattering is actually a superposition of the "potential" scattering and a scattering which may adequately be called the "resonance" scattering. That this is the case will be seen in the following manner.

In the one-level approximation (3·4) is

$$\begin{cases}(K_{00}\Omega_{10}-E_0)\Phi_1^{0}+\alpha^{*}\int K_{0\mathfrak{s}}\Phi_0^{\mathfrak{s}}d\mathfrak{s}=0,\\ (S-E_0)\Phi_0^{\mathfrak{s}}+\alpha K_{\mathfrak{s}0}\Phi_1^{0}=0.\end{cases} \tag{5·1}$$

Then the second equation gives

$$\Phi_0^{\mathfrak{s}} = \delta(\mathfrak{s}-\mathfrak{p}_0) - a\,\Phi_1^0\,\delta_+(S-E_0)K_{\mathfrak{s}0}\,, \tag{5·2}$$

where $\mathfrak{p}_0$ is a vector satisfying

$$\sqrt{\mathfrak{p}_0^2+\varkappa^2} = E_0\,. \tag{5·2'}$$

We now go back to the $\mathfrak{k}$-representation by means of (3·6). Substituting (4·5) and (5·2) into (3·6) we get

$$\begin{aligned}\Phi_0(\mathfrak{k}) &= \int[\delta_+(\mathfrak{k}-\mathfrak{s}) + K_{0\mathfrak{s}}\,\delta_+(K-S)\varphi_0(\mathfrak{k})]\cdot[\delta(\mathfrak{s}-\mathfrak{p}_0) - a\,\Phi_1^0\,\delta_+(S-E_0)K_{\mathfrak{s}0}]\,d\mathfrak{s}\\ &= \delta(\mathfrak{k}-\mathfrak{p}_0) + K_{0\mathfrak{p}_0}\,\delta_+(K-E_0)\varphi_0(\mathfrak{k}) - a\,\Phi_1^0\,\delta_+(K-E_0)K_{\mathfrak{k}0}\\ &\qquad - a\,\Phi_1^0\,\varphi_0(\mathfrak{k})\int K_{0\mathfrak{s}}K_{\mathfrak{s}0}\,\delta_+(K-S)\,\delta_+(S-E_0)\,d\mathfrak{s}\,.\end{aligned}$$

But, as will be shown in the Appendix II, we have

$$\begin{aligned}&\varphi_0(\mathfrak{k})\int K_{0\mathfrak{s}}K_{\mathfrak{s}0}\,\delta_+(K-S)\,\delta_+(S-E_0)\,d\mathfrak{s}\\ &\quad = \varphi_0(\mathfrak{k}) - \delta_+(K-E_0)K_{\mathfrak{k}0} + \frac{K_{0\mathfrak{p}_0}}{\varphi_0(\mathfrak{p}_0)}\,\delta_+(K-E_0)\varphi_0(\mathfrak{k}),\end{aligned} \tag{5·3}$$

so that

$$\begin{aligned}\Phi_0(\mathfrak{k}) &= \delta(\mathfrak{k}-\mathfrak{p}_0) + K_{0\mathfrak{p}_0}\,\delta_+(K-E_0)\varphi_0(\mathfrak{k})\\ &\quad - \frac{a\,\Phi_1^0}{\varphi_0(\mathfrak{p}_0)}\,K_{0\mathfrak{p}_0}\,\delta_+(K-E_0)\varphi_0(\mathfrak{k}) - a\,\Phi_1^0\,\varphi_0(\mathfrak{k})\,.\end{aligned} \tag{5·4}$$

In this result it is quite apparent that the total scattering actually consists of two parts, one the "potential" scattering and the other the "resonance" scattering. Namely, comparing (5·4) with (4·5), we see that the second term of the above expression in that very "potential" scattering; that the third term may adequately be regarded as the "resonance" scattering is argued by the fact that this term is proportional to Φ_1^0, i.e., to the amplitude of the resonating level, and also to a, the constant which specifies the absorption and re-emission of the incident meson into and out of the resonating level. The fourth term has nothing to do with the scattering.

The amplitude of the resonating level can be found immediately. We insert (5·2) into the first equation of (5·1). Then we get

$$\{K_{00}\,\mathcal{Q}_{10} - E_0 - |a|^2\,[\int\delta_+(S-E_0)K_{0\mathfrak{s}}K_{\mathfrak{s}0}\,d\mathfrak{s}]\}\,\Phi_1^0 = -a^*\,K_{0\mathfrak{p}_0}\,. \tag{5·5}$$

Here the integral in the square bracket is evaluated by the method given in the Appendix II:

$$\int\delta_+(S-E_0)K_{0\mathfrak{s}}K_{\mathfrak{s}0}\,d\mathfrak{s} = K_{00} - E_0 + \frac{K_{0\mathfrak{p}_0}}{\varphi_0(\mathfrak{p}_0)} \tag{5·5'}$$

with

$$K_{0\mathfrak{p}_0} = -\frac{\varphi_0(\mathfrak{p}_0)}{\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2\,d\mathfrak{k}}\,. \tag{5·5''}$$

Thus we have

$$\Phi_1^0=\frac{\frac{1}{\alpha}\varphi_0(\mathfrak{p}_0)}{1+F(E_0)\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2 d\mathfrak{k}}. \tag{5·6}$$

It is easily seen that this Φ_1^0, substituted in (5·3), yields in fact (3·12) by the help of (5·5″).

When the coupling constant vanishes, i.e., when we have (3·14) and (3·15), (5·6) is reduced into

$$\Phi_1^0=\varphi_0(\mathfrak{p}_0)$$

which clearly shows how the "potential" scattering and the "resonance" scattering cancel each other: in this case the third term of the right-hand side of (5·3) is exactly the minus of the second term.

§ 6. Effective meson-nucleon interaction

In §§ 4 and 5 we have seen that the total scattering can be described as a superposition of the "potential" and "resonance" scatterings. But this distinction is rather of mathematical nature introduced for the convenience of an approximate calculation relating closely to its zeroth approximation. This means that other possibilities for the description of the scattering process are by no means excluded. In this paragraph we shall discuss a possibility to describe the scattering as a result of some kind of "nuclear force" acting between the nucleon and colliding meson.

Also here we start from the one level equation (5·1). Reminding that, when $\alpha=\alpha^*=1$ and $\Omega_{10}=1$, no scattering takes place, we rewrite (5·1) in the form

$$\begin{cases}(K_{00}-E_0)\,\Phi_1^0+\int K_{0\mathfrak{s}}\,\Phi_0^{\mathfrak{s}}\,d\mathfrak{s}=K_{00}(1-\Omega_{10})\,\Phi_1^0+(1-\alpha^*)\int K_{0\mathfrak{s}}\,\Phi_0^{\mathfrak{s}}\,d\mathfrak{s}\\ (S-E)\,\Phi_0^{\mathfrak{s}}+K_{\mathfrak{s}0}\,\Phi_1^0=(1-\alpha)\,K_{\mathfrak{s}0}\,\Phi_1^0.\end{cases} \tag{6·1}$$

We then go back to the $\mathfrak{k}$-representation by means of

$$\Psi'(\mathfrak{k})=\Phi_1^0\varphi_0(\mathfrak{k})+\int\Phi_0^{\mathfrak{s}}\varphi_{\mathfrak{s}}(\mathfrak{k})\,d\mathfrak{s}. \tag{6·2}$$

$\Psi'(\mathfrak{k})$ defined by (6·2) differs from $\Psi(\mathfrak{k})$ of (3·6) by the term $\Phi_1^0\varphi_0(\mathfrak{k})$. But, since this term vanishes asymptotically and has nothing to do with the scattering, we can equally deal with the scattering by means of this $\Psi'(\mathfrak{k})$.

From (6·1) we obtain an equation which $\Psi'(\mathfrak{k})$ must satisfy. It is

$$\begin{aligned}(K-E_0)\,\Psi'(\mathfrak{k})&=K_{00}(\alpha+\alpha^*-\Omega_{10}-1)\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})\,\Psi'(\mathfrak{k})\,d\mathfrak{k}\\&+(1-\alpha^*)\varphi_0(\mathfrak{k})\int\varphi_0(\mathfrak{k})\,K\Psi'(\mathfrak{k})\,d\mathfrak{k}\\&+(1-\alpha)\varphi_0(\mathfrak{k})\,K\int\varphi_0(\mathfrak{k})\,\Psi'(\mathfrak{k})\,d\mathfrak{k}\,.\end{aligned} \tag{6·3}$$

The equation (6·3) shows that our process can be described by a Hamiltonian whose $(\mathfrak{k}\mathfrak{k}')$-matrix element is given by

$$H_{\mathfrak{k}\mathfrak{k}'}=K\delta(\mathfrak{k}-\mathfrak{k}')-K_{00}(\alpha+\alpha^*-\Omega_{10}-1)\varphi_0(\mathfrak{k})\varphi_0(\mathfrak{k}')$$
$$-(1-\alpha^*)\varphi_0(\mathfrak{k})\varphi_0(\mathfrak{k}')K'-(1-\alpha)\varphi_0(\mathfrak{k})K\varphi_0(\mathfrak{k}'), \qquad (6\cdot4)$$

where the first term $K\delta(\mathfrak{k}-\mathfrak{k}')$ is obviously the kinetic energy of a free meson. The effective Hamiltonian which causes the scattering is, accordingly

$$U_{\mathfrak{k}\mathfrak{k}'}=-K_0(\alpha+\alpha^*-\Omega_{10}-1)\varphi_0(\mathfrak{k})\varphi_0(\mathfrak{k}')$$
$$-(1-\alpha^*)\varphi_0(\mathfrak{k})\varphi_0(\mathfrak{k}')K'-(1-\alpha)\varphi_0(\mathfrak{k})K\varphi_0(\mathfrak{k}'). \qquad (6\cdot5)$$

Our result (6·5) shows that the scattering of the meson can not be ascribed to a force in the ordinary sense acting between meson and nucleon, because if it were the case, the matrix element would be of the form

$$U_{\mathfrak{k}\mathfrak{k}'}=\text{function of } (\mathfrak{k}-\mathfrak{k}').$$

Yet our result shows that there exists an effective interaction Hamiltonian in terms of which the meson-nucleon scattering can be described. In (6·5) it is directly clear that for $\Omega_{1'}=1$ and $\alpha=\alpha^*=1$ this interaction vanishes so that no scattering occurs when V vanishes. It can be also proved that for $\alpha=\alpha^*=0$, the scattering is reduced into the "potential" scattering discussed in § 4.

In dealing with nucleon-nucleon scattering the usual procedure is that one first calculates an interaction energy between the nucleons in some approximation, e.g., the second order nuclear potential. Then one solves the scattering problem considering that it is the result of this interaction. For the meson-nucleon scattering a corresponding method was proposed by Sawada[11]. It is now of interest to compare the result of this section with this procedure. We shall, however, leave this problem to § 8 and shall now go over to a discussion about the level scheme of the nucleon and a related discussion about matrix elements of $\boldsymbol{A}$, $\boldsymbol{A}^*$ and $\boldsymbol{B}$, $\boldsymbol{B}^*$.

§ 7. Discussions about nucleon levels and matrix elements

The energy levels (in the unit of K_{00}) of the nucleon are given by eigenvalues Ω_ν of the equation (3·2). This equation can be solved numerically as we have done in I or I′. In Fig. 1 the results of this calculation are given, where the calculation was made for three values of V^2: $V^2=0$, 1 and 4. In case of $V^2=0$ the levels lie obviously at $0, 1, 2, \cdots$. In Fig. 1 not all levels are given but only those with charge 1 and 2 (or their mirror levels with charge 0 and -1). The lowest level is the proton (or neutron) level and the next higher is the proton isobar with charge 2 (or neutron isobar with charge -1). One sees that already for $V^2=4$ the isobar state lies very closely to the proton (neutron) level as is the characteristic of the strong coupling case. We assign $\nu=0, 1, 2, \cdots$ for these levels in the order of their energy values (and $\nu=0', 1', 2', \cdots$ for their mirror levels), Ze denoting the charge of the level.

The same numerical calculation gives also eigenfunctions χ_ν. Then we can calculate the matrix elements of $\boldsymbol{A}$, $\boldsymbol{A}^*$ and $\boldsymbol{B}$, $\boldsymbol{B}^*$. These matrix elements calculated for $V^2=1$

are given in Tab. 1. Without carrying through the numerical calculation, some qualitative conclusions concerning the matrix elements can be drawn from general considerations. We can immediately show that $(\nu'|\boldsymbol{A}|\nu)$ and $(\nu'|\boldsymbol{B}^*|\nu)$ have non-vanishing values only when the charge of the state ν' is one unit smaller than the charge of the state ν, while $(\nu'|\boldsymbol{A}^*|\nu)$ and $(\nu'|\boldsymbol{B}|\nu)$ have non-vanishing values only in the reverse case. Another conclusion, which is also useful, can be drawn in the following manner.

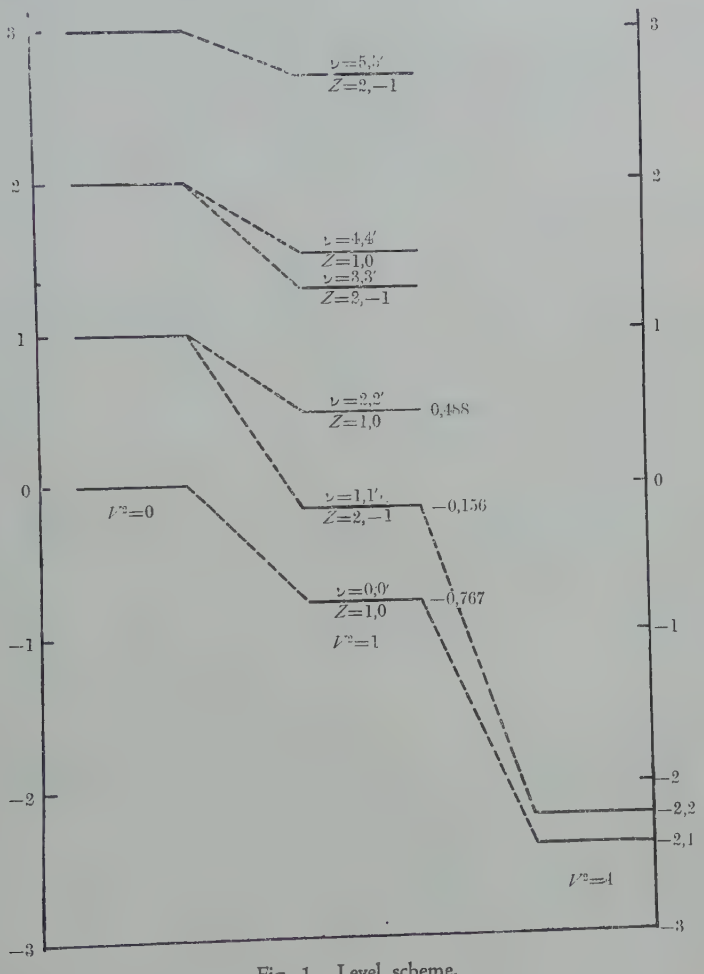

Fig. 1. Level scheme.

By the definition of Ω we have

$$A\Omega-\Omega A=A-VQ^*$$

from which we get

$$(\nu'|A|\nu)=\frac{V}{1+(\Omega_{\nu'}-\Omega_{\nu})}(\nu'|Q^*|\nu)$$

$(1\|\boldsymbol{A}^*\|0)=0.724,$	$(2'\|\boldsymbol{B}^*\|0)=-1.02,$
$(3\|\boldsymbol{A}^*\|0)=-0.293,$	$(2'\|\boldsymbol{A}\|0)=-0.10,$
$(5\|\boldsymbol{A}^*\|0)=-0.298,$	$(1\|\boldsymbol{B}\|0)=-0.17$

Table 1.

and hence

$$(\nu'|\boldsymbol{A}|\nu)=-\frac{V(\Omega_{\nu'}-\Omega_{\nu})}{1+(\Omega_{\nu'}-\Omega_{\nu})}(\nu'|Q^*|\nu). \tag{7·1}$$

The Hermitic conjugate of (7·1) is

$$(\nu'|\boldsymbol{A}^*|\nu)=\frac{V(\Omega_{\nu'}-\Omega_{\nu})}{1-(\Omega_{\nu'}-\Omega_{\nu})}(\nu'|Q|\nu). \tag{7·2}$$

In the similar way we get

$$(\nu'|\boldsymbol{B}|\nu)=-\frac{V(\Omega_{\nu'}-\Omega_{\nu})}{1+(\Omega_{\nu'}-\Omega_{\nu})}(\nu'|Q|\nu), \tag{7·3}$$

and

$$(\nu'|\boldsymbol{B}^*|\nu)=\frac{V(\Omega_{\nu'}-\Omega_{\nu})}{1-(\Omega_{\nu'}-\Omega_{\nu})}(\nu'|Q^*|\nu). \tag{7·4}$$

The relations (7·1)—(7·4) show that for transitions between levels of equal energy the matrix elements vanish. In particular the transition between proton and neutron states is prohibited. The relations (7·1)—(7·4) further imply that $(\nu'|\boldsymbol{A}^*|\nu)$ and $(\nu'|\boldsymbol{B}^*|\nu)$ have larger values, when $(\Omega_{\nu'}-\Omega_{\nu})$ is positive, than when $(\Omega_{\nu'}-\Omega_{\nu})$ is negative, while for $(\nu'|\boldsymbol{A}|\nu)$ and $(\nu'|\boldsymbol{B}|\nu)$ the reverse is the case. In other words, the operator $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ are more effective for transitions from lower to higher levels, while $\boldsymbol{A}$ and $\boldsymbol{B}$ are more effective for reverse transitions. This fact is also apparent in Table 1.

We see from the table that the matrix elements $(1|\boldsymbol{A}^*|0)$ and $(0|\boldsymbol{A}|1)$ are markedly large, comparing them with other matrix elements of $\boldsymbol{A}^*$ and $\boldsymbol{A}$. This fact means that our one-level approximation is a quite sensible one. Moreover, the fact that $(2'|\boldsymbol{A}|0)$ has small values, justifies our neglect of the states in which two unbound positive mesons are present. Because of the smallness of these matrix elements we can neglect in the Hamiltonian those matrix elements which are responsible for the emission of an unbound positive meson without accompanying the absorption of the incident meson. Similarly the smallness of the matrix element $(2|\boldsymbol{B}|0)$ justifies our neglect of unbound negative mesons in $\pi^+ + P \rightarrow \pi^+ + P$ process.

In the table we see that the matrix elements $(2'|\boldsymbol{B}^*|0)$ and $(0|\boldsymbol{B}|2')$ are almost equal to unity. Moreover, Fig. 1 shows that the level distance between the levels $\nu=0$ and $\nu=2'$ is almost unity. From these facts we can infer that a negative meson will be scattered quite little by the proton, because here we have approximately the same situation as the case of $V^2=0$ discussed in § 4. The smallness of the scattering cross-section for $\pi^- + P \rightarrow \pi^- + P$ process will be actually apparent in Fig. 2. of § 9.

§ 8. Comparison with other approximations

In this paragraph we shall investigate the relations between our approximation and various approximations currently applied to the scattering problem.

(i) *Comparison with perturbation theory*

When we expand our result in a power series of f^2 and retain the first term only, we obtain the result which coincide with the result of the perturbation theory. At first sight this seems strange because in our approximation we neglected the possibility of the emission of a meson without accompanying the absorption of the incident meson, whereas the similar process was essential in the perturbation theory. But this paradox is only apparent as can be clarified in the following manner.

The meson emitted without the absorption of the incident meson is, according to the perturbation theory, in the intermediate state whose configuration is given by

$$\varphi(\mathfrak{k}) = \frac{\text{matrix element}}{E_{\text{intermediate}} - E_{\text{initial}}}. \tag{8·1}$$

But if we notice

$$\begin{aligned} &\text{matrix element} = \frac{k}{\sqrt{K}}, \\ &E_{\text{initial}} = E_0, \\ &E_{\text{intermediate}} = E_0 + K, \end{aligned} \tag{8·2}$$

we find

$$\varphi(\mathfrak{k}) = \frac{k}{K\sqrt{K}} \propto \varphi_0(\mathfrak{k}), \tag{8·3}$$

which means that the meson in the intermediate state is just in our inner configuration $\varphi_0(\mathfrak{k})$. Now, since the inner configuration can contain any number of mesons in our intermediate coupling approximation, it does not exclude the above mentioned process in the perturbation theory. The reason why our result coincides in the limit of small f^2 with the result of the perturbation theory lies in this fact.

(ii) *Comparison with Sawada's approximation*

As was mentioned at the end of § 6, Sawada proposed an approximate method of treating the meson-nucleon scattering which corresponds to the usual procedure for the nucleon-nucleon scattering in which the scattering is ascribed to the second order nuclear forces (or more generally, second order effective Hamiltonian).

According to Sawada's method one first eliminates the term linear in the meson field by means of a suitable canonical transformation from the Hamiltonian. Then, in the transformed Hamiltonian there appears a term which is quadratic in the meson field and hence directly causes the meson scattering. Beside this term, which is second order in the coupling constant, the transformed Hamiltonian contains higher terms which could equally contribute to the scattering, but Sawada's approximation consists in the total neglect of such terms.

Applied to our simplified model, this method appears as follows: We transform the Hamiltonian (1·1) into $\bar{H}$ by means of the canonical transformation

$$\bar{H} = e^{G} H e^{-G}. \tag{8·4}$$

with

$$G=\frac{f}{\varkappa}\int\frac{k}{\sqrt{K}}\{[a(\mathfrak{k})-b^*(\mathfrak{k})]\,Q-[a^*(\mathfrak{k})-b(\mathfrak{k})]\,Q^*\}\,d\mathfrak{k}\,. \tag{8·4'}$$

Then the transformed Hamiltonian is a power series of the coupling constant f. If we neglect terms of higher power than the second, we get

$$\begin{aligned}\bar{H}=&\int K[a^*(\mathfrak{k})a(\mathfrak{k})+b^*(\mathfrak{k})b(\mathfrak{k})]\,d\mathfrak{k}\\&-\frac{1}{2}\left(\frac{f}{\varkappa}\right)^2\iint\frac{k}{\sqrt{K}}\frac{k'}{\sqrt{K'}}\left(\frac{1}{K}+\frac{1}{K'}\right)[a^*(\mathfrak{k})a(\mathfrak{k}')-b^*(\mathfrak{k})b(\mathfrak{k}')]\,\tau_3\,d\mathfrak{k}\,d\mathfrak{k}'\\&-\frac{1}{2}\left(\frac{f}{\varkappa}\right)^2\iint\frac{k}{\sqrt{K}}\frac{k'}{\sqrt{K'}}\left(\frac{1}{K}-\frac{1}{K'}\right)[a(\mathfrak{k})b(\mathfrak{k}')-a^*(\mathfrak{k})b^*(\mathfrak{k}')]\,\tau_3\,d\mathfrak{k}\,d\mathfrak{k}'\\&-\left(\frac{f}{\varkappa}\right)^2\int\frac{k^2}{K^2}\,d\mathfrak{k}\,.\end{aligned} \tag{8·5}$$

Sawada then neglects the possibility of the presence of more than one meson so that only the second term on the right-hand side of (8·5) is to be retained. In particular, for our process $\pi^+ + P\to\pi^+ + P$ the term containing b and b^* plays no rôle. So the effective Hamiltonian for our scattering process is

$$U^s=-\frac{1}{2}\left(\frac{f}{\varkappa}\right)^2\iint\frac{k}{\sqrt{K}}\frac{k'}{\sqrt{K'}}\left(\frac{1}{K}+\frac{1}{K'}\right)a^*(\mathfrak{k})a(\mathfrak{k})\,d\mathfrak{k}\,d\mathfrak{k}'. \tag{8·6}$$

Then the matrix element for the $\mathfrak{k}'\to\mathfrak{k}$ transition is in our notations (1·4) and (1·5)

$$U^s_{\mathfrak{k}\mathfrak{k}'}=-\frac{1}{2}V^2\varphi_0(\mathfrak{k})\varphi_0(\mathfrak{k}')K'-\frac{1}{2}V^2\varphi_0(\mathfrak{k})K\varphi_0(\mathfrak{k}'). \tag{8·7}$$

We now compare this matrix element with ours given by (6·5). Then we see that they do not coincide with each other. We must now investigate where this difference comes from.

Since Sawada's effective Hamiltonian is correct only up to the second power of V, it it expected that, if we use the second order approximations for α, α^* and Ω_{10} in (6·5), we shall obtain Sawada's expression (8·7).

In order to verify this, we solve the eigenvalue problem (3·2) by the perturbation theory considering V be a small perturbation. Then we get for the eigenvalues

$$\Omega_0=-V^2+\cdots \tag{8·8}$$

and

$$\Omega_1=1-2V^2+\cdots\,. \tag{8·8'}$$

For the eigenfunction $\chi_0(n^+,n^-)$, on the other hand, we find

$$\begin{Bmatrix}\chi_0(0\,0)\\ \chi_0(1\,0)\\ \cdots\cdots\\ \cdots\cdots\end{Bmatrix}=\begin{Bmatrix}1-\dfrac{V^2}{2}+\cdots\\ V+\cdots\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\end{Bmatrix} \tag{8·9}$$

and for $\chi_1(n^+, n^-)$

$$\begin{Bmatrix} \chi_1(1\,0) \\ \chi_1(2\,0) \\ \cdots\cdots \\ \cdots\cdots \end{Bmatrix} = \begin{Bmatrix} 1-V^2+\cdots \\ \sqrt{2}\,V+\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{Bmatrix}. \tag{8·9′}$$

From these results we get

$$Q_{10}=1-V^2+\cdots \tag{8·10}$$

and

$$a=a^*=1-\frac{V^2}{2}+\cdots. \tag{8·11}$$

We now insert (8·10) and (8·11) into (6·5). Then we find that Sawada's Hamiltonian (8·7) is actually obtained from our (6·5).

From this result we can conclude that our approximation contains Sawada's as a limiting case, the latter being essentially a weak coupling approximation. In the region $V^2=1$ the latter is not a consistent approximation.

(iii) *Comparison with Tamm-Dancoff's approximation*

In solving the eigenvalue problem (1·2) with the Hamiltonian (1·1), we may use the approximation proposed by Tamm and Dancoff. In this approximation we neglect the possibility of the existence of more than two mesons. We now investigate the relation of this approximation with ours.

In our problem $\pi^- + P \to \pi^+ + P$ the Tamm-Dancoff approximation means to put

$$\Phi=\begin{Bmatrix} \phi(\mathfrak{f}) \\ \psi(\mathfrak{f}\mathfrak{f}') \end{Bmatrix}, \tag{8·12}$$

where $\phi(\mathfrak{f})$ is the probability amplitude for that the nucleon is in the proton state and there is positive one meson of the momentum $\mathfrak{f}$, and $\psi(\mathfrak{f}\mathfrak{f}')$ the probability amplitude for that the nucleon is in the neutron state and there are two positive mesons of momenta $\mathfrak{f}$ and $\mathfrak{f}'$. Then according to Fock $\phi(\mathfrak{f})$ and $\psi(\mathfrak{f}\mathfrak{f}')$ satisfy

$$\begin{cases} (K-E)\phi(\mathfrak{f})=\sqrt{2}\,\dfrac{f}{\varkappa}\displaystyle\int\frac{k'}{\sqrt{K'}}\psi(\mathfrak{f}\mathfrak{f}')\,d\mathfrak{f}', \\ (K+K'-E)\psi(\mathfrak{f}\mathfrak{f}')=\dfrac{1}{\sqrt{2}}\dfrac{f}{\varkappa}\left[\dfrac{k}{\sqrt{K}}\phi(\mathfrak{f}')+\dfrac{k'}{\sqrt{K'}}\phi(\mathfrak{f})\right], \end{cases}$$

or in our notations (1·4) and (1·5″)

$$\begin{cases} (K-E)\phi(\mathfrak{f})=\sqrt{2}\,V\displaystyle\int\varphi_0(\mathfrak{f}')K'\psi(\mathfrak{f}\mathfrak{f}')d\mathfrak{f}', \\ (K+K'-E)\psi(\mathfrak{f}\mathfrak{f}')=\dfrac{V}{\sqrt{2}}\left[\varphi_0(\mathfrak{f})K\phi(\mathfrak{f}')+\varphi_0(\mathfrak{f}')K'\phi(\mathfrak{f})\right]. \end{cases} \tag{8·13}$$

In order to find the relation of this equation with equations in our theory, we now expand $\phi(\mathfrak{f})$ and $\psi(\mathfrak{f}\mathfrak{f}')$ in terms of the orthogonal system $\varphi^0(\mathfrak{f})$ and $\varphi_s(\mathfrak{f})$:

$$\begin{cases} \phi(\mathfrak{f}) = b_1^{\,0}\varphi_0(\mathfrak{f}) + \int b_0^{\,\mathfrak{F}}\varphi_{\mathfrak{F}}(\mathfrak{f})\,d\mathfrak{F}, \\ \psi(\mathfrak{f}\mathfrak{f}') = c_2^{\,0}\varphi_0(\mathfrak{f})\varphi_0(\mathfrak{f}') \\ \quad + \int c_1^{\,\mathfrak{F}}\frac{1}{\sqrt{2}}[\varphi(\mathfrak{f}')\varphi_{\mathfrak{F}}(\mathfrak{f}) + \varphi_{\mathfrak{F}}(\mathfrak{f})\varphi_0(\mathfrak{f}')]\,d\mathfrak{F} + \iint c_0^{\,\mathfrak{F}\mathfrak{F}'}\varphi_{\mathfrak{F}}(\mathfrak{f})\varphi_{\mathfrak{F}'}(\mathfrak{f}')\,d\mathfrak{F}\,d\mathfrak{F}'. \end{cases} \tag{8·14}$$

Here b's and c's are expansion coefficients, the lower suffixes being introduced to indicate explicitly the number of mesons in the $\varphi_0(\mathfrak{f})$ configuration.

We now obtain from (8·13) the following equations for b's and c's

$$\begin{cases} (K_{00}-E)b_1^{\,0} - \sqrt{2}\,VK_{00}c_2^{\,0} + \int K_{0\mathfrak{F}}(\sqrt{1}\,b_0^{\,\mathfrak{F}} - Vc_1^{\,\mathfrak{F}})\,d\mathfrak{F} = 0, \\ (2K_{00}-E)c_{\mathfrak{F}}^{\,0} - \sqrt{2}\,VK_{00}b_1^{\,0} + \int K_{0\mathfrak{F}}\sqrt{2}\,c_1^{\,\mathfrak{F}}\,d\mathfrak{F} = 0, \\ (S-E)b_0^{\,\mathfrak{F}} - \sqrt{1}\,VK_{00}c_1^{\,\mathfrak{F}} + K_{\mathfrak{F}0}\sqrt{1}\,b_1^{\,0} = -\sqrt{2}\int K_{0\mathfrak{F}'}(-V)c_0^{\,\mathfrak{F}\mathfrak{F}'}d\mathfrak{F}', \\ (K_{00}+S-E)c_1^{\,\mathfrak{F}} - \sqrt{1}\,VK_{00}b_0^{\,\mathfrak{F}} + K_{\mathfrak{F}0}(\sqrt{2}\,c_2^{\,0} - Vb_1^{\,0}) \\ \qquad\qquad = -\sqrt{2}\int K_{0\mathfrak{F}'}\sqrt{1}\,c_0^{\,\mathfrak{F}\mathfrak{F}'}d\mathfrak{F}', \\ (S+S'-E)c_0^{\,\mathfrak{F}\mathfrak{F}'} = -\frac{1}{\sqrt{2}}[K_{\mathfrak{F}0}(\sqrt{1}-V)c_1^{\,\mathfrak{F}'} + K_{\mathfrak{F}'0}(\sqrt{1}-V)c_1^{\,\mathfrak{F}}]. \end{cases} \tag{8·13}$$

Now if we want to relate our method to Tamm-Dancoff's, we expect that it will be necessary to take one step forward from our approximation I and go over to the following approximation, because meson numbers up to two are taken into account in the Tamm-Dancoff approximation. We thus introduce

Approximation I'. We neglect the possibility of the presence of more than two mesons in the unbound configurations.

This means to put

$$\Omega = \begin{Bmatrix} \Phi^0(n^+n^-) \\ \Phi^{\mathfrak{F}}(n^+n^-) \\ \Phi^{\mathfrak{F}\mathfrak{F}'}(n^+n^-) \end{Bmatrix} \tag{8·16}$$

instead of (1·11). Here $\Phi^{\mathfrak{F}\mathfrak{F}'}(n^+n^-)$ means the probability amplitude for that n^+ positive and n^- negative mesons are present in the configuration $\varphi_0(\mathfrak{f})$ while there are one meson in the $\varphi_{\mathfrak{F}}(\mathfrak{f})$ and another meson in the $\varphi_{\mathfrak{F}'}(\mathfrak{f})$ configuration; the possibility that negative mesons are present in the unbound configurations is neglected because in (8·12) no such possibility is taken into account.

Using the expression (1·8) for the Hamiltonian, we obtain the following equation for Φ^0, $\Phi^{\mathfrak{F}}$ and $\Phi^{\mathfrak{F}\mathfrak{F}'}$:

$$\begin{cases} (K_{00}\Omega - E)\Phi^0 + \int K_{0\mathfrak{F}}\boldsymbol{A}^*\Phi^{\mathfrak{F}}\,d\mathfrak{F} = 0, \\ (K_{00}\Omega + S - E)\Phi^{\mathfrak{F}} + K_{\mathfrak{F}0}\boldsymbol{A}\Phi^0 = -\sqrt{2}\int K_{0\mathfrak{F}'}\boldsymbol{A}^*\Phi^{\mathfrak{F}\mathfrak{F}'}d\mathfrak{F}', \\ (K_{00}\Omega + S + S' - E)\Phi^{\mathfrak{F}\mathfrak{F}'} = -\frac{1}{\sqrt{2}}[K_{\mathfrak{F}0}\boldsymbol{A}\Phi^{\mathfrak{F}'} + K_{\mathfrak{F}'0}\boldsymbol{A}\Phi^{\mathfrak{F}}]. \end{cases} \tag{8·17}$$

In (8·17) Φ^0, Φ^{β} and $\Phi^{\beta\beta'}$ are functions of n^+ and n^-, but, as (8·17) contains no operator which changes the number of negative mesons, we can discard n^- altogether.

Now, in the Tamm-Dancoff approximation the maximum number of mesens is limited to two without making a distinction between bound and unbound configurations. So it is expected that in this approximation (i) the number of mesons in the bound configuration is limited to two when there is no meson in the unbound configurations, (ii) the number of mesons in the bound configuration is limited to one when there is already one meson somewhere in the unbound configurations, and (iii) no meson is considered in the bound configuration when there are already two mesons somewhere in the unbound configurations.

From this fact it follows that (8·17) will be reduced into (8·15) when we neglect there

(i) $\Phi^0(n^+)$ with n^+ larger than 2,
(ii) $\Phi^{\beta}(n^+)$ with n^+ larger than 1, and
(iii) $\Phi^{\beta\beta'}(n^+)$ with n^+ larger than 0.

It is now quite easy to show that this expectation is actually fulfilled: if we put

$$\begin{cases} \Phi^0(1)=b_1^0, \quad \Phi^0(2)=c_2^0, \quad \Phi^0(n^+>2)=0 \\ \Phi^{\beta}(0)=b_0^{\beta}, \quad \Phi^{\beta}(1)=c_1^{\beta}, \quad \Phi^{\beta}(n^+>1)=0 \\ \Phi^{\beta\beta'}(0)=c_0^{\beta\beta'}, \quad \Phi^{\beta\beta'}(n^+>0)=0 \end{cases} \tag{8·19}$$

and insert them into (8·17), we immediately get (8·15).

(iv) *Comparison with strong coupling approximation*

The strong coupling approximation which was first developed by Wentzel, can also be formulated by our method. We can show that, when l is very large, the same result as Wentzel's can be obtained. But in this case some complication arises, because our approximation I does not suffice in this case. However, one can show that if one first carry through a suitable canonical transformation, the resulting theory is as simple as the present one. As the present article has become already too long, we shall treat this problem in a separate paper.

§ 9. Discussions

From our solution (3·12) we can calculate the scattering cross-section as the absolute square of the scattering amplitude. The latter is given from (3·12) by

$$R(\mathfrak{p}_0)=\frac{F(E_0)\varphi_0(\mathfrak{p}_0)^2}{1+F(E_0)\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2 d\mathfrak{k}}. \tag{9·1}$$

If the real part of the denominator of (9·1) vanishes at some value of E_0, this E_0 is a resonance energy. Because $F(E_0)$ is given by (3·12) and $\delta_+(K-E_0)$ by (3·12″), the resonance energy is given as a root of

$$(|a|^2-1)E_0+(\Omega_{10}-|a|^2)K_{00}=-\frac{|a|^2}{\int\frac{\varphi_0(\mathfrak{k})^2}{K-E_0}d\mathfrak{k}}. \tag{9·2}$$

This value of E_0 also makes the amplitude of the resonating level maximum, because Φ_1^0 of (5·6) has the same denominator.

It is to be noticed that this resonance energy does not coincide with the isobar energy $K_{00}\Omega_{10}$. Such a coincidence occurs only when $|a|$ is very small. Thus there is a shift of resonance energy from the isobar energy.

When E_0 satisfies $F(E_0)=0$ or

$$(|a|^2-1)E_0+(\Omega_{10}-|a|^2)K_{00}=0, \tag{9·3}$$

the numerator of (9·1) vanishes. This means that at this energy no scattering occurs. If the root of (9·3) is larger than $\varkappa$, such an effect should be observable, but if this is not the case no such effect can be observed. Further, if there are other collision modes than the elastic, e.g., inelastic or charge exchange collisons, such a point of zero scattering will not exist at all.

For a very low energy of the incident meson, we can expand (9·1) in a power series of $\mathfrak{p}_0$. Taking only the first term we get

$$R(\mathfrak{p}_0)=\frac{F(\varkappa)}{1+F(\varkappa)I(\varkappa)}\frac{1}{\boldsymbol{K}_2^2}\frac{\mathfrak{p}_0^2}{\varkappa^3}, \tag{9·4}$$

$I(\varkappa)$ being the value of $I(E_0)$ at $E_0=\varkappa$ (or $\mathfrak{p}_0=0$), where $I(E_0)$ is defined by

$$\begin{aligned} I(E_0)&=\int\delta_+(K-E_0)\varphi_0(\mathfrak{k})^2d\mathfrak{k}\\ &=\int\frac{\varphi_0(\mathfrak{k})^2}{K-E_0}d\mathfrak{k}-i(2\pi)^2\varphi_0(\mathfrak{p}_0)^2E_0|\mathfrak{p}_0|. \end{aligned} \tag{9·5}$$

In the region of small $\mathfrak{p}_0$ the imaginary part of (9·5) is so small that comparing it with the real part it can be neglected. This means that the damping effect is negligibly small for low energy collisions. Explicitly written down $F(\varkappa)$ in (9·4) is

$$F(\varkappa)=\frac{1}{|a|^2}\left[(|a|^2-1)\varkappa+(\Omega_{10}-|a|^2)K_{00}\right]. \tag{9·6}$$

If we use here for $|a|$ and Ω_{10} the values of (8·9) and (8·11), we get the Sawada approximation :

$$F_s(\varkappa)=-\frac{V^2\left(1-\frac{V^2}{4}\right)}{1-\frac{V^2}{2}}\varkappa-\frac{\frac{V^4}{4}}{\left(1-\frac{V^2}{2}\right)^2}K_{00}. \tag{9·7}$$

Using this $F_s(\varkappa)$ in (9·4) and expanding it in a power of V, we get

$$R(\mathfrak{p}_0)=-V^2\frac{\mathfrak{p}_0^2}{\boldsymbol{K}_2^2\varkappa^2} \tag{9·8}$$

which coincides with the result of the perturbation theory.

In any case, the scattering amplitude is proportional to $\mathfrak{p}_0^2$ for small $\mathfrak{p}_0$. But the proportionality constant depends very much on the method of calculation. Writing

$$|R(\mathfrak{p}_0)| = D\frac{\mathfrak{p}_0^2}{\boldsymbol{K}_2^2 \varkappa^2} \qquad (9\cdot9)$$

with a dimensionless constant D, we give in Tab. 2 the values of D calculated by assuming $V^2=1$. In the table also D's for the process $\pi^- + P \to \pi^- + P$ are given. Moreover, the results depend on the cut-off momentum; the calculation was carried out for two values of the cut-off momentum: $k_{\max}=6.65\,\varkappa$ and $k_{\max}=4\,\varkappa$. The factor 6.65 was taken so that the cut-off corresponds to the rest mass of the nucleon, while the factor 4 was taken quite arbitrarily.

$V^2=1$		$k_{\max}=6.65\,\varkappa$ $K_{00}=4.77\,\varkappa$	$k_{\max}=4\,\varkappa$ $K_{00}=3.11\,\varkappa$
Intermediate coupling approximation	D_+ D_-	0.119 0.706	0.516 0 431
Sawada approximation	D_+ D_-	4.48 0.026	2.126 0.185
Perturbation theory	D_+ D_-	1.000 1.000	1.000 1.000

Table 2.

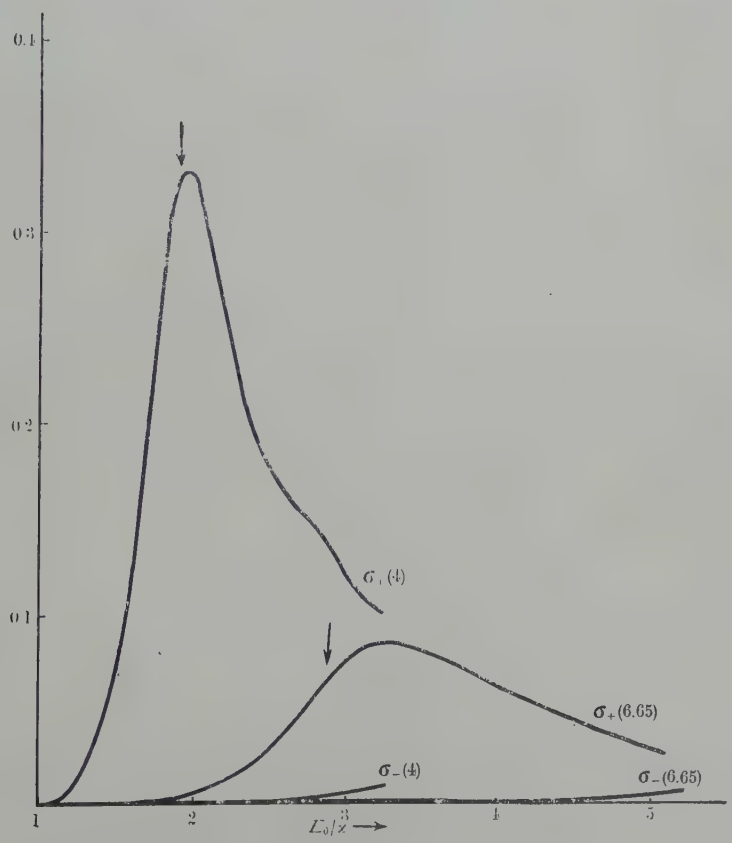

Fig. 2.

In Table 2 it is seen that D_+ for $k_{\max}=6.65\,\varkappa$ is very small in our intermediate coupling approximation. This is due to the fact that in this case the root of (9·3) lies nearly at $\varkappa$ (but a little less than $\varkappa$).

In Fig. 2 curves showing the energy dependence of scattering cross-section are plotted. They correspond to $k_{\max}=6.65\,\varkappa$ and $k_{\max}=4\,\varkappa$ respectively and are calculated also assuming $V^2=1$. In this figure also curves for the process $\pi^-+P\to\pi^-+P$ are added.

One sees that the curve for the cut-off $k_{\max}=4\,\varkappa$ shows a sharp maximum at $E_0=1.95\,\varkappa$. This is due to the resonance with the isobar state, the isobar energy being indicated by an arrow. The curve for the cut-off $k_{\max}=6.65\,\varkappa$ does not have such a sharp maximum. This depends on the fact that in this case (9·2) has no real root: no true resonance occurs in this case. The isobar energy is also here indicated by an arrow.

In the curve for the cut-off $k_{\max}=4\,\varkappa$ there is a small bump in ihe neighborhood of $E_0=2.8\,\varkappa$. This corresponds to the fact that the "potential" scattering $\varphi_{\tilde{s}}(\mathfrak{f})$ has itself a point where a resonance-like phenomenon occurs. This is the point for which

$$\int\frac{\varphi_0(\mathfrak{f})^2}{K-E_0}\,d\mathfrak{f}=0 \tag{9·10}$$

is satisfied. From (4·7) we see that here the real part of the denominator of the scattering amplitude of $\varphi_{\tilde{s}}(\mathfrak{f})$ vanishes.

Moreover we see that the curves for the process $\pi^-+P\to\pi^-+P$ rise very slowly with increasing energy, the cross-sections remaining always very small without showing any resonance. The smallness of the cross-section for this process was already anticipated in § 4. Besides we can show that no resonance at all can be obtained by the Sawada approximation for $V^2=1$ and for the cut-off momenta assumed above.

Appendix I

We shall prove that other eigenfunctions of (4·2) except $\varphi_0(\mathfrak{f})$ belong to a continuous spectrum whose domain is the same as the domain of K. The fact that the number of the eigenfunctions is of the Mächtigkeit which is equal to that of the number of the points in the $\mathfrak{f}$-space can be shown at the same time.

We first consider that the system is put in a large box and then let the box be infinitely large. Put in the box, (4·2) is replaced by

$$(K-S)\varphi(\mathfrak{f})=\varphi_0(\mathfrak{f})\sum_{\mathfrak{f}}\varphi_0(\mathfrak{f})K\varphi(\mathfrak{f}), \tag{A·1}$$

where $\sum_{\mathfrak{f}}$ means the summation over the values of $\mathfrak{f}$ at the lattice points in the $\mathfrak{f}$-space. From (A·1) we get

$$\varphi(\mathfrak{f})=\frac{\varphi_0(\mathfrak{f})}{K-S}\sum_{\mathfrak{f}}\varphi_0(\mathfrak{f})K\varphi(\mathfrak{f}). \tag{A·2}$$

Then multiplying both sides by $\varphi_0(\mathfrak{f})K$ and summing over the lattice points, we get

$$\sum_{\mathfrak{f}}\varphi_0(\mathfrak{f})K\varphi(\mathfrak{f})=\sum_{\mathfrak{f}}\frac{K\varphi_0(\mathfrak{f})^2}{K-S}\cdot\sum\varphi_0(\mathfrak{f})K\varphi(\mathfrak{f}). \tag{A·3}$$

Now this relation must be satisfied with non-vanishing $\varphi(\mathfrak{k})$. In order that this may be possible S must satisfy

$$\sum_{\mathfrak{k}} \frac{K\varphi_0(\mathfrak{k})^2}{K-S} = \sum_{\mathfrak{k}} \varphi_0(\mathfrak{k})^2 + S\sum_{\mathfrak{k}} \frac{\varphi_0(\mathfrak{k})^2}{K-S} = 1$$

or, noticing that $\sum_{\mathfrak{k}} \varphi_0(\mathfrak{k})^2 = 1$,

$$\sum_{\mathfrak{k}} \frac{\varphi_0(\mathfrak{k})^2}{K-S} = 0. \tag{A·4}$$

This is the equation which determines the eigenvalues of (A·1).

Now, considered as a function of S, the left-hand side of (A·4) becomes infinity each time when S approaches the value of K at a lattice point, and changes its sign when S passes this point. From this fact it follows that, if we arrange the values of K at the lattice points according to their magnitude, we shall find that the values of S which satisfy (A·4) lie one by one between two neighbouring values of K. This means that the eigenvalue S has the same spectrum as the spectrum of K when the box becomes infinitely large, and the number of the eigenfunctions has the Mächtigkeit which is equal to the number of points in the $\mathfrak{k}$-space.

Appendix II

Here we shall derive the relations (5·3) and (5·5′). As in (4·8) we have

$$\varphi_{\mathfrak{s}}(\mathfrak{k}) = \delta(\mathfrak{k}-\mathfrak{s}) + \delta_+(K-S)\varphi_0(\mathfrak{k})K_{0\mathfrak{s}}, \tag{A·5}$$

the conjugate complex of which is

$$\varphi_{\mathfrak{s}}{}^*(\mathfrak{k}) = \delta(\mathfrak{k}-\mathfrak{s}) - \delta_+(S-K)\varphi_0(\mathfrak{k})K_{\mathfrak{s}0}. \tag{A·6}$$

Multiplying both sides of (A·6) by $K_{0\mathfrak{s}}$ and integrating over $\mathfrak{s}$, we get

$$\int K_{0\mathfrak{s}}\varphi_{\mathfrak{s}}{}^*(\mathfrak{k})d\mathfrak{s} = K_{0\mathfrak{k}} - \varphi_0(\mathfrak{k})\int \delta_+(S-K)K_{0\mathfrak{s}}K_{\mathfrak{s}0}d\mathfrak{s}. \tag{A·7}$$

But, by help of the closure property of $\varphi_0(\mathfrak{k})$ and $\varphi_{\mathfrak{s}}(\mathfrak{k})$, we can derive from the definition of $K_{0\mathfrak{s}}$

$$\int K_{0\mathfrak{s}}\varphi_{\mathfrak{s}}{}^*(\mathfrak{k})d\mathfrak{s} = (K-K_{00})\varphi_0(\mathfrak{k}).$$

Then we get from (A·7)

$$\int \delta_+(S-K)K_{0\mathfrak{s}}K_{\mathfrak{s}0}d\mathfrak{s} = (K_{00}-K) + \frac{K_{0\mathfrak{k}}}{\varphi_0(\mathfrak{k})}. \tag{A·8}$$

If we put here $\mathfrak{k}=\mathfrak{p}_0$, this gives the required relation (5·5′).

We now proceed to the derivation of (5·3). After putting $\mathfrak{k}=\mathfrak{p}_0$ in (A·6), we multiply both sides by $\delta_+(K-S)K_{0\mathfrak{s}}$ and integrate over $\mathfrak{s}$. Then we get

$$\begin{aligned}&\int \delta_+(K-S)K_{0\mathfrak{s}}\varphi_{\mathfrak{s}}{}^*(\mathfrak{p}_0)d\mathfrak{s}\\ &\quad = \delta_+(K-E_0)K_{0\mathfrak{p}_0} - \varphi_0(\mathfrak{p}_0)\int \delta_+(K-S)\delta_+(S-E_0)K_{0\mathfrak{s}}K_{\mathfrak{s}0}d\mathfrak{s}.\end{aligned} \tag{A·9}$$

We have, on the other hand,

$$\int \varphi_{\mathfrak{F}}(\mathfrak{f})\varphi_{\mathfrak{F}}{}^{*}(\mathfrak{p}_0)d\mathfrak{F}=\varphi_{\mathfrak{f}}{}^{*}(\mathfrak{p}_0)+\varphi_0(\mathfrak{f})\int \delta_{+}(K-S)K_{0\mathfrak{F}}\varphi_{\mathfrak{F}}{}^{*}(\mathfrak{p}^0)d\mathfrak{F} \qquad (A\cdot 10)$$

which is obtained by multiplying (A·5) by $\varphi_{\mathfrak{F}}(\mathfrak{p}_0)$ and integrating over $\mathfrak{F}$. If we use here the closure property of $\varphi_0(\mathfrak{f})$ and $\varphi_{\mathfrak{F}}(\mathfrak{f})$, the left-hand side of (A·10) will be $\delta(\mathfrak{f}-\mathfrak{p}_0)-\varphi_0(\mathfrak{f})\varphi_0(\mathfrak{p}_0)$, while $\varphi_{\mathfrak{f}}{}^{*}(\mathfrak{p}_0)$ on the right-hand side is given by (A·6) by putting there $\mathfrak{f}=\mathfrak{p}_0$ and $\mathfrak{F}=\mathfrak{f}$. We get thus

$$\begin{aligned}-\varphi_0(\mathfrak{f})\varphi_0(\mathfrak{p}_0)=&-\delta_{+}(K-E_0)\varphi_0(\mathfrak{p}_0)K_{\mathfrak{f}0}\\&+\varphi_0(\mathfrak{f})\int \delta_{+}(K-S)K_{0\mathfrak{F}}\varphi_{\mathfrak{F}}{}^{*}(\mathfrak{p}_0)d\mathfrak{F},\end{aligned}$$

which yields

$$\begin{aligned}&\int \delta_{+}(K-S)K_{0\mathfrak{F}}\varphi_{\mathfrak{F}}{}^{*}(\mathfrak{p}_0)d\mathfrak{F}\\&\quad=-\varphi_0(\mathfrak{p}_0)+\delta_{+}(K-E_0)\varphi_0(\mathfrak{p}_0)\frac{K_{\mathfrak{f}0}}{\varphi_0(\mathfrak{f})}.\end{aligned} \qquad (A\cdot 11)$$

We now use this expression for the left-hand side of (A·9), then we get

$$\begin{aligned}&-\varphi_0(\mathfrak{p})+\delta_{+}(K-E_0)\varphi_0(\mathfrak{p}_0)\frac{K_{\mathfrak{f}0}}{\varphi_0(\mathfrak{f})}\\&\quad=\delta_{+}(K-E_0)K_{0\mathfrak{p}_0}-\varphi_0(\mathfrak{f}_0)\int \delta_{+}(K-S)\delta_{+}(S-E_0)K_{0\mathfrak{F}}K_{\mathfrak{F}0}d\mathfrak{F},\end{aligned} \qquad (A\cdot 12)$$

which gives immediately the required relation (5·3).

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On the Quantum Theory of Fields

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Pour obtenir la théorie quantique correspondant à une théorie des champs classique donnée, on propose ici une méthode indépendante du formalisme canonique. Au principe de correspondance entre crochets de Poisson et commutateurs, elle substitue le suivant : les opérateurs de la théorie quantique qui correspondent aux constantes de mouvement de la théorie classique forment, dans l'espace de Hilbert, une représentation des opérateurs infinitesimaux définissant les groupes continus de transformations laissant invariantes les équations de mouvement de la théorie classique. La méthode est appliquée au cas où les équations de mouvement sont supposées invariantes par rapport a une transformation continue de Lorentz et une transformation continue de jauge. Dans ce cas, et si l'on se réduit aux champs d'ondes libres, on démontre l'équivalence de la méthode proposée avec celle qui s'appuie sur le formalisme canonique. La nouvelle méthode présente entre autres les avantages suivants : un traitement unique pour les champs suivant la statistique de Bose et pour ceux suivant celle de Dirac ; une démonstration du fait que la connexion entre le spin d'une particule élémentaire et la statistique qui en régit les ensembles est une consequence de l'invariance de la théorie par rapport à une transformation continue de Lorentz. Enfin, la méthode proposée semble axiomatiquement plus simple, et plus générale d'application que la méthode de quantification qui se fonde sur la formalisme canonique.

§ 1. Introduction

The following work is an attempt to give a quantum theory of fields without any explicit or implicit reference to the canonical (Hamiltonian) formalism. Besides the requirement of invariance under infinitesimal Lorentz transformations and under infinitesimal gauge transformations[1], another requirement is introduced into the theory : at each step the theory must be independent of the particular set of space-like surfaces chosen in space-time for observing the evolution of the physical systems described by the field variables. Although it may not be necessary, such a requirement is reasonable, since it allows the exclusive use of flat three-dimensional surfaces for all practical purposes.

Quantum field theories, for the most part, start from a so-called c-number theory[2] in which the field quantities are taken to be commutable and in which the equations of motion for the fields are deduced from a variation principle under a form which is Lorentz and gauge invariant[1]. A set of space-like surfaces is necessary for defining important quantities characterizing the system as a whole : the constants of motion, such as the total energy and momentum, the total angular momentum and the total charge of the system.

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But these quantities do not depend on the set of space-like surfaces, as a consequence of the equations of motion – it is precisely this property which makes them constants of motion. Thus, the c-number theory satisfies the *surface-independence* requirement (surface-independent will be used from now on in the sense explained above).

It is the "quantization" or transition from the c-number theory to the so-called q-number theory[2] (in which the fields become non-commutative linear operators in the Hilbert space) which is usually surface dependent. For in most cases[3] it is only by introducing surface-dependent quantities, such as generalized canonical variables conjugate to the fields, that quantization is possible[4].

But when the equations of motion for the fields $\chi_\rho(x)$ are of the wave type, i.e. when it follows from them that

$$(\Box - \varkappa^2)\chi_\rho(x) = 0 \tag{1}$$

(one speaks in that case also of wave fields) it is possible to write down the commutation relations in a form which involves only the fields, without any surface dependent term. Therefore, in this case, quantum field theories may be put in a surface independent form.

This fact may lead one to think that there must be a way of quantization which uses only surface independent concepts, at least in the special case mentioned above; this case, although of minor physical interest since interacting fields are left out, is still essential since it already gives the main properties of the field operators corresponding to a given type of field functions.

In the c-number theory, one sees that a one-to-one correspondence exists between the generators (or infinitesimal operators) of the infinitesimal Lorentz and gauge groups on the one hand, and the constants of motion on the other, the constants of motion being well-defined functionals of the fields when the action density or Lagrange function is given. This correspondence is invariant under Lorentz and gauge transformations and is surface independent. Now, in the q-number theory, the constants of motion become well-defined functionals $G_{(i)}[\chi] = \boldsymbol{G}_{(i)}$ of the field operators $\chi_\rho(x)$ up to the ordering of these operators. The operators $\boldsymbol{G}_{(i)}$ are in one-to-one correspondence with the generators $g_{(i)}$

$$\boldsymbol{G}_{(i)} \overset{\rightarrow}{\leftarrow} g_{(i)}. \tag{2}$$

This correspondence, together with the analysis of quantum dynamics[5], suggests the following principle[6] for the quantization of fields: *The commutation rules for the field operators must be such that the correspondence between the generators $g_{(i)}$ and the operators $\boldsymbol{G}_{(i)}$ is an isomorphism, i.e. the $\boldsymbol{G}_{(i)}$'s are a representation of the generators in Hilbert space.* That means that between the field operators $\chi_\rho(x)$ and the $\boldsymbol{G}_{(i)}$'s the following commutation relations hold

$$\frac{1}{\hbar c}\left[G_{(i)}[\chi], \chi_\rho(x)\right]_{x^0 = \text{const}} = g_{(i)}\chi_\rho(x) \tag{3}$$

(the factor $1/\hbar c$ is necessary for dimensional reasons).

Two questions arise immediately. First, are these equations compatible? Secondly,

do they determine the commutation rules between the field operators? Concerning the first question, one has to remember that the generators $g_{(i)}$ satisfy the Lie equations of structure

$$[g_{(i)}, g_{(k)}]_- = c^l_{ik} g_{(l)} \tag{4}$$

where the *constants* c^l_{ik} are the structure constants of the groups[7]. Consequently, if the $\boldsymbol{G}$'s are to give a representation of the generators, one must have also

$$\frac{1}{\hbar c}[\boldsymbol{G}_{(i)}, \boldsymbol{G}_{(k)}]_- = c^l_{ik} \boldsymbol{G}_{(l)}. \tag{5}$$

That is to be interpreted as conditions for the structure of the Lagrange function in c-number theories which can be quantized. It may well be that these consistency conditions are too stringent, i.e. they lead to a class of Lagrange functions which cannot describe the variety of systems occurring in nature, in which case the principle stated above should be generalized. But it will be shown that the consistency conditions (5) are satisfied for the particular class of Lagrangians which correspond to wave fields.

Concerning the second question,—do the general rules (3) determine the commutation rules (in short c.r.) between the fields themselves—it is one of the purposes of this paper to show that the fundamental c.r. for the fields, i.e. the values of

$$[\chi_\rho(x), \chi_\sigma(y)]_\omega \equiv \chi_\rho(x)\chi_\sigma(y) + \omega\chi_\sigma(y)\chi_\rho(x), \qquad \omega = \pm 1 \tag{6}$$

are determined by (3) in the case of wave fields, and that they are identical with the usual c.r. obtained from the canonical formalism[2),3)]. That shows the equivalence of the method of quantization proposed above with the canonical formalism in the particular case of wave fields. Furthermore, in that case also it is possible to deduce from (3) the rule for ordering the field operators in the functionals $G_{(i)}[\chi]$ and, at the same time, the connexion between "the spin and the statistics" of the field $\chi_\rho(x)$, i.e. the relation between the value of ω in (6) and the representation of the Lorentz group according to which the fields $\chi_\rho(x)$ transform[8].

One sees that the method of quantization proposed here satisfies the requirement of invariance under Lorentz and gauge transformations and, furthermore, that it is surface independent. For, if one works in the Heisenberg picture[9], by definition the state vector $\Psi(\sum')$ describing the system is surface independent, and then the invariance of the theory with respect to infinitesimal and gauge transformations requires the operators $\boldsymbol{G}_{(i)}$ to be also constants of motion in the q-number theory, i.e. they do not depend on the set of surfaces. Consequently, the general c.r. (3) are surface independent, and so is also the method of quantization.

This method is general and is not restricted to the case of wave fields, although it is so far only in these cases that consistency is proved and that explicit c.r. between the fields operators can be given. But that is also the case with the canonical quantization as well, and the situation is due to the fact that the c-number theory for wave fields is very simple: the *linear* equations of motion can be solved exactly and it is possible to build for them a complete set of orthogonal solutions.

The systematic use of the infinitesimal gauge group in addition to the Lorentz group does not lead to new results for wave fields when the gauge is independent of space-time co-ordinates. But the fact that the generator q corresponding to the total charge Q is the simplest of the generators $g_{(i)}$ also makes it simplest to choose

$$\frac{1}{\hbar c}[\boldsymbol{Q}, \boldsymbol{\chi}_p(x)]_- = q\,\boldsymbol{\chi}_p(x) \tag{7}$$

as the basic c.r., among the relations (3). When there is an external electromagnetic field[10], however, the total charge is in general the only constant of motion; thus only (7) and not the other c.r. (3) *must* be taken if one wishes to have *a priori* surface-independent c.r. for the fields. This case of an external field must be treated with care, since, except $\boldsymbol{Q}$, the $\boldsymbol{G}_{(i)}$ are in general no longer constants of motion and therefore cannot form a representation of the generators of the Lorentz group.

The main features of the method can be stated briefly as follows: (a) It is explicitly surface-independent at each step, in contrast to the canonical formalism; (b) It is based on the principle that in the q-number theory the constants of motion are a representation in Hilbert space of the infinitesimal operators of the groups of transformations under which the theory is to be invariant; (c) It gives the connexion between spin and statistics as a consequence of the invariance under continuous Lorentz transformations; (d) It is equivalent to the canonical formalism in the case of wave fields.

§ 2. The infinitesimal groups of transformations: The Lorentz group and the gauge (or Weyl) group[11]

Throughout the work, the following notations are used: $\hbar$ is Planck's constant divided by 2π, c is the velocity of light, e the magnitude of the elementary charge, and m the mass of the particle associated with the fields; moreover, the following abbreviations are used

$$\varkappa = \frac{mc}{\hbar}, \qquad \varepsilon = \frac{e}{\hbar c}.$$

If $x^\mu = (x^1, x^2, x^3, x^4 = ct)$ are the co-ordinates of an event in space-time, an inhomogeneous Lorentz transformation is a linear transformation

$$x^{*\mu} = b^\mu{}_\nu x^\nu + a^\mu \tag{8}$$

of the co-ordinates x^μ into new co-ordinates $x^{*\mu}$, which leave invariant the metric tensor

$$g^{11} = g^{22} = g^{33} = -g^{44} = -1, \quad g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu. \tag{9}$$

In the following the properties of the infinitesimal inhomogeneous transformation

$$x^{*\mu} = x^\mu + \delta a^\mu + \delta b^{\mu\nu} x_\nu, \qquad \delta b^{\mu\nu} = -\delta b^{\nu\mu} \tag{10}$$

which leaves $g^{\mu\nu}$ invariant will only be used. (The δa's and δb's are ten independent infinitesimal constants; $x_\nu = g_{\nu\mu} x^\mu$, $x^\mu = g^{\mu\nu} x_\nu$ and the usual summation convention is employed.)

As a result of the infinitesimal Lorentz transformation (10), a general field function $\chi_\rho(x)$ is transformed into a new function $\chi_\rho^*(x^*)$ according to

$$\chi_\rho^*(x^*) = \mathfrak{L}\chi_\rho(x) \tag{11}$$

where $\mathfrak{L}$ is the linear operator

$$\mathfrak{L} = 1 + \delta a^\mu \frac{d}{dx_\mu} - \frac{1}{2}\delta b^{\mu\nu}\left(x_\mu \frac{d}{dx^\nu} - x_\nu \frac{d}{dx^\mu} + r_{\mu\nu}\right) \tag{12}$$

which may be written in the form of a unitary operator

$$\mathfrak{L} = 1 - i\delta a^\mu p_\mu + i\frac{1}{2}\delta b^{\mu\nu} m_{\mu\nu}. \tag{13}$$

In this expression, the Hermitian operators

$$p_\mu = i\frac{d}{dx^\mu} = id_\mu, \qquad m_{\mu\nu} = i(x_\mu d_\nu - x_\nu d_\mu + r_{\mu\nu}) = l_{\mu\nu} + s_{\mu\nu} = -m_{\nu\mu}, \tag{14}$$

with $s_{\mu\nu} = ir_{\mu\nu}$, represent the infinitesimal operators or the generators of the infinitesimal group $\mathcal{A}$ of Lorentz transformations. They satisfy the Lie equations of structure

$$\begin{aligned} &i[p_\mu, p_\nu]_- = 0, \\ &i[p_\mu, m_{\nu\rho}]_- = -g_{\mu\nu}p_\rho + g_{\mu\rho}p_\nu, \\ &i[m_{\mu\nu}, m_{\rho\sigma}]_- = g_{\mu\rho}m_{\nu\sigma} - g_{\nu\rho}m_{\mu\sigma} - g_{\mu\sigma}m_{\nu\rho} + g_{\nu\sigma}m_{\mu\rho}, \end{aligned} \tag{15}$$

or

$$[g_{(i)}, g_{(k)}]_- = c_{ik}^l g_{(l)}, \qquad i, k, l = 1 \cdots 10, \tag{16}$$

with $g_{(1)} \cdots g_{(4)}, g_{(5)} \cdots g_{(10)} = p_1 \cdots p_4, m_{12} \cdots m_{34}$. The c's are the structure constants of the infinitesimal Lorentz group.

In equations (12) and (14), the operator $r_{\mu\nu}$ describes the change of a constant field (i.e. independent of x) under the infinitesimal transformation (10). Its representation depends on the transformation properties of the field $\chi_\rho(x)$: e.g., for a scalar $\varphi(x)$, a vector $B^\mu(x)$, and a spinor $\psi_a(x)$, one has the following representations

$$\begin{aligned} &r_{\mu\nu}\varphi(x) = 0, \\ &r_{\mu\nu}B^\lambda(x) = r_{\mu\nu}{}^\lambda{}_\rho B^\rho(x), \qquad r_{\mu\nu}{}^\lambda{}_\rho = g_\mu{}^\lambda g_{\nu\rho} - g_\nu{}^\lambda g_{\mu\rho}, \\ &r_{\mu\nu}\psi_a(x) = r_{\mu\nu a b}\psi_b(x), \qquad r_{\mu\nu a b} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]_{-ab}, \end{aligned} \tag{17}$$

and in general

$$r_{\mu\nu}\chi_\rho(x) = r_{\mu\nu\rho\sigma}\chi_\sigma(x).$$

$\chi_\rho^+(x)$ denotes the Hermitian conjugate function of $\chi(x)$, i.e. the result of changing i into $-i$ and taking the transpose of $\chi_\rho(x)$. This means that if $\chi_\rho(x)$ is taken as a one-column matrix, then $\chi_\rho^+(x)$ is a one-row matrix. As a consequence, expressions like $\chi_\rho^+(x)\chi_\rho(x)$ are real numbers. As an illustration of the connexion between the χ's and the χ^+'s, one may define $r_{\mu\nu\rho\sigma}^+$ by

$$(r_{\mu\nu}\chi_\rho)^+ = \chi_\sigma^+ r^+_{\mu\nu\sigma\rho}. \tag{17a}$$

In (17) the four Dirac matrices $\gamma^\mu = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$ are given by

$$\gamma^i = \beta\alpha^i, \qquad i=1, 2, 3, \qquad \gamma^4 = \beta, \tag{18}$$

with[12]

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} (1) & 0 \\ 0 & (1) \end{pmatrix},$$

and satisfy the identities

$$[\gamma^\mu, \gamma^\nu]_+ = 2\delta^{\mu\nu}, \qquad \gamma^{i\,+} = -\gamma^i, \qquad \gamma^{4\,+} = \gamma^4. \tag{19}$$

A Lorentz invariant quantity I, i.e. a quantity which is left unchanged by a continuous inhomogeneous Lorentz transformation, satisfies the equations

$$p_\mu I = 0, \qquad m_{\mu\nu} I = 0,$$

expressing the invariance under the four translations and the six " rotations " in space-time. Furthermore, scalar fields of a general type can be constructed with the field functions. They are quantities invariant under the operation $r_{\mu\nu}$, like

$$r_{\mu\nu}\varphi(x) = 0, \qquad r_{\mu\nu}(A^\lambda(x) A_\lambda(x)) = 0, \qquad r_{\mu\nu}(\bar{\psi}_a(x)\psi_a(x)) = 0, \tag{21}$$

where

$$\bar{\psi}_a(x) = \psi_b^+(x)\beta_{ba} \tag{22}$$

is the spinor function adjoint to $\psi_a(x)$.

Turning now one's attention to gauge transformations, one considers a " matter field " $\chi_\rho(x)$ (by this it is meant essentially that a " zero mass " field, e.g. the electromagnetic field, is excluded). In order to emphasize its gauge dependence, it is written for the time being as $\chi_\rho(x, C)$. C is a real constant (independent of x). A gauge transformation is defined by

$$C^* = C + f. \tag{23}$$

As for Lorentz transformations, only the infinitesimal transformation

$$C^* = C + \delta f \tag{24}$$

is used here, δf being an infinitesimal constant.

The dependence of the matter fields with respect to the gauge is defined by the transformation

$$\chi_\rho{}^*(x, C^*) = \mathfrak{W}\chi_\rho(x, C) = (1 - i\varepsilon\,\delta f)\chi_\rho(x, C) \tag{25}$$

which corresponds to (24). One defines also

$$\chi_\rho^{+*}(x, C^*) = \mathfrak{W}\chi_\rho^+(x, C) = \chi_\rho^+(x, C)(1 + i\varepsilon\,\delta f). \tag{26}$$

Then $\chi_\rho^+(x, C)\chi_\rho(x, C)$ is invariant under a gauge transformation, since

$$\mathfrak{W}(\chi_\rho^+(x, C)\chi_\rho(x, C)) = \chi_\rho^+(x, C)\chi_\rho(x, C). \tag{27}$$

Now, as in the case of Lorentz transformations, it is convenient to write (25) as

$$\mathfrak{W}\chi_\rho(x, C)=\left(1+\delta f\frac{d}{dC}\right)\chi_\rho(x, C), \tag{28}$$

thereby introducing the linear operator $\mathfrak{W}=1+\delta f\frac{d}{dC}$ which can be written in the form of a unitary operator

$$\mathfrak{W}=1-i\delta f q \tag{29}$$

with the Hermitian operator

$$q=i\frac{d}{dC}. \tag{30}$$

q is the generator of the infinitesimal group Γ of gauge transformations. A gauge invariant quantity $\mathcal{J}$ satisfies the equation

$$q\mathcal{J}=0. \tag{31}$$

Comparison of (25) and (29) shows that

$$q\chi_\rho(x, C)=\varepsilon\chi_\rho(x, C), \qquad q\chi_\rho^+(x, C)=-\varepsilon\chi_\rho^+(x, C). \tag{32}$$

According to Pauli, quantities like $\chi_\rho(x, C)$ and $\chi_\rho^+(x, C)$ are called gauge-dependent of the first kind[13].

If the gauge C is independent of x, it is consistent to write

$$i[q, p_\mu]_-=0, \qquad i[q, m_{\mu\nu}]_-=0, \tag{33}$$

and one can consider the direct product $\Gamma'\times\Lambda$ of the infinitesimal Lorentz group Λ and of the infinitesimal gauge Γ' group. The Lie equations of structure of this product group $\Gamma'\times\Lambda$ are given by the union of (16) and (32), which can be written as

$$[g_{(i)}, g_{(k)}]_-=c_{ik}^l g_{(l)}, \qquad i, k, l,=0, 1\cdots 10, \tag{34}$$

with $g_{(0)}=q$ and $c_{0k}^l=0$.

However, the introduction of gauge which depends on x leads to

$$p_\mu\chi_\rho(x, C(x))=id_\mu\chi_\rho(x, C(x))=\left(i\partial_\mu+d_\mu C i\frac{d}{dC}\right)\chi_\rho(x, C(x)), \tag{35}$$

where $\partial_\mu\chi_\rho(x, C(x))$ denotes the derivative of $\chi_\rho(x, C(x))$ with respect to x^μ when $C(x)$ is kept constant. Consequently the values of the commutators $i[q, p_\mu]_-$, which can be written also as

$$-\lim_{\delta f(x)\to 0}\frac{d_\mu\delta f(x)}{\partial f(x)}q,$$

are no longer defined. All the same, by introducing an external field $A_\mu(x, C(x))$, it is possible to define the operators

$$p_\mu^\wedge=p_\mu+A_\mu(x, C(x))q=id_\mu+A_\mu(x, C(x))q \tag{36}$$

which commute with q:

$$i[q, p_\mu^A]_- = 0, \tag{37}$$

provided the external vector field $A_\mu(x, C(x))$ is a gauge dependent quantity of the kind[13], i.e. satisfying

$$\frac{d}{dC}\left(A_\mu(x, C(x)) + d_\mu C(x)\right) = 0$$

or

$$A_\mu(x, C^*(x)) = A_\mu(x, C(x)) - d_\mu[C^*(x) - C(x)]. \tag{38}$$

The electromagnetic 4-vector potential is an example of such a field.

This final remark is made mainly to emphasize the particular situation of the electromagnetic field and so to justify the restriction imposed on the fields $\chi_\rho(x)$.

From now on, and if not stated otherwise, $\chi_\rho(x)$ stands for a gauge-dependent field of the first kind $\chi_\rho(x, C)$ where C is independent of x.

§ 3. The variation principle and the constants of motion in the c-number theory

A. General Theory

The equations of motion for the field functions $\chi_\rho(x)$ can be derived from a variation principle[14]. For this purpose one sets up first a functional of the field functions $L = L[\chi_{\rho'}(x)]$, the action density or Lagrangian, a scalar having the dimension of an energy density, and invariant under gauge transformations. That means that

$$qL = 0, \qquad r_{\mu\nu} L = 0. \tag{39}$$

In the following, $\chi_{\rho'}(x)$ stands for the two sets of independent functions $\chi_\rho(x)$, $\chi_\rho'(x)$, and $\chi_{\rho'} F_{\rho'}[\chi]$ for $\chi_\rho^+ F_\rho[\chi^+] + \chi_\rho F_\rho[\chi]$, the summation convention being understood.

One then chooses two *arbitrary* space-like surfaces, $\sum_0$ and $\sum$, where $\sum_0$ is earlier than $\sum$. One considers also the surface $\sum_y$ of a cylinder y with time-like axis and a space-like radius large enough so that surface integrals like

$$\int_{\Sigma_y} d\sigma(x)\, d_\mu^{(n)} \chi_\rho^+(x)\, d_\nu^{(m)} \chi_\rho(x) \equiv \int_{\Sigma_y} d\sigma(x)\, f[\chi_{\rho'}(x)], \qquad m, n, = 0, 1, 2\cdots$$

vanish. This means that the system described by the fields is "closed". Then one can use the equality[15]

$$\int_{\Sigma} d\sigma_\mu(x)\, d_\nu f[\chi_{\rho'}(x)] = \int_{\Sigma} d\sigma_\nu(x)\, d_\mu f[\chi_{\rho'}(x)], \tag{40}$$

where $\int_\Sigma$ means that the integral is taken over the portion of $\sum$ inside the cylinder y. In (40), $d\sigma_\mu(x)$ denotes $n_\mu(x) d\sigma(x)$ where $n_\mu(x)$ is a unit time-like vector normal to $\sum$ and with positive n_4 and $d\sigma(x)$ is the differential surface element on $\sum$.

Finally, one forms the four-dimensional action integral

$$I=I[\chi_{\rho'}(x);\Sigma_0,\Sigma]=\int_{\Sigma_0}^{\Sigma}(dx)^4 L[\chi_{\rho'}(x)] \tag{41}$$

extended over the four-dimensional volume V bounded by Σ_0, Σ and Σ_y. I has the dimension of $\hbar c$, i.e. of energy times length. The equations of motion for the fields

$$\frac{\partial L}{\partial\chi_{\rho'}(x)}-d_\mu\frac{\partial L}{\partial(\partial_\mu\chi_{\rho'}(x))}\equiv E[\chi_{\rho'}(x)]=0 \tag{42}$$

follow from the variation principle

$$\delta I=0, \tag{43}$$

where δ means that the *fields* are given an arbitrary variation $\delta\chi_{\rho'}(x)$ at each point inside V but are kept constant on the boundary surface of V. In (42) it is assumed that only the field functions and their first derivatives appear in L. This assumption, although not essential here, is made from now on for the sake of brevity.

Assuming that the fields satisfy the equations of motion (42), one can obtain the constants of motion in the following way[16]. Considering the action integral I as a functional of the coordinates and of the gauge, one performs an infinitesimal Lorentz transformation and an infinitesimal gauge transformation (i.e. a transformation of the eleven parameter infinitesimal group $\Gamma\times\Lambda$) and obtains the change dI undergone by I. Using the equations of motion, one gets

$$dI=\frac{\delta I}{dC}df+\frac{\delta I}{\delta\lambda^\mu}da^\mu-\frac{1}{2}\frac{\delta I}{\delta(\partial^\mu x^\nu)}db^{\mu\nu}, \tag{44}$$

with

$$\frac{\delta I}{\delta C}=Q(\Sigma)-Q(\Sigma_0),\quad \frac{\delta I}{\delta\lambda^\mu}=P_\mu(\Sigma)-P_\mu(\Sigma_0),\quad \frac{\delta I}{\delta(\partial^\mu x^\nu)}=M_{\mu\nu}(\Sigma)-M_{\mu\nu}(\Sigma_0),$$

where the quantities $Q(\Sigma)$, $P_\mu(\Sigma)$, and $M_{\mu\nu}(\Sigma)$ are given by

$$Q(\Sigma)=\int_\Sigma d\sigma_\lambda(x)\,\mathcal{J}^\lambda[\chi_{\rho'}(x)],\qquad \mathcal{J}^\lambda[\chi]=-\Pi_{\rho'}^\lambda[\chi]\frac{d}{dC}\chi_{\rho'},$$

$$P_\mu(\Sigma)=\int_\Sigma d\sigma_\lambda(x)\,T_\mu^\lambda[\chi_{\rho'}(x)],\qquad T_\mu^\lambda[\chi]=g_\mu^\lambda L[\chi]-\Pi_{\rho'}^\lambda[\chi]\,d_\mu\chi_{\rho'}, \tag{45}$$

$$M_{\mu\nu}(\Sigma)=\int_\Sigma d\sigma_\lambda(x)\,Z_{\mu\nu}^\lambda[\chi_{\rho'}(x)],\qquad Z_{\mu\nu}^\lambda[\chi]=x_\mu T_\nu^\lambda[\chi]-x_\nu T_\mu^\lambda[\chi]-\Pi_{\rho'}^\lambda[\chi]\,r_{\mu\nu}\chi_{\rho'},$$

with

$$\Pi_{\rho'}^\lambda[\chi]\equiv\frac{\partial L}{\partial(\partial_\lambda\chi_{\rho'}(x))}. \tag{46}$$

It is an immediate consequence of the invariance properties of L and of the four-dimensional volume element $(dx)^4$ under Lorentz and gauge transformation that I itself is gauge and Lorentz invariant. Consequently

$$dI=0. \tag{47}$$

hence

$$Q(\Sigma')=Q(\Sigma_0), \qquad P_\mu(\Sigma')=P_\mu(\Sigma_0), \qquad M_{\mu\nu}(\Sigma')=M_{\mu\nu}(\Sigma_0), \tag{48}$$

which shows that the total charge of the system $Q(\Sigma')$, the total energy-momentum vector $P_\mu(\Sigma')$ and the total angular momentum tensor $M_{\mu\nu}(\Sigma')$ of the system do not depend on the particular space-like surface Σ' on which one observes them. Therefore they are constants of motion. There is an obvious one-to-one correspondence

$$q \rightleftarrows Q, \qquad p_\mu \rightleftarrows P_\mu, \qquad m_{\mu\nu} \rightleftarrows M_{\mu\nu} \tag{49}$$

between the generators of the infinitesimal transformation in $I'\times A$ and the corresponding constants of motion, which can be written as

$$g_{(i)} \rightleftarrows G_{(i)}=\int_\Sigma d\sigma_\lambda(x)\,\mathcal{G}_{(i)}^{\lambda}[\chi_{\rho'}(x)], \qquad i=0, 1\cdots 10, \tag{50}$$

with $G_{(0)}, G_{(1)}\cdots G_{(4)}, G_{(5)}\cdots G_{(10)}=Q, P_1\cdots P_4, M_{12}\cdots M_{34}$.

When the fields functions and their derivatives satisfy certain differentiality conditions, which are assumed here, it follows from (42) and (47) that the density tensors $\mathcal{G}_{(i)}^{\lambda}$, i.e. $\mathcal{J}^\lambda$, T_μ^λ, and $Z_{\mu\nu}^\lambda$ satisfy the conservation equations

$$\partial_\lambda \mathcal{G}_{(i)}^{\lambda}=0, \quad \text{i.e.} \quad \partial_\lambda \mathcal{J}^\lambda=0, \qquad \partial_\lambda T_\mu^\lambda=0, \qquad \partial_\lambda Z_{\mu\nu}^\lambda=0 \tag{51}$$

in the differential form.

B. The particular case of bilinear Lagrangians

Here the Lagrangian is assumed to be bilinear in the fields and their (first) derivatives, so that it can be written as

$$L[\chi_{\rho'}(x)]=\chi_\rho^+\{\partial_\lambda^L R\partial^\lambda+\partial_\lambda^L K^\lambda+N_\lambda\partial^\lambda+U\}\chi_\rho, \tag{52}$$

where ∂_λ^L stands for the differentiation operator ∂_λ acting only on the functions χ_ρ^+ to its left. If there are no external fields, R, K^λ, N_λ and U are constant (independent of x). The equations of motions are

$$\begin{aligned}\chi_\rho^+\{(\partial_\lambda^L R+N_\lambda)\partial^{L\lambda}-\partial_\lambda^L K^\lambda-U\}&=0,\\ \{\partial_\lambda(R\partial^\lambda+K^\lambda)-N_\lambda\partial^\lambda-U\}\chi_\rho&=0.\end{aligned} \tag{53}$$

Introducing the operators $V^{L\lambda}$ and V^λ by

$$\Pi_\rho^\lambda[\chi]=\chi_\rho^+\{\partial^{L\lambda}R+N^\lambda\}=\chi_\rho^+V^{L\lambda}, \qquad \Pi_\rho^\lambda[\chi^+]=\{R\partial^\lambda+K^\lambda\}\chi=V^\lambda\chi_\rho, \tag{54}$$

one has for the three fundamental density tensors

$$\begin{aligned}\mathcal{J}^\lambda[\chi]&=i\varepsilon\chi_\rho^+\{V^{L\lambda}-V^\lambda\}\chi_\rho,\\ T_\mu^\lambda[\chi]&=g_\mu^\lambda L[\chi]-\chi_\rho^+\{\partial_\mu^L V^\lambda+V^{L\lambda}\partial_\mu\}\chi_\rho,\\ Z_{\mu\nu}^\lambda[\chi]&=x_\mu T_\nu^\lambda[\chi]-x_\nu T_\mu^\lambda[\chi]-\chi_\rho^+\{r_{\mu\nu}^L V^\lambda+V^{L\lambda}r_{\mu\nu}\}\chi_\rho.\end{aligned} \tag{55}$$

Introducing further the important operator

$$C^\lambda = \frac{i}{\hbar c}(V'^\lambda - V^\lambda), \tag{56}$$

the current density can be written as

$$\mathcal{J}^\lambda[\chi] = e j^\lambda[\chi] = e\chi_\rho^+ C^\lambda \chi_\rho . \tag{57}$$

The density vector

$$j^\lambda[\chi] = \chi_\rho^+ C^\lambda \chi_\rho \tag{58}$$

has the dimension of the inverse of a volume, so that the integral $\int_\Sigma d\sigma_\lambda(x) j^\lambda(x)$ is dimensionless.

After some calculations involving the use of the relation (40), of the equations of motion (53) and of the invariance properties of (52), the constants of motion are found to be, in case of no external field

$$\begin{aligned}
Q &= \hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) C^\lambda(x) q \chi_\rho(x) = -\hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) q^I C^\lambda(x) \chi_\rho(x), \\
P_\mu &= \hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) C^\lambda(x) p_\mu \chi_\rho(x) = -\hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) p_\mu^I C^\lambda(x) \chi_\rho(x), \\
M_{\mu\nu} &= \hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) C^\lambda(x) m_{\mu\nu} \chi_\rho(x) = -\hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+ m_{\mu\nu}^I C^\lambda(x) \chi_\rho(x),
\end{aligned} \tag{59}$$

i.e.

$$G_{(s)} = \hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) C^\lambda(x) g_{(s)} \chi_\rho(x) = -\hbar c \int_\Sigma d\sigma_\lambda(x) \cdot \chi_\rho^+(x) g_{(s)}^I C^\lambda(x) \chi_\rho(x),$$

with

$$p_\mu^I = i d_\mu^I , \qquad m_{\mu\nu}^I = i(d_\nu^I x_\mu - d_\mu^I x_\nu) + s_{\mu\nu}^I ,$$

and where in general the operator g^I is defined by

$$\cdots f(x) g^I \cdots \equiv \cdots (g f(x)) \cdots . \tag{60}$$

In expressions like (52), it is understood that $\chi_\rho A \chi_\rho = \mathcal{A}$ stands for $\chi_\rho^+ A_{\rho\sigma} \chi_\sigma$, where $A_{\rho\sigma}$ is a matrix operator giving to $\mathcal{A}$ its correct transformation properties with respect to Lorentz and gauge transformations. For instance, one has

$$\chi_\rho^+ V^\lambda \chi_\rho \equiv \chi_\rho^+ V_{\rho\sigma}^\lambda \chi_\sigma , \qquad V_{\rho\sigma}^\lambda = \tau'_{0\,\rho\sigma}{}^\lambda + \tau'_{1\,\rho\sigma}{}^{\lambda\tau} \partial_\tau , \tag{61}$$

the τ's being invariant matrices if there is no external field.

The wave fields belong to the class described by bilinear Lagrangians. As an illustration, and in order to prepare for later considerations, the cases of spin 0, 1/2, and 1 are briefly sketched here[17]. (Use is made here of the restriction to matter fields ($\varkappa \neq 0$), so that by proper factorization of $\varkappa$ in L, the matter field functions $\varphi(x)$, $\psi'_\alpha(x)$, $B^\alpha(x)$ are dimensionless.)

Spin zero: scalar field $\varphi(x)$. The Lagrangian is

$$L[\varphi(x)] = \hbar c \varkappa^2 \varphi^+ \{ i\partial_\lambda^I \cdot i\partial^\lambda + \varkappa^2 \} \varphi ;$$

Hence

$$V^{L\lambda}=-\hbar c\varkappa^2\partial^{L\lambda},\qquad V^{\lambda}=-\hbar c\varkappa^2\partial^{\lambda},\qquad C^{\lambda}=-i\varkappa^2(\partial^{L\lambda}-\partial^{\lambda}). \tag{62}$$

The equations of motion are

$$\varphi^+(\Box^L-\varkappa^2)=0,\qquad (\Box-\varkappa^2)\varphi=0,\qquad \Box=i\partial^{\lambda}i\partial_{\lambda}.$$

One verifies easily that T_{44} is positive definite, but j_4 is not.

Spin 1/2: spinor field $\psi_a(x)$. Here the Lagrangian may be taken as

$$L[\psi_a(x)]=\hbar c\varkappa^3\psi^+\beta\left\{\frac{1}{2}(i\partial^L\gamma-\gamma i\partial)+\varkappa\right\}\psi,\qquad a\gamma=a^{\mu}\gamma_{\mu}.$$

Hence

$$V^{L\lambda}=-\frac{i}{2}\hbar c\varkappa^3\beta\gamma^{\lambda},\qquad V^{\lambda}=\frac{i}{2}\hbar c\varkappa^3\beta\gamma^{\lambda},\qquad C^{\lambda}=\varkappa^3\beta\gamma^{\lambda}. \tag{63}$$

The equations of motion are

$$\psi^+\beta(i\partial^L\gamma+\varkappa)=0,\qquad (\gamma i\partial-\varkappa)\psi=0.$$

Here j_4 is positive definite and T_{44} is not; furthermore, it follows from the equations of motion that

$$\psi^+(\Box-\varkappa^2)=0,\qquad (\Box-\varkappa^2)\psi=0. \tag{63b}$$

Spin 1: vector field $B^{\alpha}(x)$. One can choose for the Lagrangian

$$L[B^{\lambda}(x)]=\hbar c\varkappa^2\left\{\frac{1}{2}F^{+\lambda\tau}F_{\lambda\tau}+\varkappa^2B^{+\lambda}B_{\lambda}\right\},\qquad F^{\lambda\tau}=\partial^{\lambda}B^{\tau}-\partial^{\tau}B^{\lambda},$$

or, in a form closer to (52)

$$L[B^{\lambda}(x)]=\hbar c\varkappa^2B^{+\sigma}\{i\partial^{L\lambda}r^{\tau\rho}_{\lambda\sigma}i\partial_{\tau}+\varkappa^2\cdot g^{\rho}_{\sigma}\}B_{\rho}. \tag{64}$$

In this case

$$V^{L\lambda\rho}_{\sigma}=-\hbar c\varkappa^2\partial^{L\tau}r^{\lambda\rho}_{\tau\sigma},\qquad V^{\lambda\rho}_{\sigma}=-\hbar c\varkappa^2r^{\lambda\tau\rho}_{\sigma}\partial_{\tau},\qquad C^{\lambda\rho}_{\sigma}=-i\varkappa^2(\partial^{L\tau}r^{\lambda\rho}_{\tau\sigma}-r^{\lambda\tau\rho}_{\sigma}\partial_{\tau}),$$

and equations of motion are

$$B^{+\sigma}(\partial^{L\lambda}\partial^{L}_{\tau}r^{\tau\rho}_{\lambda\sigma}+\varkappa^2g^{\rho}_{\sigma})=0,\qquad (r^{\tau\rho}_{\lambda\sigma}\partial^{\lambda}\partial_{\tau}+\varkappa^2g^{\rho}_{\sigma})B_{\rho}=0,$$

or

$$B^{+\sigma}(\Box^L-\varkappa^2)=0,\quad B^{+\sigma}\partial^L_{\sigma}=0,\qquad (\Box-\varkappa^2)B_{\rho}=0,\quad \partial^{\rho}B_{\rho}=0.$$

Here, as in the case of spin 0, T_{44} is positive definite and j_4 is not.

For later use, an important property of the linear systems (i.e. described by a bilinear Lagrangian) is proved. If one defines $\delta/\delta\sigma(x)$ by[18] ($f(x)$ is assumed to be differentiable)

$$\frac{\delta}{\delta\sigma(x)}\int_{\Sigma}d\sigma_{\lambda}(y)f(y)=\lim_{V\to 0}\frac{1}{V(x)}\left\{\int_{\Sigma'}d\sigma_{\lambda}(y)f(y)-\int_{\Sigma}d\sigma_{\lambda}(y)f(y)\right\}=\partial_{\lambda}f(x), \tag{65}$$

(where $\sum$ and $\sum'$ coincide except near the point x and are the boundary of the volume

V) any of the constants of motion $G_{(i)}$ satisfies the equation

$$\frac{\delta}{\delta\sigma(x)} G_{(i)} = \frac{\delta}{\delta\sigma(x)} \int_\Sigma d\sigma_\lambda(y) \cdot \chi_\rho^+(y) I'^\lambda_{(i)}(y) \chi_\rho(y) = 0 \tag{66}$$

for $\delta/\delta\sigma(x) \cdot G_{(i)} = 0$ expresses precisely the surface-independence of $G_{(i)}$ (see 51). In particular

$$\frac{\delta}{\delta\sigma(x)} \begin{pmatrix} Q \\ e \end{pmatrix} = \frac{\delta}{\delta\sigma(x)} \int_\Sigma d\sigma_\lambda(y) \cdot \chi_\rho^+(y) C^\lambda(y) \chi_\rho(y) = 0. \tag{66a}$$

If now χ_ρ'' and χ_ρ' are two different solutions of the *linear* equations of motion (53), then $\chi_\rho = \alpha\chi_\rho'' + \beta\chi_\rho'$ is also a solution for arbitrary α, β. Substituting in (66), one gets the important relation

$$\frac{\delta}{\delta\sigma(x)} \int_\Sigma d\sigma_\lambda(y) \cdot \chi_\rho^{+\prime\prime}(y) I'^\lambda_{(i)}(y) \chi_\rho'(y) = 0. \tag{67}$$

In particular

$$\frac{\delta}{\delta\sigma(x)} \int_\Sigma d\sigma_\lambda(y) \cdot \chi_\rho^{+\prime\prime}(y) C^\lambda(y) \chi_\rho'(y) = 0. \tag{67a}$$

§ 4. Covariant generalizations of Dirac's function solutions of the wave equation

A solution of the wave equation can be developed as a superposition[19] of plane waves of the two types

$$\varphi(x/\boldsymbol{k}, +) = \frac{1}{\sqrt{2}} (2\pi)^{-3/2} e^{-i(kx)}, \qquad \varphi(x/\boldsymbol{k}, -) = \frac{1}{\sqrt{2}} (2\pi)^{-3/2} e^{+i(kx)}, \tag{68}$$

which can be referred to as "positive" and "negative" frequency waves. In (68) the customary notation

$$k = (k^1, k^2, k^3, k^4) = (\boldsymbol{k}, k^4), \qquad k^4 = + \left(\sum_1^3 k_i^2 + \varkappa^2\right)^{1/2} \tag{69}$$

is used. At a given time t_0, each set of solution, the $\varphi(x/\boldsymbol{k}, +)$'s and the $\varphi(x/\boldsymbol{k}, -)$'s satisfy the *orthogonality relation*

$$2\int_{-\infty}^{+\infty} (dy)^3 \varphi^+(\boldsymbol{k}'', \tau/\boldsymbol{y}, t_0) \varphi(\boldsymbol{y}, t_0/\boldsymbol{k}', \tau) = \delta(\boldsymbol{k}'' - \boldsymbol{k}'),$$
$$\tau = \pm 1, \qquad \varphi^+(\boldsymbol{k}, \tau/y) = \varphi(y/\boldsymbol{k}, \tau)]^+, \tag{70}$$

and the *completeness relation*

$$2\int_{-\infty}^{+\infty} (dk)^3 \varphi(x'', t_0/\boldsymbol{k}, \tau) \varphi^+(\boldsymbol{k}, \tau/x', t_0) = \delta(x'' - x'), \tag{71}$$

where $\delta(\boldsymbol{a}) = \delta(a^1)\delta(a^2)\delta(a^3)$ is the three-dimensional generalization of Dirac's δ-function (henceforth called distribution[20]) which satisfies the identity

$$\chi_\rho(x, t_0) = \int_{-\infty}^{+\infty} (dy)^3 \delta(x-y)\chi_\rho(y, t_0). \tag{72}$$

In a covariant theory, the relations (70), (71), (72) are not suitable since they are not explicitly covariant. The purpose of this section is to give a covariant form to these relations. This form depends on the transformation properties of the fields.

In order to give a covariant form to the orthogonality relation and to the completeness relation, it is necessary first to define covariant generalizations of the volume elements $(dy)^3$ and $(dk)^3$. They are $d\sigma_\lambda(y)$ and $d\Gamma(k) = (dk)^3/\varkappa^2 k^4$, the integration in the k-space being over the hypersurface (69) (denoted H_+) and the scalar $d\Gamma(k)$ being invariant and dimensionless. In space-time, the domain which generalizes the plane $t = t_0$ is one of the space-like surfaces Σ. But at this stage, the *surface-independence* requirement of the theory must be remembered. If one writes the left-hand side of the orthogonality relation for two field functions $\chi_\rho''(x)$ and $\chi_\rho'(x)$

$$\int_\Sigma d\sigma_\lambda(y)\chi_\rho^{+\prime\prime}(y)\cdots\chi_\rho'(y),$$

it must be required also that (see 65)

$$\frac{\delta}{\delta\sigma(x)}\int_\Sigma d\sigma_\lambda(y)\chi_\rho^{+\prime\prime}(y)\cdots\chi_\rho'(y) = 0.$$

The remark at the end of the preceding section shows that the simplest expression to satisfy this condition is

$$\int_\Sigma d\sigma_\lambda(y)\cdot\chi_\rho^{+\prime\prime}(y)C^\lambda(y)\chi_\rho'(y)$$

provided the functions $\chi_\rho''(x)$ and $\chi_\rho'(x)$ are solutions of the equations of motion (53), of which the wave equation is in general only a consequence (see 62, 63, 64). This shows that the orthogonality relation and the completeness relation are, in their covariant form, intimately linked to the equations of motion and therefore will depend on the transformation properties of the field $\chi_\rho(x)$.

Consequently, given a wave field $\chi_\rho(x)$ satisfying the equation of motion (53), derived from the Lagrangian (52), one looks for two sets of solutions

$$\chi_\rho(x/k', n', \tau'), \chi_\rho(x/k'', n'', \tau''), \cdots \qquad \tau = \pm 1 \tag{73}$$

corresponding, in the case of a scalar field $\varphi(x)$, to the positive and negative frequency plane waves (68), and satisfying the relations[21]

$$\int_\Sigma d\sigma_\lambda(y)\cdot\chi_\rho^+(k'', n'', \tau''/y)C^\lambda(y)\chi_\rho(y/k', n', \tau') = \Delta(k'', n'', \tau''/k', n', \tau') \tag{74}$$

and

$$\sum_{\tau=+1,-1}\nu(\tau)\sum_n\int_{H_+} dV(k)\chi_\rho(x''/k, n, \tau)\chi_\sigma^+(k, n, \tau/x') = F_{\rho\sigma}(x'', x'), \tag{75}$$

which will be referred to still as orthogonality relation and completeness relation. In (73),

n denotes the independent polarizations of plane waves having the same k, τ. The construction of these solutions $\chi_\rho(x/\boldsymbol{k}, n, \tau)$ in each particular case (spin 0, 1/2, 1,$\cdots$) is straightforward, since the solutions for spin 0, namely the plane waves (68), are known. And once the $\chi_\rho(x/\boldsymbol{k}, n, \tau)$'s are known, it is easy to evaluate the right-hand side of (74) and (75), which are defined up to a normalization factor. Here, the normalization factor is such that

$$\Delta(\boldsymbol{k}'', n'', \tau''/\boldsymbol{k}', n', \tau') = \nu(\tau'')\,\delta(\boldsymbol{k}'', n'', \tau''/\boldsymbol{k}', n', \tau'), \tag{74a}$$

where $\delta(\boldsymbol{k}'', n'', \tau''/\boldsymbol{k}', n', \tau')$ is defined by

$$\sum_{n,\tau}\int_{H_+} dV(\boldsymbol{k})\,\delta(\boldsymbol{k}', n', \tau'/\boldsymbol{k}, n, \tau)\,f(\boldsymbol{k}, n, \tau) = f(\boldsymbol{k}', n', \tau'). \tag{74b}$$

In (74a) and (75) $\nu(\tau)$ is defined as follows: for spin 0 and 1, $\nu(\tau)=\tau$; for spin 1/2, $\nu(\tau)=+1$. This spin dependence of $\nu(\tau)$ is directly connected with the definite or indefinite character of j_4 (see 58). When there are no external fields, the distribution $F_{\rho\sigma}(x'', x')$ is invariant under a translation and can be written as

$$F_{\rho\sigma}(x'', x') = F_{\rho\sigma}(x''-x') = \sum_{\tau=+1,-1} \tau F^{(\tau)}_{\rho\sigma}(x''-x'), \tag{75b}$$

where $F^{(+)}_{\rho\sigma}(x''-x')$ contains only positive frequency waves, and $F^{(-)}_{\rho\sigma}(x''-x')$ only negative frequency waves.

The covariant generalization of (72) is found simply by developing a general solution $\chi_\rho(x)$ of the equations of motion (53) in terms of the plane waves (73)

$$\begin{aligned}
\chi_\rho(x) &= \sum_{n,\tau}\nu(\tau)\int_{H_+} dV(\boldsymbol{k})\,\chi_\rho(x/\boldsymbol{k}, n, \tau)\,a(\boldsymbol{k}, n, \tau),\\
\chi^+_\rho(x) &= \sum_{n,\tau}\nu(\tau)\int_{H_+} dV(\boldsymbol{k})\,a^+(\boldsymbol{k}, n, \tau)\,\chi^+_\rho(\boldsymbol{k}, n, \tau/x),
\end{aligned} \tag{76}$$

and by computing the "Fourier coefficients" $a(\boldsymbol{k}, n, \tau)$ and $a^+(\boldsymbol{k}, n, \tau)$ by means of the orthogonality relations. One then finds

$$\begin{aligned}
a(\boldsymbol{k}, n, \tau) &= \int_\Sigma d\sigma_\lambda(y)\cdot\chi^+_\sigma(\boldsymbol{k}, n, \tau/y)\,C^\lambda(y)\,\chi_\sigma(y),\\
a^+(\boldsymbol{k}, n, \tau) &= \int_\Sigma d\sigma_\lambda(y)\cdot\chi^+_\sigma(y)\,C^\lambda(y)\,\chi_\sigma(y/\boldsymbol{k}, n, \tau).
\end{aligned} \tag{77}$$

Now one replaces in the developments (76) the coefficients $a(\boldsymbol{k}, n, \tau)$, $a^+(\boldsymbol{k}, n, \tau)$ by their values in (77). One gets, using the completeness relation

$$\begin{aligned}
\chi_\rho(x) &= \int_\Sigma d\sigma_\lambda(y)\cdot F_{\rho\sigma}(x-y)\,C^\lambda(y)\,\chi_\sigma(y),\\
\chi^+_\rho(x) &= \int_\Sigma d\sigma_\lambda(y)\cdot\chi^+_\sigma(y)\,C^\lambda(y)\,F_{\sigma\rho}(y-x),
\end{aligned} \tag{78}$$

which is the desired generalization of (72)[22]. It is important to notice here that (71)

is not only a generalization of (72) in the sense of covariance, but also in the fact that whereas in (72) $\chi_\rho(x, t_0)$ and $\chi_\rho(y, t_0)$ on both sides are evaluated at the same time t_0 (on the same space-like surface), in (78) the two points x and y are not necessarily on the same space-like surface, but may define a time-like vector as well. Consequently, (78) gives the value $\chi_\rho(x)$ of the field function in terms of its value on an arbitrary space-like surface Σ, which may be earlier or later than x. When multiplied by $C^\lambda(x)$, the distribution $F_{\rho\sigma}(x)$ may be considered as a covariant generalization of Dirac's distribution $\delta(x)$. It is different from zero only on and inside the light cone; it satisfies the equations of motion, and the following identities

$$F^+_{\rho\sigma}(x)=F_{\rho\sigma}(-x), \quad qF_{\rho\sigma}(x)=0, \quad m_{\mu\nu}F_{\rho\sigma}(x)=0, \quad p_\mu F_{\rho\sigma}(x-y)=0, \qquad (79)$$

so that $F_{\rho\sigma}(x-y)$ is *invariant* in the group $(\Gamma\times\Lambda)$ in case of no external fields.

The relations (74), (75) and (78) in the particular cases of spin 0, 1/2, and 1 are given below.

Spin 0

$$\int_\Sigma d\sigma_\lambda(y)\cdot\varphi^+(\boldsymbol{k}''\tau''/y)\,C^\lambda(y)\varphi(y/\boldsymbol{k}'\tau')=\tau''\,\delta(\boldsymbol{k}''\tau''/\boldsymbol{k}'\tau'),$$

$$\sum_\tau \tau\int_{H_+} dV(\boldsymbol{k})\varphi(x''/\boldsymbol{k},\tau)\varphi^+(\boldsymbol{k},\tau/x')=F(x''-x')=\sum\tau F^{(\tau)}(x''-x'),$$

$$\varphi(x)=\int_\Sigma d\sigma_\lambda(y)\cdot F(x-y)C^\lambda(y)\varphi(y),$$

with[23] (80)

$$2\varkappa^2 F^{(\tau)}(x)=D^\tau(x)=(2\pi)^{-3}\int_{H_+}\frac{(dk)^3}{k^4}e^{-i\tau(kx)},$$

$$i\varkappa^2 F(x)=D^0(x)=\frac{i}{2}(D^+(x)-D^-(x)),$$

so that one can write, using (62)

$$\varphi(x)=-\int_\Sigma d\sigma_\lambda(y)\cdot D^0(x-y)\,(\partial'^\lambda-\partial^\lambda)\,\varphi(y).$$

Spin 1/2

$$\int_\Sigma d\sigma_\lambda(y)\cdot\psi^+_a(\boldsymbol{k}'',n'',\tau''/y)\,C^\lambda(y)\psi_a(y/\boldsymbol{k}',n',\tau')=\delta(\boldsymbol{k}'',n'',\tau''/\boldsymbol{k}',n',\tau'),$$

$$\sum_\tau\int_{H_+} dV(\boldsymbol{k})\psi_a(x''/\boldsymbol{k},n,\tau)\psi^+_b(\boldsymbol{k},n,\tau/x')=F_{ab}(x''-x')=\sum_\tau\tau F^{(\tau)}_{ab}(x''-x'),$$

$$\psi_a(x)=\int_\Sigma d\sigma_\lambda(y)\cdot F_{ab}(x-y)C^\lambda(y)\psi_b(y),$$

with (81)

$$2\varkappa^3 F^\tau_{ac}(x)\beta_{cb}=S^\tau_{ab}(x)=-(\gamma i\partial+\varkappa)_{ab}D^\tau(x),$$

$$-i\varkappa^3 F_{ac}(x)\beta_{cb}=S^0_{ab}(x)=-(\gamma i\partial+\varkappa)_{ab}D^0(x),$$

so that one has

$$\psi_a(x) = i\int_\Sigma d\sigma_\lambda(y)\, S^0_{ab}(x-y)\gamma^\lambda \psi_b(y).$$

Spin 1

$$\int_\Sigma d\sigma_\lambda(y)\cdot B^+_\alpha(\boldsymbol{k}'', n'', \tau''/y) C^\lambda(y) B^\alpha(y/\boldsymbol{k}', n', \tau') = \tau''\,\delta(\boldsymbol{k}'', n'', \tau''/\boldsymbol{k}', n', \tau'),$$

$$\sum_\tau \tau \int_{H_+} dV(\boldsymbol{k}) B_\alpha(x''/\boldsymbol{k}, n, \tau) B^+_\beta(\boldsymbol{k}, n, \tau/x') = F_{\alpha\beta}(x''-x') = \sum_\tau \tau F^{(\tau)}_{\alpha\beta}(x''-x'),$$

$$B_\alpha(x) = \int_\Sigma d\sigma_\lambda(y)\cdot F_{\alpha\beta}(x-y) C^\lambda(y) B^\beta(y),$$

with (82)

$$2\varkappa^2 F^{(\tau)}_{\alpha\beta}(x) = \Delta^\tau_{\alpha\beta}(x) = (\varkappa^{-2}\partial_\alpha\partial_\beta + g_{\alpha\beta}) D^\tau_{\alpha\beta}(x),$$

$$i\varkappa^2 F_{\alpha\beta}(x) = \Delta^0_{\alpha\beta}(x) = (\varkappa^{-2}\partial_\alpha\partial_\beta + g_{\alpha\beta}) D^0_{\alpha\beta}(x),$$

which gives

$$B_\alpha(x) = -\int_\Sigma d\sigma_\lambda(y)\cdot \Delta^0_{\alpha\beta}(x-y)\,(\partial^{\prime\lambda} - \partial^\lambda) B^\beta(y).$$

Like the fields themselves, the distributions $F_{\rho\sigma}(x)$ are dimensionless.

§ 5. The transition to the q-number theory

Both the c-number theory and the q-number theory define the state of the system described by the fields on a space-like surface $\sum$, e.g. at a given time[21]. But whereas the c-number theory defines the state of the system by the values on $\sum$ of the fields functions $\chi_\rho(x)$ and of the maximum number of their derivatives compatible with the equations of motion, the q-number theory defines the state of the system by a probability amplitude function $\Psi(\sum)$, a vector in Hilbert space and in general surface-dependent, i.e. the state $\Psi(\sum)$ evolves with the "time $\sum$".

With each observable quantity, and in particular with each constant of motion $G_{(i)}$, the q-number theory associates a Hermitian operator $\boldsymbol{G}_{(i)}(\sum)$ in Hilbert space. The number

$$\overline{G_{(i)}}(\sum) = (\Psi(\sum)\, \boldsymbol{G}_{(i)}(\sum)\, \Psi(\sum)) \tag{83}$$

is the expectation value, i.e. the probable result of the measurement of the quantity $G_{(i)}$ on the surface $\sum$ when the system is supposed to be in the state $\Psi(\sum)$.

As in the c-number theory, the q-number theory is assumed to be invariant under infinitesimal Lorentz and gauge transformations. That is to say, the expectation values of the constants of motion are surface independent

$$\frac{\delta}{\delta\sigma(x)}\,\overline{G_{(i)}}(\sum) = 0, \tag{84}$$

as the constants of motion themselves are in the c-number theory, as a consequence of the invariance (see § 3).

The problem of quantization is that of building the q-number theory corresponding to a given c-number theory. This problem is twofold : first, one must know how to construct the operators $\boldsymbol{G}_{(i)}(\sum)$ which correspond to the quantities $G_{(i)}$; secondly, one must find the equations of motion for the state vector $\Psi(\sum)$ and for the operators $\boldsymbol{G}_{(i)}(\sum)$ in such a way that the relations (84) are satisfied. That must be done with the idea of exploiting the c-number theory to the limit. It is therefore natural to work in the Heisenberg picture of the equations of motion[25], in which, by definition, the state vector $\Psi(\sum)$ is assumed to follow the very simple equation of motion

$$\frac{\delta}{\delta\sigma(x)}\Psi(\sum)=0, \tag{85}$$

i.e. it is assumed to be surface independent so that the relation (84) can be written as

$$\frac{\delta}{\delta\sigma(x)}\boldsymbol{G}_{(i)}(\sum)=0, \tag{86}$$

which is identical with (66) in the c-number theory, except that the operator $\boldsymbol{G}_{(i)}$ replaces the function $G_{(i)}$.

In order to define the correspondence $G_{(i)}\rightarrow\boldsymbol{G}_{(i)}$, this identity between (66) and (86) suggests the introduction of a more general correspondence

$$\chi_{\rho'}(x)\rightarrow\chi_{\rho'}(x) \tag{87}$$

between the field functions $\chi_{\rho'}(x)$ of the c-number theory on the one hand and the linear field operators $\chi_{\rho'}(x)$ in Hilbert space of the q-number theory on the other. The essentially new feature of the field operators is that in general

$$\chi_{\rho'}(x)\chi_{\sigma'}(y)\neq\chi_{\sigma'}(y)\chi_{\rho'}(x). \tag{88}$$

It is worthwhile to notice here that one assumes that the operators $\chi_{\rho'}(x)$, as the functions $\chi_{\rho'}(x)$, *do not depend on the surface* $\sum$.

As soon as the correspondence (87) is fixed, to each functional of the field functions, in particular to each constant of motion $G_{(i)}[\chi]$, corresponds in the Hilbert space the functional operator $G_{(i)}[\chi]$ of the field operators $\chi_{\rho'}(x)$. But from (88) this correspondence is defined only up to an arbitrary order of the field operators which, incidentally, has to assure the Hermiticity of the operator $G_{(i)}[\chi]$. The same is true also for the density operator $\mathcal{G}^{\lambda}_{(i)}[\chi]$ corresponding to $G_{(i)}[\chi]$. That allows us to write the relation (86) in the differential form (see 50, 51, and 65)

$$\partial_{\lambda}\mathcal{G}^{\lambda}_{(i)}[\chi]=0 \tag{89}$$

up to the ordering of the χ's, which is exactly similar to equation (51) in the c-number theory. In (89) it is assumed that the operators $\chi_{\rho'}(x)$ satisfy the necessary conditions of differentiability as the functions $\chi_{\rho'}(x)$ do in the c-number theory. It must be noted that the transition from (86) to (89) also makes explicit use of the *surface-independence of the field operators* $\chi_{\rho'}(x)$.

On the other hand, it also follows from the correspondence (87) that, to the left-hand side of the equations of motions (42)

$$\frac{\partial L}{\partial \chi_{\rho'}(x)} - d_\mu \frac{\partial L}{\partial(\partial_\mu \chi_{\rho'}(x)} \equiv E[\chi_{\rho'}(x)] \tag{90}$$

for the field functions, there corresponds the same functional $E[\chi_{\rho'}(x)]$ in the q-number theory, up to the order of the field operators. Therefore, in the Heisenberg picture the equations of motion for the field operators are

$$E[\chi_{\rho'}(x)]=0 \tag{91}$$

up to the ordering of the χ's.

The three relations (85), (86), and (87) form the first part of the foundations on which it is proposed here to build the q-number theory. (85) gives the framework (the Heisenberg picture), (86) expresses the condition of invariance under Lorentz and gauge transformations in this framework, and (87) expresses the link between the q-number theory to be built and the c-number theory to which it must correspond.

But so far these relations do not define completely the q-number theory; in general even their consistency, i.e. the consistency between their consequences (89) and (91), cannot be checked: for the algebra of the linear operators $\chi_{\rho'}(x)$ is not completely defined (see 88). To fill the gap should be the task of a *principle of quantization*, the second part of the foundations on which the q-number theory is to be built. From this principle it should be possible to deduce first the complete algebra of the operators and secondly a rule for ordering the operators $\chi_{\rho'}(x)$ so that the correspondence between the c-number functionals $G_{(i)}[\chi]$ and the q-number operators $G_{(i)}[\chi]$ is uniquely defined. Then only can the consistency of the theory be checked in general. By the algebra of the field operators is meant essentially relations of the type

$$[\chi_{\rho'}(x), \chi_{\sigma'}(y)]_\omega = f[\chi_{\rho'}(x), \chi_{\sigma'}(x), x, y], \tag{92}$$

necessary for working with these non-commutative q-numbers.

For reasons given in the introduction, the following principle of quantization is used here: *The operators $\boldsymbol{G}_{(i)}$ corresponding to the constants of motion $G_{(i)}$ are a representation in Hilbert space of the infinitesimal operators $g_{(i)}$ corresponding to the constants of motion $G_{(i)}$.* With the notation of section 3, that means that[27]

$$\frac{1}{\hbar c}[G_{(i)}[\chi], \chi_{\rho'}(x)]_- \equiv g_{(i)}\chi_{\rho'}(x), \tag{93}$$

where the factor $1/\hbar c$ is introduced for obvious dimensional reasons since I has the dimension of $\hbar c$ (see equation 44). Furthermore, since the infinitesimal operators $g_{(i)}$ are Hermitian, that is

$$g_{(i)}^+ = g_{(i)}, \tag{94}$$

and since the structural equations of Lie[26]

$$[g_{(i)}, g_{(k)}]_- = c_{ik}^l g_l \tag{95}$$

hold, it is also a necessary consequence of the principle stated above that the operators $\boldsymbol{G}_{(i)}$ must satisfy the corresponding relations

$$G_{(i)}^{+} = G_{(i)}\,, \tag{96a}$$

$$\frac{1}{\hbar c}[G_{(i)}, G_{(k)}]_{-} = c_{ik}^{l} G_{(l)}. \tag{96b}$$

These relations may be considered consequences of (93). For, from (93)

$$[G_{(i)}^{+} - G_{(i)}, \chi_{\rho'}(x)]_{-} \equiv 0, \tag{97a}$$

$$\left[\frac{1}{\hbar c}[G_{(i)}, G_{(k)}]_{-} - c_{ik}^{l} G_{(l)}, \chi_{\rho'}(x)\right]_{-} \equiv 0, \tag{97b}$$

and if operators $\chi_{\rho'}(x)$ are assumed to form a complete set of variables for the system, i.e. if it is assumed that for an operator O the relations

$$[O, \chi_{\rho'}(x)]_{-} \equiv [O, \chi_{\rho}(x)]_{-} \equiv [O, \chi_{\rho}^{+}(x)]_{-} \equiv 0$$

are equivalent to the statement that O is a c-number function, then (96) follows from (97) and from the fact that the G's are independent of x.

Conversely, if the relations (96) are satisfied, the G's can be taken as a representation of the generators g's, and the relations (93) are consistent. But the G's are already defined by the c-number theory, up to the ordering of the field operators, and it is not known *a priori* whether the relations (96) can be satisfied by a suitable ordering of the operators. The relations (96) appear therefore as necessary conditions for the *consistency* of the q-number theory.

Consider a c-number theory leading to functionals $G_{(i)}[\chi]$ which, by the correspondence (87), do not lead to operators $G_{(i)}[\chi]$ satisfying the consistency conditions (96), for any ordering of the field operators $\chi_{\rho}(x)$. Such a c-number theory would not be "quantizable" according to the principle given above.

Suppose now that for a given c-number theory the consistency conditions are shown to be satisfied. Then, from equations (93) it is possible to deduce general commutation rules for the field operators of the form

$$\int_{\Sigma} d\sigma_{\lambda}(y)[\mathcal{G}_{(i)}^{\lambda}[\chi_{\sigma'}(y)], \chi_{\rho'}(x)]_{-} \equiv g_{(i)}\chi_{\rho'}(x) \tag{98}$$

which are surface independent because of (89). These are the relations which must lead in general both to the fundamental commutation rules (92) for the field operators (in short f.c.r.) and to the rule for ordering the operators in the densities $\mathcal{G}_{(i)}^{\lambda}[\chi]$. In the next section it will be shown how the f.c.r. and the rule of ordering can in fact be deduced from (98) in the case of the wave fields. But at this stage it may be pointed out that the f.c.r. and the rule of ordering are intimately related through the relations (98) from which they both follow.

§ 6. The q-number theory for the wave fields

As an illustration of the general theory given in the preceding section, the case of wave fields is treated here.

In the c-number theory for linear systems (see § 3b), the Lagrangian and the constants of motion are functionals, bilinear in the field functions and their (first) derivatives. When there are no external fields, the constants of motion themselves are given by

$$G_{(i)}[\chi]=\int_{\Sigma} d\sigma_\lambda(y)\cdot\chi_\rho^+(y)\,\Gamma'^{\lambda}_{(i)}(y)\chi_\rho(y), \qquad i=0, 1, \cdots 10, \tag{99}$$

with the correspondences

$$G_{(i)}[\chi]\rightleftarrows \Gamma'^{\lambda}_{(i)}(y)=\hbar c\, C^\lambda(y) g_{(i)}(y)=-\hbar c\, g^{\prime}_{(i)}(y)\, C^\lambda(y). \tag{100}$$

It was shown also in § 4 that in the c-number theory of wave fields, to each particular kind of the field $\chi_\rho(x)$, correspond first a complete set of orthogonal solutions of the equations of motion

$$\chi_\rho(x/\boldsymbol{k}, n, \tau)\equiv\chi_\rho(x/u),\ \chi_\rho(x/\boldsymbol{k}', n', \tau')\equiv\chi_\rho(x/u'),\ \cdots \tag{101}$$

satisfying the orthogonality relation and the completeness relation (75), and secondly an invariant distribution $F_{\rho\sigma}(x-y)$ such that

$$\chi_\rho(x)=\int_{\Sigma} d\sigma_\lambda(y)\cdot F_{\rho\sigma}(x-y)\,C^\lambda(y)\chi^\sigma(y). \tag{102}$$

These two important features enable one to give a proof of the consistency of the quantization of wave fields, in the sense explained in the last section. For this purpose, to each constant of motion $G_{(i)}$ one associates a generalized matrix

$$(u''/G_{(i)}/u')=\int_{\Sigma} d\sigma_\lambda(y)\cdot\chi_\rho^+(u''/y)\,\Gamma'^{\lambda}_{(i)}(y)\chi_\rho(y/u'), \tag{103}$$

which is surface indenpent according to equation (67). The multiplication of $(u''/G_{(k)}/u')$ by $(u''/G_{(i)}/u')$ is denoted by (see 75)

$$(u''/G_{(i)}\,G_{(k)}/u')=\int_{H_+} dV(u)\,(u''/G_{(i)}/u)\,(u/G_{(k)}/u'),$$

$$\text{with}\qquad \int_{H_+} dV(u)\equiv\sum_{\tau=+1,-1}\nu(\tau)\sum_{n}\int_{H_+} d V(\boldsymbol{k}). \tag{104}$$

Using (75), one gets

$$(u''/G_{(i)}\,G_{(k)}/u')=\int_{\Sigma} d\sigma_\lambda(y)\int_{\Sigma} d\sigma_\theta(x)\cdot\chi_\rho^+(u''/y)\,\Gamma'^{\lambda}_{(i)}(y)\,F_{\rho\sigma}(y-x)\,\Gamma'^{0}_{(k)}(x)\,\chi_\sigma(x/u'). \tag{105}$$

One defines now $K(A)=A_{\rho\sigma}(x-y)$ as the kernel of the matrix

$$(u''/A/u')=\int_{\Sigma} d\sigma_\lambda(y)\int_{\Sigma} d\sigma_\theta(x)\cdot\chi_\rho^+(u''/y)C^\lambda(y)\,A_{\rho\sigma}(y-x)\,C^0(x)\chi_\sigma(x/u').$$

Using (102), one sees that the kernel of the matrix $(u''/G_{(i)}/u')$ can also be written as

$$K(G_{(i)})=\frac{\hbar c}{2}\{g_{(i)}(y)F_{\rho\sigma}(y-x)-F_{\rho\sigma}(y-x)g^{\prime}_{(i)}(x)\}. \tag{108}$$

Moreover, one has for any of the operators $g_{(i)}$, $i=0, 1, \cdots 10$

$$K(0)=\frac{\hbar c}{2}\{g_{(i)}(y)F_{\rho\sigma}(y-x)+F_{\rho\sigma}(y-x)g_{(i)}^{I}(x)\}\,. \tag{109}$$

Finally

$$K([G_{(i)}, G_{(k)}]_-)=-(\hbar c)^2\{g_{(i)}(y)F_{\rho\sigma}(y-x)g_{(k)}^{I}(x)-g_{(k)}(y)F_{\rho\sigma}(y-x)g_{(i)}^{I}(x)\}\,. \tag{110}$$

In the preceding formulae, note that in $g_{(i)}(y)F_{\rho\sigma}(y-x)$ the operator $g_{(i)}(y)$ operates only on the variable y and on the left index ρ of $F_{\rho\sigma}(y-x)$; similarly, in $F_{\rho\sigma}(y-x)g_{(i)}^{I}(x)$, $g_{(i)}^{I}(x)$ operates only on the variable x and on the right index σ of $F_{\rho\sigma}(y-x)$. Now, as a consequence of the invariance of the distribution $F_{\rho\sigma}(y-x)$ (see 79 or 109), one has

$$g_{(i)}(y)F_{\rho\sigma}(y-x)=-F_{\rho\sigma}(y-x)g_{(i)}^{I}(x)\,. \tag{111}$$

Consequently, the kernel of $(u''/[G_{(i)}, G_{(k)}]_-/u')$ can be written as

$$K([G_{(i)}, G_{(k)}]_-)=(\hbar c)^2\{g_{(i)}(y)g_{(k)}(y)F_{\rho\sigma}(y-x)-F_{\rho\sigma}(y-x)g_{(k)}^{I}(x)g_{(i)}^{I}(x)\},$$

or, using (111) once more,

$$K([G_{(i)}, G_{(k)}]_-)=\frac{(\hbar c)^2}{2}\{[g_{(i)}(y), g_{(k)}(y)]_-F_{\rho\sigma}(y-x)$$

$$-F_{\rho\sigma}(y-x)[g_{(k)}^{I}(x), g_{(i)}^{I}(x)]_-\}\,.$$

But the generators $g_{(i)}$ satisfy the structure equations of Lie

$$[g_{(i)}, g_{(k)}]_-=c_{ik}^{l}g_{(l)}\,.$$

Therefore one has

$$K([G_{(i)}, G_{(k)}]_-)=c_{ik}^{l}\frac{(\hbar c)^2}{2}\{g_{(l)}(y)F_{\rho\sigma}(y-x)-F_{\rho\sigma}(y-x)g_{(l)}^{I}(x)\}, \tag{112}$$

which shows that the kernel of the matrix $(u''/[G_{(i)}, G_{(k)}]/u')$ is equal to that of the matrix $\hbar c\cdot c_{ik}^{l}(u''/G_{(l)}/u')$. Therefore one has

$$\left(u''\left|\frac{1}{\hbar c}[G_{(i)}, G_{(k)}]_-\right|u'\right)=c_{ik}^{l}(u''/G_{(l)}/u')\,. \tag{113b}$$

Furthermore, one sees easily from (59) that the matrices $(u''/G_{(i)}/u')$ are Hermitian since

$$(u''/G_{(i)}^{+}/u')=(u''/G_{(i)}/u')\,. \tag{113a}$$

Consequently, the structure of the constants of motion $G_{(i)}[\chi]$ is such that the matrices $(u''/G_{(i)}/u')$ deduced from them by (103) form a representation of the infinitesimal group $\Gamma\times\Lambda$ in the linear space spanned by the vectors $\chi_\rho(x/u')$, $\chi_\rho(x/u'')\cdots$. It is very important to note that this representation is invariant with respect to the Lorentz group and gauge group, and is also surface-independent.

Returning to the quantization of the c-number theory for wave fields, one has, according to the principle given in the preceding section, to set up the correspondence (87) in such a way that the resulting correspondence

$$G_{(i)}[\chi] \rightarrow G_{(i)}[\chi] \equiv \boldsymbol{G}_{(i)} \tag{114}$$

defines operators $\boldsymbol{G}_{(i)}$ which are constants of motion and which satisfy the consistency conditions

$$\boldsymbol{G}_{(i)}^{+} = \boldsymbol{G}_{(i)}, \tag{115a}$$

$$\frac{1}{\hbar c}[\boldsymbol{G}_{(i)}, \boldsymbol{G}_{(k)}]_{-} = c_{ik}^{l}\boldsymbol{G}_{(l)}. \tag{115b}$$

Yet, it was just shown that the structure of the functionals $G_{(i)}[\chi]$, together with the existence of the complete set (101) of solutions of the equations of motion, allows one to set up a matrix representation $(u''/G_{(i)}/u')$ of the operators $\boldsymbol{G}_{(i)}$, and that this representation satisfies the relations (113) identical to (115). This fact shows that the consistency equations (115) for the operators $\boldsymbol{G}_{(i)}$ are themselves consequences of the particular structure of the constants of motion (99) and of the equations of motion of the wave fields. But this structure itself is completely determined by the structure of the Lagrangian. Therefore, in the case of wave fields, the consistency conditions (115) are satisfied as a consequence of the structure of the Lagrangian.

One can note also that, in the case of wave fields, the existence of the matrix representation (103), which is a feature of the c-number theory only, leads directly to the principle of quantization stated in the preceding section.

In order to complete the q-number theory for wave fields, it is now possible to write down consistently the general c.r.'s.

$$\int_{\Sigma} d\sigma_{\lambda}(y) \cdot [(\chi_{\sigma}^{+}(y)\Gamma_{(i)}^{\lambda}(y)\chi_{\sigma}(y))_{or}, \chi_{\rho'}(x)]_{-} = g_{(i)}\chi_{\rho'}(x), \tag{116}$$

from which one must deduce both the f.c.r.'s for the field operators (92) and the rule of ordering. In (116) $(\ )_{or}$ means that the field operators $\chi_{\sigma'}(y)$ are ordered according to the rule to be found below. With the help of (102), the relations (116) can be written as

$$\begin{cases} \displaystyle\int_{\Sigma} d\sigma_{\lambda}(y) \cdot [\chi_{\rho}(x), (\chi_{\sigma}^{+}(y) g_{(i)}^{L}(y) C^{\lambda}(y)\chi_{\sigma}(y))_{or}]_{-} \\ \qquad \equiv \equiv \displaystyle\int_{\Sigma} d\sigma_{\lambda}(y) \cdot g_{(i)}(x) F_{\rho\sigma}(x-y) C^{\lambda}(y)\chi_{\sigma}(y), \\ \displaystyle\int_{\Sigma} d\sigma_{\lambda}(y) \cdot [(\chi_{\sigma}^{+}(y) C^{\lambda}(y) g_{(i)}(y)\chi_{\sigma}(y))_{or}, \chi_{\rho}^{+}(x)]_{-} \\ \qquad \equiv \equiv \displaystyle\int_{\Sigma} d\sigma_{\lambda}(y) \cdot \chi_{\sigma}^{+}(y) C^{\lambda}(y) F_{\rho\sigma}(y-x) g_{(i)}^{L}(x). \end{cases} \tag{117}$$

Since, as a consequence of the equations of motion, $u_{\lambda}(y) C^{\lambda}(y)\chi_{\sigma}(y)$ may be given arbitrarily on the space-like surface $\sum_{y}$ (117) is equivalent to

$$[\chi_\rho(x), (\chi_\sigma^+(y) g_{(i)}^I(y) C^\lambda(y) \chi_\sigma(y))_{or}]_- \equiv g_{(i)}(x) F_{\rho\sigma}(x-y) C^\lambda(y) \chi_\sigma(y).$$

The second identity is not considered further, since it is the Hermitian conjugate of the first (see 79). Finally, the use of (111) gives the general c.r. (116) in the form

$$[\chi_\rho(x), (\chi_\sigma^+(y) g_{(i)}^I(y) C^\lambda(y) \chi_\sigma(y))_{or}]_- \equiv -F_{\rho\sigma}(x-y) g_{(i)}^I(y) C^\lambda(y) \chi_\sigma(y). \quad (118)$$

The commutator of the left-hand side involves an expression bilinear in the field operators, which is a property of linear systems. The identity

$$[A, BC]_- \equiv [A, B]_\omega C - \omega B [A, C]_\omega, \qquad \omega = \pm 1 \quad (119)$$

shows that in order to satisfy the identity (118), one must have

$$\begin{aligned} [\chi_\rho(x), \chi_\sigma^+(y)]_\omega &= c_{\rho\sigma}(x, y) = c\text{-number}, \\ [\chi_\rho(x), \chi_\sigma(y)]_\omega &= c'_{\rho\sigma}(x, y) = c\text{-number}, \end{aligned} \quad (120)$$

whatever may be the rule of ordering. This result shows that the ordered expression $(\chi_\sigma^+(y)\chi_\sigma(y))_{or}$ can always be written as

$$(\chi_\sigma^+(y)\chi_\sigma(y))_{or} = \eta(\omega)\frac{1}{2}[\chi_\sigma^+(y), \chi_\sigma(y)]_{-\omega} + \zeta(\omega)\frac{1}{2}c_{\sigma\sigma}(y, y), \quad (121)$$

where $\eta(\omega)$ and $\zeta(\omega)$ are numbers. For, if one puts $[A, B]_\omega = c$, then from the identity

$$AB = \frac{1}{2}[A, B]_\omega + \frac{1}{2}[A, B]_{-\omega} \quad (122)$$

and the symmetry of the c-number AB with respect to the two factors A and B one must have

$$(AB)_{or} = \eta(\omega)\frac{1}{2}[A, B]_{-\omega} + \zeta(\omega)\frac{c}{2}, \quad (123)$$

with $\eta(+1) = \pm 1$ and $\eta(-1) = 1$.

In (121), $\eta(\omega) \cdot 1/2 \cdot [\chi_\sigma^+(y), \chi_\sigma(y)]_{-\omega}$ will be called the *principal part* of $(\chi_\sigma^+(y)\chi_\sigma(y))_{or}$. The relation (121) gives the rule of ordering[28] up to the factors $\eta(\omega)$ and $\zeta(\omega)$.

In order to get the actual value of the commutators (120), one replaces $(\chi_\sigma^+(y)\chi_\sigma(y))_{or}$ in (118) by its principal part given by (121), the c-number $\zeta(\omega) \cdot c/2$ being irrelevant here. With the help of (119), one gets

$$\begin{aligned} \eta(\omega)\{[\chi_\rho(x), \chi_\sigma^+(y)]_\omega g_{(i)}^I(y) C^\lambda(y)\chi_\sigma(y) - \omega\chi_\sigma^+(y) g_{(i)}^I(y) C^\lambda(y)[\chi_\rho(x), \chi_\sigma(y)]_\omega\} \\ \equiv -F_{\rho\sigma}(x-y) g_{(i)}^I(y) C^\lambda(y)\chi_\sigma(y). \end{aligned} \quad (124)$$

Since this identity must be valid for any $g_{(i)}^I C^\lambda(y)\chi_\sigma(y)$, it follows from the independence of the χ's and the χ^+'s that

$$\begin{cases} \eta(\omega)[\chi_\rho(x), \chi_\sigma^+(y)]_\omega = -F_{\rho\sigma}(x-y), \\ [\chi_\rho(x), \chi_\sigma(y)]_\omega = [\chi_\rho^+(x), \chi_\sigma^+(y)]_\omega = 0. \end{cases} \quad (125)$$

These are the f.c.r.'s.

Using the particular definitions given in (4) for the distribution $F_{\rho\sigma}(x-y)$ in the particular cases of spin 0, 1/2, and 1, one gets

$$\eta(\omega)[\varphi(x), \varphi^+(y)]_\omega = -F(x-y) = i\varkappa^{-2} D^0(x-y), \tag{125a}$$

$$\eta(\omega)[\psi_a(x), \psi_b^+(y)]_\omega = -F_{ab}(x-y) = -i\varkappa^{-3}((\gamma i\partial + \varkappa)\beta)_{ab} D^0(x-y), \tag{125b}$$

$$\eta(\omega)[\boldsymbol{B}_\alpha(x), \boldsymbol{B}_\beta^+(y)]_\omega = -F_{\alpha\beta}(x-y) = i\varkappa^{-4}(\partial_\alpha\partial_\beta + \varkappa^2 g_{\alpha\beta}) D^0(x-y). \tag{125c}$$

The last step consists now in showing that the values of ω and $\eta(\omega)$ in these relations are determined in each particular case by the distribution $F_{\rho\sigma}(x-y)$ of the right-hand side. The proof will be given in the cases of spin 0, 1/2, and 1, but its extension to higher spin value is straightforward.

Taking first the case of spin 0, following Pauli[30], one forms the expression

$$[\varphi(x), \varphi^+(y)]_\omega + [\varphi(y), \varphi^+(x)]_\omega = \eta(\omega) i\varkappa^{-2}(D^0(x-y) + D^0(y-x)) = 0 \tag{126}$$

which vanishes as a consequence of the property of the distribution $D^0(x-y)$. One sees that in general the left-hand side of (126) vanishes only if $\omega = -1$; for with $\omega = +1$, it is essentially positive, unless the φ's vanish identically. Consequently, in the case of spin 0, one must have $\omega = -1$, and therefore $\eta = \eta(-1) = 1$. The f.c.r.'s are[35]

$$[\varphi(x), \varphi^+(y)]_- = i\varkappa^{-2} D^0(x-y), [\varphi(x), \varphi(y)]_- = [\varphi^+(x), \varphi^+(y)]_- = 0. \tag{127a}$$

The same argument is valid in the case of spin 1, since only $D^0(x-y)$ and its second-order derivatives occur in the right-hand side of (125c), so that the f.c.r.'s in this case are

$$[\boldsymbol{B}_\alpha(x), \boldsymbol{B}_\beta^+(y)]_- = i\varkappa^{-3}(\partial_\alpha\partial_\beta + \varkappa^2 g_{\alpha\beta}) D^0(x-y),$$

$$[\boldsymbol{B}_\alpha(x), \boldsymbol{B}_\beta(y)]_- = [\boldsymbol{B}_\alpha^+(x), \boldsymbol{B}_\beta(y)]_- = 0. \tag{127c}$$

Turning now to the case of spin 1/2, one computes the trace of each side of the expression

$$\eta(\omega)([\psi_a(x), \psi_b^+(y)]_\omega + [\psi_a(y), \psi_b^+(x)]_\omega) = 2\varkappa^{-3}(\gamma^\mu\beta)_{ab}\partial_\mu D^0(x-y),$$

and integrates over the particular space-like surface $x^4 - y^4 = 0$. One gets

$$\eta(\omega)\int_{-\infty}^{+\infty}(dx)^3([\psi_a(x), \psi_a^+(y)]_\omega + [\psi_a(y), \psi_a^+(x)]_\omega)_{x^4=y^4}$$

$$= 8\varkappa^{-3}\int_{-\infty}^{+\infty}(dx)^3\partial_4 D^0(x-y) = 8\varkappa^{-3} > 0,$$

if the following properties of the distribution $D^0(x)$ are used

$$\partial^t D(x)|_{x^4=0} = 0, \qquad \partial_4 D^0(x)|_{x^4=0} = \delta(\boldsymbol{x}).$$

One sees here that, whereas the right-hand side is positive definite (the mass of the particle is positive), the left-hand side is in general not definite for $\omega = -1$, since $\psi_a(x)$ can be given arbitrarily on the space-like surface $x_4 = y_4$. For instance, if $\psi_a(x)$ is chosen to be real, the sum of commutators $[\psi_a(x), \psi_a(y)]_-$ and $[\psi_a(y), \psi_a(x)]_-$ vanishes.

Therefore, the only possible value for ω is $+1$, with $\gamma_j = \gamma_j(+1) = +1$, since the right-hand side is positive. The f.c.r.'s in the case of spin $1/2$ are consequently

$$[\psi_a(x), \psi_b^+(y)]_+ = -ix^{-3}((\gamma i\partial + \varkappa)\beta)_{ab} D^0(x-y),$$
$$[\psi_a(x), \psi_b(y)]_+ = [\psi_a^+(x), \psi_b^+(y)]_+ = 0. \tag{127b}$$

Comparing the relations (127) among themselves, one sees that for integral spins they involve the commutator, whereas for half integral spins they involve the anticommutator. That is the connexion between spin and statistics as it was given first by Pauli[30]. Furthermore, (127) are exactly the c.r.'s which are obtained with the canonical quantization. The equivalence between the two methods is therefore established.

Finally, a word is necessary on the rule of ordering as given by (121). From (120) and (127), $c_{\sigma\sigma}(y,y)$ vanishes in the cases of spin 0 and 1; in the case of spin $1/2$, however, it is not defined; the factor $c_{\sigma\sigma}(y,y)$ must be defined in such a way that the operators $G_{(\omega)}[\chi]$ are well defined. One can therefore write in every case

$$(\chi_\sigma^+(y)\chi_\sigma(y))_{or} = \frac{1}{2}[\chi_\sigma^+(y), \chi_\sigma(y)]_{-\omega}, \tag{128}$$

where the right-hand side is to be understood as a *principal value* in the sense given above. That is the rule of ordering in its complete form.[28]

§ 7. Discussion

It must be emphasized first that only the case of matter field has been treated here, i.e. the case of fields which are gauge dependent of the first kind, as characterized by

$$q\chi_\rho(x) = \varepsilon\chi_\rho(x), \qquad q\chi_\rho^+(x) = -\varepsilon\chi_\rho^+(x). \tag{129}$$

Neither the electromagnetic field nor more generally real fields can be treated by the preceding method without further specifications concerning their gauge dependence. For instance, for a gauge independent field $\xi(x)$, one has

$$q\xi(x) = 0. \tag{129a}$$

Or the field can be gauge dependent of the second kind, as the electromagnetic field $A^\mu(x)$, with (see § 2)

$$qA^\mu(x) = -qd^\mu C(x). \tag{129b}$$

But there are other cases. For instance, it is possible to describe the matter fields $\chi_\rho(x), \chi_\rho^+(x)$ by two real fields $\chi_\rho^{(1)}(x), \chi_\rho^{(2)}(x)$ characterized by the gauge dependence

$$q\chi_\rho^{(1)}(x) = i\varepsilon\chi_\rho^{(2)}(x), \qquad q\chi_\rho^{(2)}(x) = -i\varepsilon\chi_\rho^{(1)}(x) \tag{129c}$$

instead of (129). The general theory can be developed similarly and is equivalent to the theory with the components $\chi_\rho(x), \chi_\rho^+(x)$. For instance, in the case of wave fields, the Lagrangian and the constants of motion are of the form[31]

$$L=\frac{1}{2}\sum_{\alpha=1}^{2}\chi_\rho^{(\alpha)}(x)I\chi_\rho^{(\alpha)}(x), \qquad G_{(\sigma)}=\frac{\hbar c}{2}\sum_{\alpha=1}^{2}\int_\Sigma d\sigma_\lambda(y)\chi_\rho^{(\alpha)}(y)C^\lambda(y)g_{(\sigma)}\chi_\rho^{(\alpha)}(y), \tag{130}$$

with

$$C^\lambda=\frac{i}{\hbar c}(W^{L\lambda}-W^\lambda), \qquad \frac{\partial L}{\partial(\partial_\lambda\chi_\rho^{(\alpha)}(x))}=W^\lambda\chi_\rho^{(\alpha)}(x)=\chi_\rho^{(\alpha)}(x)W^{L\lambda},$$

and the f.c.r.'s for the q-number theory are

$$\eta(\omega)[\chi_\rho^{(\alpha)}(x),\chi_\rho^{(\beta)}(y)]_\omega=-\delta^{\alpha\beta}F_{\rho\sigma}(x-y), \tag{131}$$

as can be derived from the general relation (93).

A second remark deals with the case of the presence of an external field. When this external field is arbitrarily given, the equations of motion are no longer invariant under a Lorentz transformation. But it is possible to introduce the external field in such a way that they are still gauge invariant; then, in general, only the total charge of the system is a constant of motion. That is precisely what happens in the theory of wave fields when the gauge is assumed to be a function of x. If the fields $\chi_\rho^A(x)$ are gauge dependent of the first kind, one has (see § 2)

$$p_\mu\chi_\rho^A(x)=\left(i\partial_\mu+d_\mu C(x)i\frac{d}{dC}\right)\chi_\rho^A(x), \tag{132}$$

and the Lagrangian (52) is no longer gauge invariant. In order to have it still gauge invariant, one makes in L the substitution

$$i\partial_\mu\to id_\mu+\varepsilon A_\mu(x)\equiv id_\mu^A, \qquad i\partial_\mu^L\to id_\mu^L-\varepsilon A_\mu(x)\equiv id_\mu^{LA}, \tag{133}$$

i.e.

$$p_\mu\to p_\mu+A_\mu(x)q\equiv p_\mu^A, \qquad p_\mu^L\to p_\mu^L+q^LA_\mu(x)\equiv p_\mu^{LA} \tag{133a}$$

where $A^\mu(x)$ is an external field which is gauge dependent of the second kind. Using the modified action principle[32], one obtains the equations of motion for the fields

$$E^A[\chi_{\rho'}^A(x)]=0, \tag{134}$$

where the index A in E^A expresses in general the fact that the substitution (133) has been carried out in E. From (134) the wave equation

$$(\Box^A-\varkappa^2)\chi_{\rho'}^A(x)=0, \qquad \begin{matrix}\Box^A=id^{A\lambda}\cdot id_\lambda^A, & \text{for spin 0, 1,}\\ \Box^A=(\gamma id^A)(\gamma id^A), & \text{for spin 1/2}\end{matrix} \tag{135}$$

follows. The only constant of motion is in general the total charge

$$Q=e\int_\Sigma d\sigma_\lambda(y)\chi_\rho^{A+}(y)C^{\lambda A}(y)\chi_\rho(y). \tag{136}$$

The method of quantization gives the general c.r.

$$[Q,\chi_{\rho'}^A(x)]_-\equiv q\chi_{\rho'}^A(x), \tag{137}$$

which leads consistently to the f.c.r.'s for the field operators[33]

$$\begin{cases} \eta(\omega)[\chi_\rho^A(x), \chi_\rho^{A+}(y)]_\omega = -F_{\rho\sigma}^A(x,y), \\ [\chi_\rho^A(x), \chi_\sigma^A(y)]_\omega = [\chi_\rho^{A+}(x), \chi_\sigma^{A+}(y)]_\omega = 0. \end{cases} \tag{138}$$

The distribution $F_{\rho\sigma}^A(x,y)$ is defined by

$$\int_{H_+} dV(u)\chi_\rho^A(x/u)\chi_\sigma^{A+}(u/y) = F_{\rho\sigma}^A(x,y), \tag{139}$$

where the functions $\chi_\rho^A(x/u')$, $\chi_\rho^A(x/u'')$, $\cdots$ form a complete set of orthonormal solutions of (134). They are defined such that they coincide on a given space-like surface Σ with the complete set of solutions $\chi_\rho(x/u')$, $\chi_\rho(x/u'')$, $\cdots$ of the equations of motion $L[\chi_\rho(x)] = 0$. The rule of ordering is still given by

$$(\chi_\sigma^{A+}(y)\chi_\sigma^A(y))_{op} = \frac{1}{2}[\chi_\sigma^{A+}(y), \chi_\sigma^A(y)]_{-\omega}, \tag{140}$$

but it must be borne in mind that the right-hand side of (140) has the meaning of a principal value, so that the operator Q, for instance, is a well-defined operator.

Turning now one's attention to the comparison between the method of quantization given in the preceding sections with that based on the canonical formalism, one sees that in the case of wave fields, with or without an external field, both methods lead to the same q-number theory corresponding to a given c-number theory. But from the point of view of self-consistency and economy in postulates, the method given here presents some advantages. First, it does not require that the equations of motion of the c-number theory shall be expressible in the canonical form. The only requirements are that the equations of motion of the c-number theory shall be invariant under a certain group $\mathcal{G}$ of infinitesimal transformations representable in a Hilbert space $\mathcal{H}$, and that the structure of the constants of motion (which is already restricted by the first requirement) considered as functionals of the fields permits one to build with them a representation of the infinitesimal group $\mathcal{G}$ in $\mathcal{H}$. Consequently, the problem of the quantization of a given c-number theory can be reduced to that of finding a Lagrangian with an adequate structure.

A second advantage of the method of quantization proposed here is the fact that it brings out the connexion between spin and statistics as a straightforward consequence of the requirement of invariance under continuous Lorentz transformations. Moreover, fields belonging to both kinds of statistics are treated on the same footing, no hypothesis being necessary about the fields as physical quantities. Only the constants of motion are assumed to be observable quantities, and the fields themselves appear only as auxiliary mathematical quantities.

Finally, there are difficulties linked to the canonical formalism, as, for instance, those connected with the existence of *identities*, i.e. equations of motion which do not involve time-derivatives of the fields[34]. From the points of view adopted here, one sees that these difficulties disappear with the canonical formalism itself.

However, a much severer test for the preceding method of quantization will lie in its ability to throw some light on the quantum theory of interacting systems, especially on Quantum Electrodynamics. For a consistent foundation of a theory of interacting wave

fields which would provide a rigorous explanation of the successes of the so-called renormalization techniques is still to be given.

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References

1) Concerning gauge transformations, see H. Weyl, *Raum-Zeit-Materie* (Springer, Berlin, 1923) §§ 40, 41; *The Theory of Groups and Quantum Mechanics* (Methuen, London, 1931) pp. 100, 213 ff.; W. Heisenberg and W. Pauli, ZS. f. Phys. **59** (1929), 168; an infinitesimal gauge transformation is introduced in the latter paper, p. 172. Furthermore, throughout this paper only continuous transformatinns are considered; Lorentz invariance means, therefore, invariance under continuous Lorentz transformations.

2) W. Pauli, Rev. of Mod. Phys. **13** (1941), 203, § 2. For the definition of c-numbers and q-numbers, see P.A.M. Dirac, Proc. Roy. Soc. A **111** (1926), 281.

3) W. Heisenberg and W. Pauli, ZS. f. Phys. **56** (1929), 1; Gregor Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (Deuticke, Wien, 1943) § 1; J. Schwinger, Phys. Rev. **82** (1951), 914 esp. formulae (2·61) and (2·81).

4) Pauli (see ref. in footnote 2, p. 210) does not use explicitly the canonical formalism, but rather a generalization of a method introduced by Jordan and Pauli, ZS. f. Phys. **47** (1928), 151, and developed by Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1935), 2nd ed., § 62. But this method makes full use of the canonical quantization of the harmonic oscillator.

5) P.A.M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed., §§ 25 and 27.

6) Compare P.A.M. Dirac, Rev. Mod. Phys. **21** (1949), 392. In this paper,only the ten constants of motion (called the ten fundamental quantities) related to the Lorentz group are considered. But Dirac's use of the structure of the Lorentz group for building up the theory of a dynamical system is closely related to the method of quantization proposed here (see esp. § 2 and § 6).

7) E. Cartan, *La Théorie des groupes finis et continus et la Géométrie différentielle* (Gauthier-Villars, Paris, 1937) §§ 157, 210.

8) W. Pauli, Phys. Rev. **58** (1940), 716; Prog. Theor. Phys. **5** (1950), 526. Part of the argument which will be used for establishing the connexion between spin and statistics is due to Pauli (see § 4 of the first quoted paper).

9) P.A.M. Dirac, ref. footnote 5, § 28.

10) W. Pauli, see reference in footnote 2, § 2b.

11) For more details on the results given in this section, see H. Weyl, *Theory of Groups and Quantum Mechanics* (see ref. in footnote 1), p. 175 ff; and V. Bargmann and E. P. Winger, Proc. Nat. Acad. Sci. **34** (1948), 211. For the definition of an infinitesimal group of a Lie group, see, for instance, Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, 1946) p. 238.

12) See for instance W. Pauli, *Handbuch der Physik* (Springer, Berlin, 1933), Bd. XXIV/1, p. 219.

13) See W. Pauli, first ref. in footnote 8, p. 718.

14) See for instance W. Pauli, ref. in footnote 2, I § 2.

15) See J. Schwinger, Phys. Rev. **74** (1948), 1439, p. 1447.

16) This method is due to E. Cartan, *Leçons sur les Invariants Intégraux* (Hermann, Paris, 1922) pp. 3, 9. It has been used by Weiss (Proc. Roy. Soc. A **156** (1936), 192, and recently by J. Schwinger (see ref. footnote 3, p. 918).

17) For more details and references to original papers, see G. Wentzel, ref. in footnote 3, §§ 8, 12, 20; or W. Pauli, ref. 2, II.

18) See J. Schwinger, ref in footnote 15, p. 1446.

19) See W. Pauli, ref. in footnote 2, II.
20) For a definition of distributions, see Laurent Schwartz, *Théorie des Distributions* I and II (Hermann, Paris, 1950, 1951), chap. 1.
21) Orthogonality relation and completeness relation of this kind have been considered by Stueckelberg, (see E.C.G. Stueckelberg and D. Rivier, Helvetica Physica Acta **18** (1950), 216.
22) Formulae of this type have been already used by J. Schwinger, see ref. in footnote 15, p. 1451, in the case of spin 1/2, and of spin 0 with $\kappa=0$.
23) $D^0(x)$ is the distribution $D(x)$ of Pauli (see ref. in footnote 2, p. 211).
24) The validity of such a definition has been discussed from the point of view of the S-matrix by E.C.G. Stueckelberg, Phys. Rev. **81** (1951), 130.
25) See P.A.M. Dirac, ref. in footnote 5, § 28.
26) See for instance E. Cartan, ref. in footnote 7, § 210
27) These relations have been considered by H. Weyl, second ref. in footnote 1, Chap. IV, § 13, and by others, in particular J. Schwinger, last ref. in footnote 3, p. 921 ; but they appear there as *consequences* of the f.c.r.'s. However, W. Pauli, ref. in footnote 2, uses the relation $dF/dx^4=i[H, F]$ (H is the total energy divided by $\hbar c$) as a starting point for his quantization; but the method uses implicitly the canonical quantization of the harmonic oscillator (see footnote 4).
28) This rule of ordering was introduced in the case of spin 1/2 by Heisenberg (ZS. f. Phys. **90** (1934), 209, p. 224).
29) See first ref. in footnote 8, p. 722.
30) See first ref. in footnote 8. It is worth noticing here that the proof given above of the connexion between spin and statistics uses only the invariance of the theory with respect to the *continuous* Lorentz group, as Pauli's original proof did. The point of view adopted here is different from that adopted by Schwinger, who considers the connexion between statistics and spin as a consequence of the invariance of the theory under time inversion combined with charge conjugation (see last ref. in footnote 3, p. 920).
31) For the description of matter with real field functions, see for instance W. Pauli, ref. in footnote 2, pp. 213, 219, 225.
32) See W. Pauli, ref. in footnote 2, I 2b.
33) See W. H. Furry, Phys. Rev. **81** (1951), 115.
34) See for instance P.A.M. Dirac, Can. J. Math. **2** (1950), 129.
35) (Added in proof) S. Schweber, Phys. Rev. **78** (1950), 613, generalizing an argument given by Wigner, Phys. Rev. **77** (1950), 711, has shown that, in the particular case of a real scalar field, the right-hand side of (127a) is determined by the Hamiltonian, the equations of motion and the requirement of relativistic invariance only up to the multiplication by a matrix $(1+W)$, where W is a scalar invariant independant of x and of $\boldsymbol{k}$, and satisfies certain commutation relations. However, the proof of the existence of such a matrix $W\neq 0$ which satisfies all these conditions is not given. Furthermore, this questionable lack of uniqueness seems linked to the canonical formalism.

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On the Origin of the Anisotropy Energy of $CuCl_2 \cdot 2H_2O$

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The anisotropy energy of $CuCl_2 \cdot 2H_2O$ having an orthorhombic symmetry can be expressed by $K_1\beta^2 + K_2\gamma^2$, where β and γ are the direction cosines of the magnetic moment with respect to the b-axis and the c-axis. The calculated value of K_2 arising from the magnetic dipole interaction nearly accords with the experimental value, but that of K_1 becomes much smaller. Since the magnitude of the spin of $CuCl_2 \cdot 2H_2O$ is equal to 1/2, the crystalline field can not produce any anisotropy energy, so that we must consider that the other part of the anisotropy energy than that from the magnetic dipole interaction arises from the anisotropic exchange interaction which has originally been proposed by Van Vleck. Therefore, we derive the anisotropic exchange interaction on the basis of the perturbation method and estimate the anisotropy constants K_1 and K_2 due to this interaction by use of the experimental g-values which are relating to the energy differences between the ground state and the excited states of Cu^{2+} ions. Thus it is shown that the sum of theoretical anisotropy constants due to both origins gives the values of the same order with that of the experimental values.

§ 1. Introduction

The anisotropy energy in an antiferromagnetic substance may be considered to consist of three parts : the first part arises from interactions of individual magnetic ions with their surrounding anisotropic crystalline field, the second part from magnetic dipole interactions between pairs of ions and the third part from the anisotropic exchange interactions between them. Magnetic dipole interactions do not produce any anisotropy in the first order approximation for crystals of cubic symmetry and the anisotropy arising from the crystalline field vanishes for crystals whose magnetic ions have a spin of 1/2.

According to the calculation by Keffer,[1] the anisotropy energy of MnF_2 having a crystal structure of rutile type arises mainly from the magnetic dipole interaction which makes spins point in the c-axis and those parts from the anisotropy of the crystalline field and the anisotropic exchange interaction amount only to ten percents of the experimental value. In the case of MnO_2, however, which is similar to MnF_2 in the crystal structure but different from it in the superstructure due to the antiferromagnetic spin orientation, the magnetic dipole interaction keeps the direction of spins in the plane perpendicular to the c-axis, but this interaction does not contribute to the anisotropy in this plane which may be considered to be arising from the crystalline field.[2] Therefore, both anisotropy from the magnetic dipole interaction and that from the anisotropy of the crystalline field are dominant in this substance. Thus, which is predominant among above mentioned three parts of anisotropy depends not only upon the crystal structure, but also upon the

superstructure due to the spin orientation, and so the mechanism of anisotropy in antiferromagnetic substances might distinctively be different from one substance to another.

Experimental determination of anisotropy constants in an antiferromagnetic substance is not so easy as ferromagnetic cases. One method for it consists of applying a sufficient field in the direction of the preferred axis and measuring the critical field strength under which spins rotate from the preferred axis to the direction perpendicular to it. Another method is by means of antiferromagnetic resonance experiments. Unfortunately, the critical field as well as the antiferromagnetic resonance absorption can not be observed by use of a magnetic field whose strength is less than 10^5 oersteds for ordinary antiferromagnetic substances having a high Néel temperature such as MnF_2, MnO_2 and MnO. Therefore, there is no other way to know the anisotropy constants for such substances below the Néel temperature than to estimate them by measuring the anisotropy of the susceptibility above the Néel temperature.

$CuCl_2 \cdot 2H_2O$ is only one substance for which the anisotropy constants below the Néel temperature are experimentally known. This substance has a very low Néel temperature and so the critical field and the antiferromagnetic resonance field can be observed under an ordinary field strength. These experiments have been made by Gorter and his collaborators[3] from whose experimental data we can deduce the anisotropy constants and so it would be interesting to calculate the anisotropy constants theoretically and to compare them with the experimental values for this substance.

The purpose of this paper is to calculate each part of the anisotropy energy separately from the view point of the molecular theory and to discuss its origin. Since a magnetic Cu^{2+} ion has a spin of 1/2, the anisotropy energy arising from the crystalline field does not exist and so we shall begin with calculation of the anisotropy arising from the magnetic dipole interaction, and then turn to the consideration for the anisotropic exchange interaction.

§ 2. Anisotropy arising from magnetic dipole interactions

$CuCl_2 \cdot 2H_2O$ has an orthorhombic symmetry and its lattice parameters are $a=7.38$ Å, $b=8.04$ Å and $c=3.72$ Å, and each copper ion has a spin orientation shown by Fig. 1 which has been determined from the nuclear resonance experiments by Poulis and Hardeman.[1]

We shall divide the lattice consisting of copper ions into four sublattices such that all the spins of each sublattice are parallel, and we shall denote the total magnetizations per unit volume of these four sublattices by $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$ and $\boldsymbol{M}_4$ as assigned in Fig. 1. Then x-, y- and z-components of the magnetic dipole field $\boldsymbol{H}_1$ produced at a lattice point of sublattice 1 by the magnetic ions on the other lattice points can be expressed by

$$\begin{aligned} H_{1x} &= \Phi_{a1} M_{1x} + \Phi_{a2} M_{2x} + \Phi_{a3} M_{3x} + \Phi_{a4} M_{4x}, \\ H_{1y} &= \Phi_{b1} M_{1y} + \Phi_{b2} M_{2y} + \Phi_{b3} M_{3y} + \Phi_{b4} M_{4y}, \\ H_{1z} &= \Phi_{c1} M_{1z} + \Phi_{c2} M_{2z} + \Phi_{c3} M_{3z} + \Phi_{c4} M_{4z}, \end{aligned} \tag{2·1}$$

and $\boldsymbol{H}_2$, $\boldsymbol{H}_3$ and $\boldsymbol{H}_4$ are respectively derived from $\boldsymbol{H}_1$ by an appropriate permutation

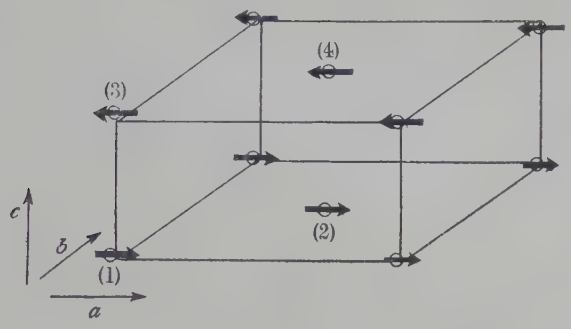

Fig. 1. A spin orientation in $CuCl_2 \cdot 2H_2O$

among $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$ and $\boldsymbol{M}_4$. In (2·1) Φ_{ai}, Φ_{bi} and Φ_{ci} are the following dipole sums :

$$\begin{aligned}
\Phi_{ai} &= -\frac{4}{N}\sum_j^{(i)}\left[1-3\cos^2(x, r_{1j})\right]r_{1j}^{-3}, \\
\Phi_{bi} &= -\frac{4}{N}\sum_j^{(i)}\left[1-3\cos^2(y, r_{1j})\right]r_{1j}^{-3}, \qquad (2\cdot 2) \\
\Phi_{ci} &= -\frac{4}{N}\sum_j^{(i)}\left[1-3\cos^2(z, r_{1j})\right]r_{1j}^{-3},
\end{aligned}$$

where $\sum_j^{(i)}$ is taken over all the distances r_{1j} between one lattice point of sublattice 1 nap lattice points j surrounding it situated on the i-th sublattice.

The anisotropy energy arising from the magnetic dipole interaction can easily be derived as the magnetic energy possessed by $\boldsymbol{M}_i$'s subjected to the magnetic dipole field $\boldsymbol{H}_i$,

$$F = -\frac{1}{2}\sum_i\{M_{ix}H_{ix}+M_{iy}H_{iy}+M_{iz}H_{iz}\}. \qquad (2\cdot 3)$$

According to the proton resonance absorption experiment by Poulis and Hardeman,[4] the magnetic moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, which are parallel to each other, point in the opposite direction to the magnetic moments $\boldsymbol{M}_3$ and $\boldsymbol{M}_4$ which are also parallel and so if we put $\boldsymbol{M}_1 = \boldsymbol{M}_2 = (\alpha M, \beta M, \gamma M)$ and $\boldsymbol{M}_3 = \boldsymbol{M}_4 = (-\alpha M, -\beta M, -\gamma M)$, where α, β and γ are the direction cosines of the magnetic moment $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$, and use (2·1) and the relations similar to it for $\boldsymbol{H}_i$ in (2·3), we obtain

$$\begin{aligned}
F = -2M^2\big[&(\Phi_{a1}-\Phi_{a3}+\Phi_{a2}-\Phi_{a4})\alpha^2+(\Phi_{b1}-\Phi_{b3}+\Phi_{b2}-\Phi_{b4})\beta^2 \\
&+(\Phi_{c1}-\Phi_{c3}+\Phi_{c2}-\Phi_{c4})\gamma^2\big] \qquad (2\cdot 4)
\end{aligned}$$

or apart from a term independent of the direction cosines,

$$\begin{aligned}
F = -2M^2\big[&\{(\Phi_{b1}-\Phi_{a1})+(\Phi_{b2}-\Phi_{a2})-(\Phi_{b3}-\Phi_{a3})-(\Phi_{b4}-\Phi_{a4})\}\beta^2 \\
&+\{(\Phi_{c1}-\Phi_{a1})+(\Phi_{c2}-\Phi_{a2})-(\Phi_{c3}-\Phi_{a3})-(\Phi_{c4}-\Phi_{a4})\}\gamma^2\big]. \qquad (2\cdot 4')
\end{aligned}$$

Therefore, if we write the anisotropy energy as

$$F=K_1\beta^2+K_2\gamma^2=N(\varkappa_1\beta^2+\varkappa_2\gamma^2), \tag{2·5}$$

where N is the total number of ions included in unit volume, the anisotropy constants per ion $\varkappa_1$ and $\varkappa_2$ arising from the magnetic dipole interaction become

$$\frac{K_1}{N}=\varkappa_1=-\frac{2M^2}{N}\{(\Phi_{b1}-\Phi_{a1})+(\Phi_{b2}-\Phi_{a2})-(\Phi_{b3}-\Phi_{a3})-(\Phi_{b4}-\Phi_{a4})\}, \tag{2·6a}$$

$$\frac{K_2}{N}=\varkappa_2=-\frac{2M^2}{N}\{(\Phi_{c1}-\Phi_{a1})+(\Phi_{c2}-\Phi_{a2})-(\Phi_{b3}-\Phi_{a3})-(\Phi_{b4}-\Phi_{a4})\}. \tag{2·6b}$$

The dipole sums of Φ_{ai}, Φ_{bi} and Φ_{ci} can be calculated by Ewald-Kornfeld's method.[5] The results of numerical calculations for them are as follows:

$$\begin{aligned}
&\Phi_{b1}-\Phi_{a1}=-1.61\,, && \Phi_{c1}-\Phi_{a}=-0.16\\
&\Phi_{b2}-\Phi_{a2}=2.52\,, && \Phi_{c2}-\Phi_{a2}=-10.99\\
&\Phi_{b3}-\Phi_{a3}=-1.20\,, && \Phi_{c3}-\Phi_{a3}=48.03\\
&\Phi_{b4}-\Phi_{a4}=1.56\,, && \Phi_{c4}-\Phi_{a4}=0.13\,.
\end{aligned} \tag{2·7}$$

Inserting (2·7) into (2·6 a, b), we obtain

$$\frac{K_1}{N}=\varkappa_1=-1.10\,\frac{M^2}{N}, \tag{2·8a}$$

$$\frac{K_2}{N}=\varkappa_2=118.6\,\frac{M^2}{N}. \tag{2·8b}$$

If we denote the saturation magnetization of one sublattice per unit volume at the absolute zero of temperature by M_0, which is equal to $N\beta/4$, β being the Bohr magneton, (2·8 a, b) are expressed by

$$\varkappa_1=-2.70\times10^{-4}\left(\frac{M}{M_0}\right)^2\mathrm{cm}^{-1},\quad \varkappa_2=2.91\times10^{-2}\left(\frac{M}{M_0}\right)^2\mathrm{cm}^{-1}. \tag{2·9 a, b}$$

Here $(M/M_0)^2$ may be considered to be nearly equal to unity in a sufficiently low temperature region, and so in its region (2·9 a, b) taken $(M/M_0)^2$ as 1, become the anisotropy constants per ion arising from the magnetic dipole interaction. On the other hand, the values of $\varkappa_1$ and $\varkappa_2$ deduced from the experimental results of antiferromagnetic resonance absorption by Ubbink, Poulis, Gerritsen, and Gorter[6),7)] are

$$\varkappa_1\cong 8\times10^{-3}\,\mathrm{cm}^{-1},\quad \varkappa_2\cong 2.6\times10^{-2}\,\mathrm{cm}^{-1} \tag{2·10 a, b}$$

at the lowest temperature at which these experiments were made. Comparing (2·9 a, b) with (2·10 a, b), it would be seen that the theoretical value of the anisotropy constant in the ac-plane $\varkappa_2$ arising from the magnetic dipole interaction is of the same order of magnitude with the experimental value, but that in the ab-plane $\varkappa_1$ is not only too small to explain the experimental one, but also has an opposite sign. Therefore, the anisotropy coming from the other origin, i.e., the anisotropic exchange interaction must be considered to be important, especially for $\varkappa_1$, in the case of $CuCl_2\cdot 2H_2O$.

Since the anisotropic exchange interaction is dependent upon the excited states as well as the ground state of a magnetic ion, we shall consider the energy states of a Cu^{2+} ion under the influence of the surrounding crystalline field before turning to the discussion on the anisotropic exchange interaction.

§ 3. Energy levels and the anisotropy of the g-factor of a Cu^{2+} ion

A Cu^{2+} ion of $CuCl_2 \cdot 2H_2O$ is subjected to the crystalline field from the surrounding ions. One of the principal axes of this field having a nearly orthorhombic symmetry the η-axis coincides with the b-axis and the other two, i.e., the ξ-axis and the ζ-axis lie in the ac-plane and make an angle α for an ion on the corner site and $-\alpha$ for one on the face center site with the a-axis and the c-axis, respectively.

The precise calculation of the energy levels and the anisotropy of the g-factor of Cu^{2+} ion under such a crystalline field has been made by Itoh and Fujimoto.[8] However, on account of simplicity, we shall here adopt a simpler assumption that the crystalline field can be represented by

$$V(\xi, \eta, \zeta) = D(\xi^4 + \eta^4 + \zeta^4) + A(\xi^2 + \eta^2 - 2\zeta^2) + C(\xi^2 - \eta^2), \qquad (3 \cdot 1)$$

referring to the principal axes of the crystalline field, where the first term represents a cubic part and the second and the third terms are tetragonal and orthorhombic parts.

The lowest state of a free Cu^{2+} ion is ^2D-state, corresponding to a closed $3d$ shell minus one electron. Ignoring the spin, the D state is fivefold degenerate in the free state, and these five normalized wave functions with $L=2$ may, as usual, be denoted by φ_2, φ_1, φ_0, φ_{-1}, φ_{-2} corresponding to $m=2, 1, 0, -1, -2$, m representing the component of L with respect to the ζ-axis. This fivefold degenerate state is splitted into five non-degenerate states by the orthorhombic crystalline field of $(3 \cdot 1)$. These splitted five wave functions and their energy levels can be calculated as follows:

wave function	energy level
$\psi_g = C_2 \frac{1}{\sqrt{2}}(\phi_2 + \phi_{-2}) + C_0 \phi_0$,	$E_g = \frac{5}{7}D - \frac{4}{21}(3\bar{C}^2 + 9\bar{A}^2)^{1/2}$,
$\psi_h = C_2' \frac{1}{\sqrt{2}}(\phi_2 + \phi_{-2}) + C_0' \phi_0$,	$E_h = \frac{5}{7}\bar{D} + \frac{4}{21}(3\bar{C}^2 + 9\bar{A}^2)^{1/2}$,
$\psi_4 = \frac{1}{\sqrt{2}\,i}(\phi_2 - \phi_{-2})$,	$E_4 = -\frac{11}{21}\bar{D} + \frac{4}{7}\bar{A}$,
$\psi_2 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_{-1})$,	$E_2 = -\frac{11}{21}\bar{D} - \frac{2}{7}\bar{A} - \frac{2}{7}\bar{C}$,
$\psi_5 = \frac{1}{\sqrt{2}\,i}(\phi_1 - \phi_{-1})$,	$E_5 = -\frac{11}{21}\bar{D} - \frac{2}{7}\bar{A} + \frac{2}{7}\bar{C}$,

$(3 \cdot 2)$

where $\bar{D}$, $\bar{A}$ and $\bar{C}$ represent $eD\langle r^4 \rangle_{av.}$, $eA\langle r^2 \rangle_{av.}$ and $eC\langle r^2 \rangle_{av.}$ respectively, and C_2, C_0, C_2' and C_0' are the constants determined by the following relation:

$$\frac{C_0}{C_2}=\frac{3\bar{A}+(3\bar{C}^2+9\bar{A}^2)^{1/2}}{\sqrt{3}\,\bar{C}}, \quad \frac{C_0'}{C_2'}=\frac{3\bar{A}-(3\bar{C}^2+9\bar{A}^2)^{1/2}}{\sqrt{3}\,\bar{C}}, \qquad (3\cdot3\,\mathrm{a,b})$$

and the normalization conditions,

$$|C_2|^2+|C_0|^2=1\,, \quad |C_2'|^2+|C_0'|^2=1\,. \qquad (3\cdot4\,\mathrm{a,b})$$

From (3·3 a, b) and (3·4 a, b), we can obtain the real solutions of C_2, C_0, C_2' and C_0', and so we can select the wave functions as real.

Polder[9] has shown that for the case of Cu-Tutton salt where a Cu^{2+} ion may be approximated to be subjected to the tetragonal crystalline field, corresponding to the case of vanishing $\bar{C}$ or $\bar{A}=\pm(1/3)\bar{C}$ in our case, the lowest state is corresponding to ψ_g or ψ_h. Therefore, we shall assume that the lowest state of a Cu^{2+} ion of $CuCl_2\cdot 2H_2O$ is also ψ_g in this case.

If we now introduce spin functions α and β corresponding to the components of S, $+(1/2)$ and $-(1/2)$ with respect to the ζ–axis, the ground state ψ_g is still doubly degenerate, and these two wave functions $\psi_g\cdot\alpha$ and $\psi_g\cdot\beta$ including spin functions are modified by the $L-S$ coupling in the first order as the following forms:

$$f_g^+=\{\frac{C_2}{\sqrt{2}}(\phi_2+\phi_{-2})+C_0\phi_0\}\alpha+\frac{\gamma_2}{2}(\sqrt{3}\,C_0-C_2)\frac{1}{\sqrt{2}}(\phi_1+\phi_{-1})\beta$$
$$-i\gamma_4 C_2\frac{1}{\sqrt{2}\,i}(\phi_2-\phi_{-2})\alpha+i\frac{\gamma_5}{2}(\sqrt{3}\,C_0+C_2)\frac{1}{\sqrt{2}\,i}(\phi_1-\phi_{-1})\beta\,, \qquad (3\cdot5\mathrm{a})$$

$$f_g^-=\{\frac{C_2}{\sqrt{2}}(\phi_2+\phi_{-2})+C_0\phi_0\}\beta+\frac{\gamma_2}{2}(C_2-\sqrt{3}\,C_0)\frac{1}{\sqrt{2}}(\phi_1+\phi_{-1})\cdot\alpha$$
$$+i\gamma_4 C_2\frac{1}{\sqrt{2}\,i}(\phi_2-\phi_{-2})\beta+i\frac{\gamma_5}{2}(C_2+\sqrt{3}\,C_0)\frac{1}{\sqrt{2}\,i}(\phi_1-\phi_{-1})\alpha\,, \qquad (3\cdot5\mathrm{b})$$

where

$$\gamma_i=\frac{\lambda}{E_i-E_g}\,. \qquad (3\cdot6)$$

The degeneracy of these two states is not still lifted until the external field is introduced.

The paramagnetic susceptibility χ sufficiently above the Néel temperature is given by the following formula:

$$\chi_i=\frac{M_i}{H_i}=N\beta^2\left\{\sum_{mm'}\frac{|(L_i+2S_i)_{mm'}|^2}{g_m kT}+\sum_{mn}\frac{2|(L_i+2S_i)_{mn}|^2}{g_m(E_n-E_m)}\right\}, \qquad (3\cdot7)$$

where i means ξ, η and ζ. In our case, m and m' represent two degenerate ground states and n represents the higher states, and the number of degeneracy of the ground state g_m must be taken as 2. The matrix elements of the first term of (3·7) expressing the temperature dependent part of the susceptibility become, using (3·5 a, b) for the wave functions of the ground states,

$$
\begin{aligned}
&(f_g^+|L_\zeta+2S_\zeta|f_g^+)=1-4\gamma_4 C_2^2,\\
&(f_g^+|L_\zeta+2S_\zeta|f_g^-)=0\,,\\
&(f_g^-|L_\zeta+2S_\zeta|f_g^-)=-(1-4\gamma_4 C_2^2),
\end{aligned}
\tag{3·8a}
$$

$$
\begin{aligned}
&(f_g^+|L_\xi+2S_\xi|f_g^+)=0\,,\\
&(f_g^+|L_\xi+2S_\xi|f_g^-)=1-\gamma_5(C_2+\sqrt{3}\,C_0)^2,\\
&(f_g^-|L_\xi+2S_\xi|f_g^-)=0\,,
\end{aligned}
\tag{3·8b}
$$

$$
\begin{aligned}
&(f_g^+|L_\eta+2S_\eta|f_g^+)=0\,,\\
&(f_g^+|L_\eta+2S_\eta|f_g^-)=-i\{1-\gamma_2(C_2-\sqrt{3}\,C_0)^2\},\\
&(f_g^-|L_\eta+2S_\eta|f_g^-)=0\,.
\end{aligned}
\tag{3·8c}
$$

If we use (3·8 a, b, c) in the first term of (3·7), we obtain, as the temperature dependent part of three principal susceptibilities, the following relations:

$$\chi_\zeta=\frac{N\beta^2}{kT}(1-4\gamma_4 C_2^2)^2, \tag{3·9a}$$

$$\chi_\xi=\frac{N\beta^2}{kT}\{1-\gamma_5(C_2+\sqrt{3}\,C_0)^2\}^2, \tag{3·9b}$$

$$\chi_\eta=\frac{N\beta^2}{kT}\{1-\gamma_2(C_2-\sqrt{3}\,C_0)^2\}^2. \tag{3·9c}$$

From these relations, the principal values of the g-tensor become

$$g_\zeta=2(1-4\gamma_4 C_2^2), \tag{3·10a}$$

$$g_\xi=2\{1-\gamma_5(C_2+\sqrt{3}\,C_0)^2\}, \tag{3·10b}$$

$$g_\eta=2\{1-\gamma_2(C_2-\sqrt{3}\,C_0)^2\}, \tag{3·10c}$$

which are related to the principal g-values g_z, g_x and g_y with respect to the crystal axes z, x and y, by the following relations:

$$g_z=(g_\xi^2\sin^2\alpha+g_\zeta^2\cos^2\alpha)^{1/2}, \tag{3·11a}$$

$$g_x=(g_\xi^2\cos^2\alpha+g_\zeta^2\sin^2\alpha)^{1/2}, \tag{3·11b}$$

$$g_y=g_\eta\,, \tag{3·11c}$$

where α is an angle between the ξ-axis and the x-axis.

According to the experiments of the paramagnetic resonance absorption at room temperature by Itoh, Fujimoto and Ibamoto,[10] $g_z=2.260$, $g_x=2.195$ and $g_y=2.075$ and, on the other hand, according to the measurements of the susceptibilities above the Néel temperature by Van den Handel, Gijsman and Poulis,[11] $g_z=2.24$, $g_x=2.20$ and $g_y=2.03$. Since the anisotropy of the g-values obtained from the susceptibility measurements possibly includes also the anisotropy from the other origins than the proper anisotropy of g-factor, we shall here adopt the values from the magnetic resonance absorption, and assume that the ξ-axis coincides with the axis joining a Cu^{2+} ion and its nearest neighboring Cl ion.

Then, using $\cos^2\alpha=0.6425$, we obtain from (3·11 a, b, c), $g_\xi=2.339$, $g_\xi=2.111$ and $g_\eta=2.075$ and from these g-values and (3·10 a, b, c)

$$4\gamma_4^2 C_2^2=\frac{7.174\times10^{-3}}{C_2^2}, \tag{3·12a}$$

$$\gamma_5^2(C_2+\sqrt{3}\,C_0)^2=\frac{3.056\times10^{-3}}{(C_2+\sqrt{3}\,C_0)^2}, \tag{3·12b}$$

$$\gamma_2^2(C_2-\sqrt{3}\,C_0)^2=\frac{1.406\times10^{-3}}{(C_2-\sqrt{3}\,C_0)^2}. \tag{3·12c}$$

The values of γ_i estimated in such a way will later be used for the estimation of the anisotropy constants from the anisotropic exchange interaction.

§ 4. Formulation of the anisotropic exchange interaction

As has been pointed out by Van Vleck,[12), 13)] the anisotropic exchange interaction appears between two magnetic ions through their excited states. Van Vleck[13)] has derived this interaction by the second order perturbation calculation. In this section, however, we shall give a little more general formulation of this interaction.

We shall now consider a pair of two Cu ions, take five states represented by (3·2) as the zeroth order eigenfunctions of each Cu ion and calculate the energy of the lowest state of two ion system by a perturbation method, considering the following energy:

$$V=\lambda(L_1\cdot S_1)+\lambda(L_2\cdot S_2)+V_{\text{exch.}} \tag{4·1}$$

as a perturbation, where the first and the second terms are respectively representing the $L-S$ coupling of Cu ion 1 and Cu ion 2, and the last term represents the exchange operator between these two ions. In this case, the perturbation energy of the first order becomes the ordinary exchange energy for the case where both ions are in their ground states.

The second order perturbation energy can be expressed by

$$\Delta_2 V=-\sum_{i,s}\frac{(g,r|V|i,s)(i,s|V|g,r)}{(H_i-H_g)}, \tag{4·2}$$

where i and g denote the excited states and the ground state of two ion system respectively and r and s denote the spin states which are degenerate in the zeroth approximation. We shall hereafter take out only the diagonal terms and omit the off-diagonal terms of $V_{\text{exch.}}$ with respect to the orbital functions. Then, since the diagonal elements of (L, S) with respect to the orbital parts are zero for the real orbital wave functions, any linear term with respect to the exchange integral does not appear from (4·2). For the term which does not include the exchange integral, we may consider, as the excited states for two ion system, only the cases where the one of two ions is excited and the other remains in the ground state, as the $\lambda(LS)$ is relating to only one ion. Since each element of the matrix of (L_1S_1) inclusive of the part of spin states is equal to that of (L_2S_2), (4·2) becomes

$$-\sum_{i,s}\frac{(g_1 g_2 r|2\lambda(L_1\cdot S_1)|i_1 g_2, s)\,(i_1 g_2, s|2\lambda(L_1 S_1)|g_1 g_2, r)}{E_i - E_g}, \qquad (4\cdot3)$$

where i_1 represents the excited states of ion 1 and s represents the spin states of two spin system and E_i and E_g are respectively the energies of the excited states and that of the ground state expressed by $(3\cdot2)$. If we remain $\boldsymbol{S}$ as an operator, we can express $(4\cdot3)$ in the spin variables as follows:

$$-\sum_{i_1}\frac{4\lambda^2}{E_i - E_g}\{L_x(g_1, i_1)S_{1x} + L_y(g_1, i_1)S_{1y} + L_z(g_1, i_1)S_{1z}\}$$
$$\times\{L_x(i_1, g_1)S_{1x} + L_y(i_1, g_1)S_{1y} + L_z(i_1, g_1)S_{1z}\}, \qquad (4\cdot4)$$

where $L_x(g_1, i_1)$, etc., are the matrix elements of L with respect to the orbital wave functions of one ion. For the case of $S = 1/2$, $(4\cdot4)$ becomes a constant value independent of spin variables. Therefore, the perturbation of the second order does not produce a spin dependent energy term.

The perturbation energy of the third order can be expressed by

$$\Delta_3 V = -\sum_{i,s}\sum_{j,t}\frac{(g, r|V|i, s)\,(i, s|V|j, t)\,(j, t|V|g, r)}{(H_i - H_g)(H_g - H_j)}$$
$$-\sum_{i,s}\frac{(g, r|V|i, s)\,(i, s|V|g, r)\,(g, r|V|g, r)}{(H_i - H_g)^2}, \qquad (4\cdot5)$$

where i and j again represent the orbital excited states of two ion system and s and t represent their spin states. Also in this case we shall intend to express this energy in spin variables. The second term of $(4\cdot5)$ has a diagonal element with respect to the orbital state only in the third factor and so for this factor the part from $V_{\text{exch.}}$ of the perturbation energy remains but the part from $\lambda(LS)$ disappears, and the latter part remains but the former disappears for the other two factors so far as the off-diagonal elements of $V_{\text{exch.}}$ with respect to the orbital wave functions are omitted. Hence, confining to the linear parts with respect to the exchange integral, the second term of $(4\cdot5)$ can be expressed by

$$-\sum_{i,s}\frac{(g, r|\lambda(L_1 S_1) + \lambda(L_2 S_2)|i, s)\,(i, s|\lambda(L_1 S_1) + \lambda(L_2 S_2)|g, r)\,(g, r|V_{\text{exch.}}|g, r)}{(H_i - H_g)^2}. \qquad (4\cdot6)$$

If we take into consideration the fact that the matrix elements of $(L_1 S_1)$ are equal to those of $(L_2 S_2)$ and that the diagonal elements of $V_{\text{exch.}}$ with respect to the orbital states can be expressed by $-(1/2)J_{g_1 g_2}\{1 + 4(S_1 S_2)\}$, where $J_{g_1 g_2}$ is an exchange integral between two ions which are both in each ground state, $(4\cdot6)$ can be written, in spin variables, as

$$\sum_{i_1}\frac{2J_{g_1 g_2}\lambda^2}{(E_i - E_g)^2}\{L_{1x}(g_1, i_1)S_{1x} + L_{1y}(g_1, i_1)S_{1y} + L_{1z}(g_1, i_1)S_{1z}\}$$
$$\times\{L_{1x}(i_1, g_1)S_{1x} + L_{1y}(i_1, g_1)S_{1y} + L_{1z}(i_1, g_1)S_{1z}\}(1 + 4S_1\cdot S_2)$$

$$= \frac{J_{i_1 g_2}}{2} \lambda^2 \sum_{i_1} \frac{1}{(E_i - E_g)^2} \{|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2\} (1 + 4 S_1 \cdot S_2), \qquad (4 \cdot 7)$$

where $L_{1x}(g_1, i_1) \cdot L_{1x}(i_1, g_1)$, etc., are written as $|L_{1x}|^2$, etc. Thus, from the second term of (4·5) appears only an isotropic exchange coupling.

The first term of (4·5) can be written as

$$\sum_{i,s} \sum_{t} \frac{(g, r|V|i, s)(i, s|V|i, t)(i, t|V|g, r)}{(H_i - H_g)^2}$$

$$- \sum_{i,s} \sum_{\substack{j,t \\ (i \neq j)}} \frac{(g, r|V|i, s)(i, s|V|j, t)(j, t|V|g, r)}{(H_i - H_g)(H_g - H_j)} . \qquad (4 \cdot 8)$$

We can, here, leave out of consideration the second term, so far as we omit the off-diagonal elements of $H_{\text{exch.}}$ and pick out only the linear part with respect to the exchange integral. Replacing the middle factor in the first term of (4·8) with $-(1/2) J_{i_1 g_2}(1 + 4 S_1 S_2)$ where $J_{i_1 g_2}$ is an exchange integral between two ions, one being in an excited state i_1 and the other in the ground state g_2, and remaining the spin as an operator, we obtain

$$-2\lambda^2 \sum_{i_1} \frac{J_{i_1 g_2}}{(E_i - E_g)^2} \{L_{1x}(g_1, i_1) S_{1x} + L_{1y}(g_1, i_1) S_{1y} + L_{1z}(g_1, i_1) S_{1z}\}$$

$$\times (1 + 4 S_1 \cdot S_2) \{L_{1x}(i_1, g_1) S_{1x} + L_{1y}(i_1, g_1) S_{1y} + L_{1z}(i_1, g_1) S_{1z}\} . \qquad (4 \cdot 9)$$

For $S = 1/2$, this can be rewritten as

$$-\sum_{i_1} \frac{J_{i_1 g_2} \lambda^2}{2(E_i - E_g)^2} \Bigg[\frac{1}{4} (|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2) - \frac{1}{3} (|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2)(S_1 \cdot S_2)$$

$$+ \frac{i}{2} \{(L_{1y} L_{1z}^* - L_{1z} L_{1y}^*)(S_{1z} - S_{2z}) + (L_{1z} L_{1x}^* - L_{1x} L_{1z}^*)(S_{1y} - S_{2y})$$

$$+ (L_{1x} L_{1y}^* - L_{1y} L_{1x}^*)(S_{1z} - S_{2z})\} - \frac{2}{3} \{(|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2 - 3|L_{1x}|^2) S_{1x} \cdot S_{2x}$$

$$+ (|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2 - 3|L_{1y}|^2) S_{1y} S_{2y}$$

$$+ (|L_{1x}|^2 + |L_{1y}|^2 + |L_{1z}|^2 - 3|L_{1z}|^2) S_{1z} S_{2z} - \frac{3}{2} (L_{1x} L_{1y}^* + L_{1y} L_{1x}^*)(S_{1y} S_{2x} + S_{1x} S_{2y})$$

$$- \frac{3}{2} (L_{1y} L_{1z}^* + L_{1z} L_{1y}^*)(S_{1z} S_{2y} + S_{1y} S_{2z}) - \frac{3}{2} (L_{1z} L_{1x}^* + L_{1x} L_{1z}^*)(S_{1x} S_{2y} + S_{1y} S_{2x})\} \Bigg], \qquad (4 \cdot 10)$$

where we have written $L(g_1, i_1)$ simply as L and $L(i_1, g_1)$ as L^*. In (4·10), the first term gives only a constant term independent of spin variables, the second term gives an isotropic coupling and the third term vanishes for the real orbital wave functions. The anisotropic coupling between two spins arises from the last part of (4·10). This part would coincide with the expression given by Van Vleck[13] when the states of two ions are invariant under the rotation around the axis joining two ions.

§ 5. Estimation of the anisotropy constants arising from the anisotropic exchange interaction

It is seen from (4·10) that the following anisotropic exchange interaction exists between two ions:

$$\frac{1}{3}\sum_{i_1}\frac{J_{i_1 0 2}\lambda^2}{(E_i-E_g)^2}\{(|L_{1x}|^2+|L_{1y}|^2+|L_{1z}|^2-3|L_{1x}|^2)S_{1x}S_{2x}$$
$$+(|L_{1x}|^2+|L_{1y}|^2+|L_{1z}|^2-3|L_{1y}|^2)S_{1y}S_{2y}+(|L_{1x}|^2+|L_{1y}|^2+|L_{1z}|^2-3|L_{1z}|^2)S_{1z}S_{2z}$$
$$-3L_{1x}L_{1y}^*(S_{1y}S_{2x}+S_{1x}S_{2y})-3L_{1y}L_{1z}^*(S_{1z}S_{2y}+S_{1y}S_{2z})-3L_{1z}L_{1x}^*(S_{1x}S_{2z}+S_{1z}S_{2x})\}. \tag{5·1}$$

This form of the anisotropic exchange interaction would also be valid for the superexchange interaction, so far as $V_{\text{exch.}}$ can be expressed by $-(1/2)J(1+4S_1S_2)$.

In order to obtain the anisotropy constants arising from the interaction of (5·1), we shall denote the components of the spins on $+$sublattice by αS_0, βS_0 and γS_0 and those on $-$sublattice by $-\alpha S_0$, $-\beta S_0$ and $-\gamma S_0$, where α, β and γ are the direction cosines of $\boldsymbol{M}^+$. Then from (5·1) the anisotropy energy per ion can be obtained, apart from the constant term independent of α, β and γ, as

$$f_a=S_0^2\sum_{i_1}[\sum_{++}J_{i_1 0 2}-\sum_{+-}J_{i_1 0 2}]\frac{\lambda^2}{(E_i-E_g)^2}\{(|L_{1x}|^2-L_{1y}|^2)\beta^2$$
$$+(|L_{1x}|^2-|L_{1z}|^2)\gamma^2-2L_{1x}L_{1y}^*\alpha\beta-2L_{1y}L_{1z}^*\beta\gamma-2L_{1z}L_{1x}^*\gamma\alpha\}, \tag{5·2}$$

where $\sum_{++}$ is taken over the ions with the spin of the same direction and $\sum_{+-}$ is taken over those with the spin of the opposite direction. To estimate the coefficients of β^2, γ^2, $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$, we shall first consider L_{1x}, L_{1y}, and L_{1z}. There are the following relations between L_x, L_y, L_z and L_ξ, L_η, L_ζ:

$$\begin{aligned} L_x&=L_\xi\cos\alpha\mp L_\zeta\sin\alpha,\\ L_y&=L_\eta,\\ L_z&=\pm L_\xi\sin\alpha+L_\zeta\cos\alpha, \end{aligned} \tag{5·3}$$

where the upper sign is taken for an ion on the corner site and the lower sign is taken for that on the face-center site. Using the wave functions of (3·2), the matrix elements of $L_{1\xi}$, $L_{1\eta}$, $L_{1\zeta}$ can be calculated as follows:

$$\begin{aligned} &(g|L_{1\xi}|2)=0, \quad (g|L_{1\xi}|4)=0, \quad (g|L_{1\xi}|5)=i(C_2+\sqrt{3}C_0),\\ &(g|L_{1\eta}|2)=i(C_2-\sqrt{3}C_0), \quad (g|L_{1\eta}|4)=0, \quad (g|L_{1\eta}|5)=0,\\ &(g|L_{1\zeta}|2)=0, \quad (g|L_{1\zeta}|4)=-2C_2i, \quad (g|L_{1\zeta}|5)=0. \end{aligned} \tag{5·4}$$

From (5·4) and (5·3) we obtain

$$\begin{aligned} L_{1x}L_{1z}^*&=\pm(|L_{1\xi}|^2-|L_{1\zeta}|^2)\sin\alpha\cos\alpha,\\ L_{1y}L_{1x}^*&=0\\ L_{1z}L_{1y}^*&=0 \end{aligned} \tag{5·5}$$

$$|L_{1x}|^2-|L_{1y}|^2=|L_{1\xi}|^2\cos^2\alpha+|L_{1\zeta}|^2\sin^2\alpha-|L_{1\eta}|^2,$$
$$|L_{1x}|^2-|L_{1z}|^2=(|L_{1\xi}|^2-|L_{1\zeta}|^2)(\cos^2\alpha-\sin^2\alpha). \qquad (5\cdot6)$$

If we put (5·5) and (5·6) into (5·2) and take an average over two kinds of ions corresponding to $\pm\alpha$, the anisotropy energy can be written in the form like (2·5), where $\varkappa_1$ and $\varkappa_2$ are

$$\varkappa_1=S_0^2\sum_{i_1}\gamma_i^2\Big[\sum_{++}J_{i_1 0 i_2}-\sum_{+-}J_{i_1 0 i_2}\Big]\{|L_{1\xi}|^2\cos^2\alpha+|L_{1\zeta}|^2\sin^2\alpha-|L_{1\eta}|^2\}, \qquad (5\cdot7a)$$

$$\varkappa_2=S_0^2\sum_{i_1}\gamma_i^2\Big[\sum_{++}J_{i_1 0 i_2}-\sum_{+-}J_{i_1 0 i_2}\Big]\{(|L_{1\xi}|^2-|L_{1\zeta}|^2)(\cos^2\alpha-\sin^2\alpha)\}. \qquad (5\cdot7b)$$

$J_{i_1 0 i_2}$ is very much difficult to be estimated and so we shall replace it with $J_{g_1 0 i_2}$ which is the exchange integral between two ions being in their ground states, and take it out before the summation and use (5·4) for $L_{1\xi}$, $L_{1\eta}$ and $L_{1\zeta}$. Then (5·7 a, b) become

$$\varkappa_1=S_0^2\Big\{\sum_{++}J_{g_1 0 i_2}-\sum_{+-}J_{g_1 0 i_2}\Big\}\{\gamma_5^2(C_2+\sqrt{3}\,C_0)^2\cos^2\alpha+4\gamma_4^2C_2^2\sin^2\alpha-\gamma_2^2(C_2-\sqrt{3}\,C_0)^2\}, \qquad (5\cdot8a)$$

$$\varkappa_2=S_0^2\Big\{\sum_{++}J_{g_1 0 i_2}-\sum_{+-}J_{g_1 0 i_2}\Big\}\{\gamma_5^2(C_2+\sqrt{3}\,C_0)^2-4\gamma_4^2C_2^2\}(\cos^2\alpha-\sin^2\alpha). \qquad (5\cdot8b)$$

The values of the coefficients of $\sin^2\alpha$ and $\cos^2\alpha$ in (5·8 a, b) can be estimated by (3·12 a, b, c) but C_0 and C_2 remain still undeterminate. Therefore, we assume that the denominators in the right hand side of (3·12) are of the order of unity. Then (5·8 a, b) become

$$\varkappa_1=S_0^2\Big[\sum_{++}J_{g_1 0 i_2}-\sum_{+-}J_{g_1 0 i_2}\Big]\times3.127\times10^{-3}, \qquad (5\cdot9a)$$

$$\varkappa_2=-S_0^2\Big[\sum_{++}J_{g_1 0 i_2}-\sum_{+-}J_{g_1 0 i_2}\Big]\times1.176\times10^{-3}. \qquad (5\cdot9b)$$

Estimating $\sum_{++}J_{g_1 0 i_2}-\sum_{+-}J_{g_1 0 i_2}$ as $2kT_N$ where T_N is 4.3°K and putting $S_0=1/2$, the anisotropy constants per ion $\varkappa_1$ and $\varkappa_2$ are obtained as

$$\varkappa_1\cong4.51\times10^{-3}\,\mathrm{cm}^{-1},\quad \varkappa_2\cong-1.70\times10^{-3}\,\mathrm{cm}^{-1}, \qquad (5\cdot10\,a, b)$$

$\varkappa_1$ obtained in such a way is of the same order with the experimental value and on the other hand $\varkappa_2$ is much smaller than an experimental one which is of the same order with that from the magnetic dipole interaction.

§ 6. Concluding remarks

The anisotropy energy in $CuCl_2\cdot2H_2O$ is arising from the magnetic dipole interaction and the anisotropic exchange interaction. The anisotropy constants per ions $\varkappa_1$ and $\varkappa_2$ from these two origins estimated above are shown in the following table:

	Magnetic dipole interaction	Anisotropic exchange interaction	Experimental value
$\varkappa_1$	$-2.7\times10^{-4}\,\mathrm{cm}^{-1}$	$4.51\times10^{-3}\,\mathrm{cm}^{-1}$	$8\times10^{-3}\,\mathrm{cm}^{-1}$
$\varkappa_2$	$2.91\times10^{-2}\,\mathrm{cm}^{-1}$	$-1.70\times10^{-3}\,\mathrm{cm}^{-1}$	$2.6\times10^{-3}\,\mathrm{cm}^{-1}$

It may be concluded from this table that the magnetic dipole interaction contributes mainly to $\varkappa_2$ and the anisotropic exchange interaction contributes mainly to $\varkappa_1$.

It is, however, noted that in estimating the anisotropy constants $\varkappa_1$ and $\varkappa_2$ arising from the anisotropic exchange interaction we have made the several rather crude assumptions. The main ones are the followings:

(1) The off-diagonal elements of the exchange operator with respect to the orbital wave functions have been omitted in deriving the anisotropic exchange interaction.

(2) The exchange integrals between one ion in its excited state and the other in its ground state have been replaced with that between ions both being in their ground states.

(3) We have assumed that the denominators of the right hand side of (3·12) are of the order of unity, on account of the difficulty in estimating them.

(4) The ξ-axis has been assumed to coincide with the axis joining a Cu ion and its nearest neighboring Cl ion.

It is hoped that these assumptions are got rid of by richer experimental data or more precise calculations and a more rigorous estimation is made for the anisotropy energy arising from the anisotropic exchange interaction.

In conclusion, the authors are indebted to Mr. Kanamori and Miss Motizuki for checking the numerical calculation in § 2.

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Letters to the Editor

On the Evaluation of the Operator Function $\log(e^y e^z)$

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Hitherto we have met a great difficulty in obtaining an explicit expression of the operator function $\log(e^y e^z)$ of two arbitrary non-commuting operators y and z. Intending to work out this problem the author was previously[1] led to rather a disappointing result of introducing the so-called expansional operator. However some half a century ago, the same problem had already been attacked by Hausdorff,[2] who had attained a correct result after lengthy and rather clumsy procedures. Suggested by his work the author has recently arrived at the following simple solution which is based exclusively on the commutator operation. Although the present one is nothing but a reformulation of Hausdorff's result, the author wishes to make up the preceding paper by touching afresh upon the problem in which quantum-physicists have in general been so much interested.

The key to the present solution consists in assuming that $\log(e^y e^z)$ is simply given by the following transformation of y in terms of an auxiliary operator u:

$$\log(e^y e^z) = e^{-u} y e^u = \{y, e^u\}, \qquad (1)$$

where use is made of the abbreviation that

$$\{y, u^n\} = [\{y, u^{n-1}\}, u]$$

with $\{y, 1\} = y$ and $[y, u] = yu - uy$. By exponentiation we get from eq. (1)

$$e^y e^z = e^{-u} e^y e^u = \{e^y, e^u\}, \qquad (2)$$

on the basis of which we are then only to determine the generating operator u. There is no need however to find out the u itself that should not appear in the final result and we are solely interested in its commutators with y and z which are the only ones essentially operative in eq. (1). Although eq. (2) apparently implies that the u should fulfil an infinite number of simultaneous equations

$$e^y z^n = \{e^y, u^n\} \quad (n = 1, 2, \cdots),$$

these are readily seen to reduce to only the two conditions

$$[e^y, u] = e^y z$$

and

$$[z, u] = 0. \qquad (3)$$

The former equation can further be transcribed as

$$z = e^{-y}[e^y, u] = \{u, (1 - e^y)\} = \{[y, u], (e^y - 1)/y\},$$

from which we have at once

$$[y, u] = \{z, y/(e^y - 1)\} = z - (1/2)[z, y] - \sum_{n=1}^{\infty} (-1)^n B_n \{z, y^{2n}\}/(2n)!, \qquad (4)$$

where B_n is the n-th Bernoullian number. In quite the same way the alternative assumption

$$\log(e^y e^z) = e^{-v} z e^v = \{z, e^v\}$$

leads to the commutators

$$[y, v] = 0 \text{ and } [z, v] = \{y, z/(1 - e^{-z})\}.$$

So far as we are concerned with an abstract scheme as the above, the generating operator u remains completely undetermined playing no more than a subsidiary part to effectuate a consistently operational treatment of the present problem. It is important however to observe that, provided that the u is a well-defined one, eq. (2) offers an essentially new type of relationship that is expected to be helpful in the calculations of the quantized field theory. In order that eqs. (3) and (4) may further be soluble with respect to u, it is necessary not only to define the operators y and z but also to investigate the whole gamut of operators related to them. In this connection it may be useful to notice that in terms of an operator u' satisfying the relation

$$[y, u'] = z \qquad (4')$$

the u can be expressed as

$$u = \{u', y/(e^y - 1)\}. \qquad (5)$$

Finally the author wishes to express his sincere thanks to Mr. S. Tani for having presented Hausdorff's paper to him.

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On the Electronic Polaron State

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A conduction electron in insulating or semiconducting crystal accompanies electronic polarization around itself, so that the true (or observed) bottom of the conduction band must be lower than the fictitious position which should be obtained without interaction. As the electronic polarization is to be caused, according to the band picture of crystal electrons, by the virtual excitation of filled band electrons to the conduction band, the above self-energy is analogous to that of the Dirac electron due to the vacuum polarization. On the other hand, the electronic polarization induced by the conduction electron cannot perfectly follow the motion of the latter, especially for high velocities.

Abstracting the most essential aspects of these situations we can simplify the problem by assuming the Hamiltonian of the total system (an additive electron+crystal electrons) in the form:

$$H=\frac{1}{2m}\boldsymbol{p}^2+\varepsilon\Sigma_{\boldsymbol{w}}a_{\boldsymbol{w}}^*a_{\boldsymbol{w}}+\frac{e\gamma}{\sqrt{L^3}}\Sigma_{\boldsymbol{w}}\frac{i}{w}\times\{a_{\boldsymbol{w}}\exp(i\boldsymbol{w}\cdot\boldsymbol{r})-a_{\boldsymbol{w}}^*\exp(-i\boldsymbol{w}\cdot\boldsymbol{r})\} \quad (1)$$

where $\boldsymbol{p}$ is the momentum of the conduction electron, ε and $a_{\boldsymbol{w}}$, $a_{\boldsymbol{w}}^*$ mean the energy (assumed not to depend on $\boldsymbol{w}$) and the operators of second quantization for the production of an exciton with wave number $\boldsymbol{w}$, $a_{\boldsymbol{w}}^*a_{\boldsymbol{w}}$ taking values $n_{\boldsymbol{w}}=1$ or 0 according as the exciton exists or not. The interaction constant γ is to be related with high-frequency dielectric constant $\varkappa_0$ as follows:

$$\gamma^2=2\pi\varepsilon\left(1-\frac{1}{\varkappa_0}\right).$$

Taking advantage of the commutativity of total momentum with (1), the characteristic wave function $\Psi(\boldsymbol{k}\,|\,\boldsymbol{r};\cdots,m_{\boldsymbol{w}},\cdots)$ and energy $E(\hbar\boldsymbol{k})$ of (1) have been calculated ($\hbar\boldsymbol{k}$ being the value of the total momentum) both to the second order of perturbation, starting from the non-perturbed state in which the electron has momentum $\hbar\boldsymbol{k}$ and no exciton exists. For instance the energy becomes

$$E(\hbar\boldsymbol{k})=-a\varepsilon+\frac{\hbar^2k^2}{2m^*}+0(k^4), \quad (2)$$

where a and m^* are given by

$$a=\frac{1}{2}\left(1-\frac{1}{\varkappa_0}\right)\frac{e^2u}{\varepsilon},\quad \frac{\hbar^2u^2}{2m}=\varepsilon,$$

$$\frac{1}{m^*}=\frac{1}{m}\left(1-\frac{a}{6}\right).$$

The first term means the self-energy of the conduction electron due to the interaction with electronic polarization. Though the above treatment is mathematically quite parallel to the case of the interaction of a conduction electron with lattice vibrations[1] (known as a dolaron), the production of phonons in the latter is to be replaced by that of excitons in the former. The stationary state of (1) will be called "electronic polaron state" in the following.

When there is a perturbing potential in the crystal due to an impurity ion or a vacancy, or applied electric field, we must solve the Schrödinger equation

$$\{H+V(\boldsymbol{r})+V_c\}\Psi(|\,\boldsymbol{r};m_{\boldsymbol{w}},\cdots;t)=i\hbar\frac{\partial}{\partial t}\Psi, \quad (3)$$

where $V(\boldsymbol{r})$ and V_c are the interactions of the potential with the conduction electron (which may be trapped or not) and the crystal electrons, respectively, their relation being given by

$$V_c=\frac{1}{4\pi e}\frac{\gamma}{\sqrt{L^3}}\Sigma_{\boldsymbol{w}}\frac{i}{w}\left\{a_{\boldsymbol{w}}\int\exp(i\boldsymbol{w}\cdot\boldsymbol{r})\Delta V\,d\boldsymbol{r}-a_{\boldsymbol{w}}^*\int\exp(-i\boldsymbol{w}\cdot\boldsymbol{r})\Delta V\,d\boldsymbol{r}\right\}. \quad (4)$$

We first transform (3) into k-space by the expansion

$$\Psi(|\,\boldsymbol{r};\cdots,m_{\boldsymbol{w}},\cdots;t)=\Sigma_{\boldsymbol{k}}f(\boldsymbol{k};t)\,\Psi(\boldsymbol{k}\,|\,\boldsymbol{r};m_{\boldsymbol{w}},\cdots) \quad (5)$$

and then come back to r-space again by the Fourier transformation

$$\Sigma_{\boldsymbol{k}}f(\boldsymbol{k};t)\frac{1}{\sqrt{L^3}}\exp(i\boldsymbol{k}\cdot\boldsymbol{r})=\psi(\boldsymbol{r};t), \quad (6)$$

thus obtaining the equation

$$E(-i\hbar\,\mathrm{grad})\psi+\Sigma_{\boldsymbol{k}}\Sigma_{\boldsymbol{k}'}(V+V_c)_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{k}'}\times\exp(i\boldsymbol{k}\cdot\boldsymbol{r})f(\boldsymbol{k}';t)\frac{1}{\sqrt{L^3}}\exp(i\boldsymbol{k}'\cdot\boldsymbol{r})=i\hbar\frac{\partial}{\partial t}\psi, \quad (7)$$

where E is an operator which is obtained by replacing $\hbar\boldsymbol{k}$ with $-i\hbar$ grad in the band structure function $E(\hbar k)$ of the electronic polaron given by (2). $(V+V_c)_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{k}'}$, the matrix elements in regard to the basic system $\Psi(\boldsymbol{k}\,|\,\boldsymbol{r};\cdots m_{\boldsymbol{w}},\cdots)$, can be calculated explicitly if we expand the various integrands in terms of $(\boldsymbol{k}/u)$ and take into account to the second

power of them, on the assumption that only the terms of relatively small k appear in the expansion (5). Making use of the correspondence theorem:

$$k\times \longrightarrow -i\hbar \operatorname{grad}_r$$

in the transformation (6), we finally get to the equation

$$\left\{E(-i\hbar \operatorname{grad})+\frac{1}{\varkappa_0^*}V(r)+\mu\alpha\frac{1}{\varepsilon}\frac{\hbar^2}{2m}\Delta V(r)\right.$$
$$\times\psi(r;t)+\frac{1}{8}\alpha\left(1-\frac{1}{\varkappa_0}\right)\frac{1}{\varepsilon}\frac{\hbar^2}{2m}$$
$$\left.\times\{\operatorname{grad}V(r)\cdot\operatorname{grad}\psi(r;t)+V(r)\Delta\psi(r;t)\}\right.$$
$$=i\hbar\frac{\partial}{\partial t}\psi(r;t), \qquad (8)$$

where

$$\frac{1}{\varkappa_0^*}=\frac{1}{\varkappa_0}-\frac{1}{4}\alpha\left(1-\frac{1}{\varkappa_0}\right),$$
$$\mu=\frac{1}{24}+\left(\frac{27}{16}-\frac{7\sqrt{2}}{6}\right)\left(1-\frac{1}{\varkappa_0}\right).$$

The second term of (8) shows that the potential $V(r)$ is shielded, not by the macroscopic dielectric constant $\varkappa_0$, but by the effective dielectric constant $\varkappa_0^*$ which is characteristic of the particle considered (an electron in the present case) as well as of the crystal. The last two terms on the left hand side of (8) represents the retardation effects of the electronic polarization behind the motion of the additive electron, which may be important for the large velocities of the latter.

In order that the solutions of (8) satisfy the equation of continuity:

$$\frac{\partial}{\partial t}(\psi^*\psi)+\operatorname{div}J=0,$$

one must define the current vector J as follows:

$$J=\frac{\hbar}{2m^*i}\left\{1-\frac{1}{8}\alpha\left(1-\frac{1}{\varkappa_0}\right)\frac{m^*}{m}\frac{V(r)}{\varepsilon}\right\}$$
$$\times(\psi^*\operatorname{grad}\psi-\psi\operatorname{grad}\psi^*). \qquad (9)$$

The probability distribution of the additive electron is given by

$$\Sigma\cdots\Sigma m_w\cdots|\Psi(|r;\cdots m_w,\cdots;t)|^2$$
$$=|\psi(r;t)|^2+\frac{\alpha}{24}\frac{1}{\varepsilon}\frac{\hbar^2}{2m}\Delta|\psi(r;t)|^2, \qquad (10)$$

thus being different from that of the electronic polaron: $|\psi(r;t)|^2$.

As the interaction constant α approaches to zero, the equation (8) tends to the form

$$\left\{-\frac{\hbar^2}{2m}\Delta+\frac{1}{\varkappa_0}V(r)\right\}\psi=i\hbar\frac{\partial}{\partial t}\psi, \qquad (11)$$

which is the intuitive classical equation generally used in treating such problems. As we have shown previously,[2] (11) can be derived on the continuum model of dielectrics if we apply the adiabatic approximation between the additive and crystal electrons on the ground that the former moves much more slowly than the latter. In fact, the condition $\alpha\to 0$ is attained when ε tends to infinity, that is, when the electronic polarization can follow any rapid motion of the additive electron, the time required to polarize the crystal being of the order of $\hbar/\varepsilon$.

In place of the simplified model formulated in the equation (1), one can take more elaborate one in which the excitation consists of the ionization of filled band electrons with wave numbers w' to the levels of conduction band with wave number w, the energy required thereby depending on w and w'. Such treatment, mathematically more complicated but physically more comprehensive, is expected to lead to essentially the same results as have been derived above. The author wishes to express his sincere gratitude to Prof. T. Inui for his kind encouragements and to Mr. Y. Uemura, Mr. T. Yokota, Mr. F. Takano and Mr. T. Okabayashi for their valuable discussions.

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A Note on the Elastic Frequency Distribution Function of Crystal

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In this note, we attempt to predict the location of the peaks in the elastic frequency distribution function of a simple cubic lattice, together with some remarks on the validity of the Houston approximation.[1]

Our conjectures are based on the examination of the topological nature of the equi-frequency surfaces

in Brillouin zone. We quote our results directly here, without describing the details of the analysis.

It is convenient to use the quantities $f=\nu/\nu_L$ (ν_L is the largest frequency) and $\tau=[2+(a/2\gamma)]^{-1}$ (a is a force constant derived from the interaction of nearest neighbors, and γ is that derived from the interaction of next nearest neighbors). Note that $0\leqq\tau\leqq 1/2$.

When the lattice parameters are such as to make $\tau=1/3$, the frequency distribution has a remarkable correspondence to the one of a square lattice studied by Montroll.[2] There are three branches in the distribution function and the total distribution function is given by their sum. The first branch is extended over the frequency range $f^2=0\sim 2/3$, and has one peak at $f^2=1/3$. The second branch is extended over the frequency range $f^2=0\sim 2/3$. The frequencies are heaped in the region between $f^2=1/3$ and $f^2=2/3$. It was impossible for the author to decide the location of the peaks definitely. It is supposed that the dominant peak would be located at $f^2=1/3$. The third branch is extended over the frequency range $f^2=0\sim 1$. The frequencies are heaped in the region between $f^2=2/3$ and $f^2=3/4$, and its peak will be found at $f^2=3/4$.

Starting from the analysis for the case of $\tau=1/3$, we can draw some predictions for the case of general values of τ. The first dominant peak would be located at $f^2=\tau$, and the second peak at $f^2=(1-\tau)$ for $\tau<1/5$, or at $f^2=(1+3\tau)^2/16\tau$ for $\tau<1/5$. Moreover, in the case of $\tau>1/(2\sqrt{2})$, there would be a steep rise in the frequency spectrum at $f^2=1/8\tau$.

Generally, the use of the Houston formula[3] should be confined to the low frequency region below the first peak of the frequency spectrum.[4] For example, the smaller one between $1/8\tau$ and τ decides the limit of validity of the Houston formula in the case of a simple cubic lattice. There are some numerical errors in the original form of the Houston formula and we present here the corrected formula suitable for practical purposes:

$$g_H(f\nu_L)=(4\pi/105\,V)(10F_1+16F_2+9F_3) \quad \text{(A)}$$

with $V=8\pi^3$, for a simple cubic lattice,

or $V=4\pi^3$, for a face centered cubic lattice.

The definitions of F_1, F_2, and F_3 are found in the reference (1).

The above formula (A) proves to be an excellent approximation for the actual distribution function of the low frequency region below the first peak. It is correctly normalized in the meaning that, if $g(f\nu_L)$ ($\int_0^\infty g(f\nu_L)\,d\nu=1$) is an exact distribution function, $g_H(f\nu_L)$ coincides closely with $g(f\nu_L)$ in the low frequency region below the first peak.

Confining ourselves in the low frequency region, we can obtain approximate formulas of practical convenience as follows:

$$\nu_L g_H(f\nu_L)\cong S(\tau)f^2+P(\tau)f^4,$$

for a simple cubic lattice, (B)

$$g_H(\lambda)\cong F(\varkappa)\lambda^2+H(\varkappa)\lambda^4,$$

for a face centered cubic lattice ($\varkappa=\gamma/a$). (C)

The notations used in the second formula are in accord with the ones in the Leighton's paper.[5] The coefficients $S(\tau)$, $P(\tau)$, $F(\varkappa)$ and $H(\varkappa)$ are certain expressions in terms of τ or $\varkappa$. These are easily obtained, but not listed here. It may be of interest to compare the formula (C) with the result obtained from the laborious numerical calculation.[5] The function $G(\lambda)$ of the Leighton's paper corresponds to $3g_H(\lambda)$:

$$G(\lambda)=0.99\lambda^2+0.35\lambda^4+\cdots, \quad (\varkappa=0)$$

$$3g_H(\lambda)=1.015\lambda^2+0.488\lambda^4+\cdots. \quad (\varkappa=0)$$

Full account, together with other calculations on the related subjects, will be presented in a future publication.

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Numerical Work on the Fluctuation Problem of Electron Cascades

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The problem of obtaining the probability distribution function for the number of electrons or photons above a certain energy in a cosmic-ray shower initiated by a primary electron or photon after passage through

a given thickness of matter has received much attention since the first development of the cascade theory by Carlson and Oppenheimer and independently by Bhabha and Heitler in 1937. Though the formal mathematical treatment of the problem is now complete reliable numerical work has been done only in relation to the first moments of the distribution. Numerical results relating to the second moments were first obtained by Scott and Uhlenbeck and later by Janossy and Messel.

Explicit analytical formulae for the second moments based upon the same assumptions as those made in the original cascade theory were obtained by Janossy and Messel and independently by Bhabha and Remakrishnan by different methods. But numerical computations were made only by Janossy and Messel and the results obtained by them were neither confirmed nor contradicted by subsequent calculations. One of us (R) however felt that those numerical results were open to doubt, but criticism at that stage was not possible since alternative numerical data were not available. The authors have now completed numerical calculations on the second moments of the electron distribution based on the analytical formulae of Bhabha and Ramakrishnan and they find that their results are at variance with those of Janossy and Messel and the earlier calculations of Scott and Uhlenbeck.

If $\bar{N}(E, t)$ and $\overline{N^2(E, t)}$ represent the mean and mean square number of electrons above the energy E, the simple Furry Model required that $\bar{N}^2-(\bar{N})^2=\bar{N}(\bar{N}-1)$ while the Poisson distribution predicted by Bhabha and Heitler in 1937 required that $\bar{N}^2-(\bar{N})^2=\bar{N}$. Janossy and Messel predicted a Poisson deviation in the neighbourhood of the cascade maximum and a greater deviation before and after the cascade maximum. We contend that there seems to be no logical justification from the point of view of stochastic theory why the particles should be distributed in the energy range according to the Poisson law as if they were statistically uncorrelated with one another. This seems to have originated probably from the idea that in a stationary state, correlation tends to zero. *It does not seem to be logically tenable to extend this idea to the cascade maximum, since a maximum of the first moment is not a stationary state in the stochastic sense.*

We have now computed the second moments and the results obtained by us show that the fluctuation is of the same order as the Furry fluctuation—*even in the neighbourhood of the cascade maximum.* The calculations are based upon the explicit Mellin's Transform solution obtained by Bhabha and Ramakrishnan (Equations 41 of their paper). The assumptions made are those of the original cascade theory i.e., we neglect ionisation loss and the lateral spread of showers. We assumed the exact Bethe-Heitler cross-sections for radiation and pair creation processes.

The integral equations for the second moments of Janossy and Messel are mathematically equivalent to those of Bhabha and Ramakrishnan. But those equations are in such a form that an explicit Mellin's transform solution is not possible. In fact, equation 22 of Janossy and Heitler is not an explicit Mellin's transform solution. On the contrary the equations of Bhabha and Remakrishnan have yielded exact Mellin's transform solutions. (Equations 41 of their paper).

We give here a brief summary of the numerical results. A detailed account of the numerical calculations in support of these results and a critical, comparative study of the various attempts to solve the fluctuation problem will be given in a later paper in this journal. The mathematical theory on which the equations of Bhabha and Ramakrishnan are based was formulated earlier by Ramakrishnan as the theory of product-densities.

The mean number of electrons above the energy E is given for all except small t (t is the thickness of matter traversed) by $\bar{N}(y, t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{(s-1)y}}{(s-1)}\times\frac{(D-\lambda_s)}{(\mu_s-\lambda_s)}\cdot e^{-\lambda_s t}\,ds$, where $y=\log\frac{E_0}{E}$ and E_0 is the energy of the initial electron. The functions D, λ_s, μ_s, are defined in the paper of Bhabha and Chakrabarthy who have computed $\bar{N}(y, t)$ for various thicknesses and different values of y.

The mean square number of particles with energy above E is given for all but small t by (for details and notation, see the paper of Bhabha and Ramakrishnan)

$$\overline{N^2(y, t)}=\bar{N}(y, t)+\frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{y(s+r-2)}}{(s-1)(r-1)}\times[(D-\lambda_r)(D-\lambda_s)F_1(r, s)\,e^{-(\lambda_r+\lambda_s)t}+F_2(r, s)\,e^{-\lambda_{r+s-1}t}]\,dr\,ds,$$

where $F_1(r, s)$ is the first element in the matrix $v_1(r, s)$ given by equation (35) of Bhabha and Ramakrishnan, and $F_2(r, s)$ is the first element of

the matrix $v_\lambda(r, s)$ given by (37) of the same paper.* We computed $\overline{N^2}(y, t)$ by the saddle-point method and obtained the following results.

Mean and mean square number of electrons in cosmic ray showers

$t=2$

y	$\bar{N}$	$(\bar{N})^2$	$\overline{N^2}$
3.881	5.8	33.6	60.2
4.356	7.6	57.8	133.0
4.889	10.0	100.0	292.0
5.548	13.8	190.0	667.0

$t=4$

y	$\bar{N}$	$(\bar{N})^2$	$\overline{N^2}$
5.534	24.2	502	922
6.295	42.0	1760	3710
7.186	80.0	6400	16200
8.297	162.0	26200	81100

$t=6$

y	$\bar{N}$	$(\bar{N})^2$	$\overline{N^2}$
7.188	96.1	9.24×10^3	1.71×10^4
8.249	247.0	6.10×10^4	1.30×10^5
9.500	672.0	4.52×10^5	1.13×10^6
11.065	2050.0	4.19×10^6	1.22×10^7

$t=8$

y	$\bar{N}$	$(\bar{N})^2$	$\overline{N^2}$
8.859	436	1.90×10^5	3.62×10^5
10.213	1560	2.43×10^6	5.14×10^6
11.826	6010	3.62×10^7	8.81×10^7
13.840	26900	7.24×10^8	2.05×10^9

We find according to the above calculation that the mean square deviation is of the order of the square of the mean. The full details of our calculations and a comparison with previous results will be given in a later paper. In the above tabulation $\bar{N}$ was obtained by simple interpolation from the cascade tables. In the later paper we propose to give more detailed results and study the behaviour of the fluctuation for a given y with increasing t.

* In that paper, there is a misprint. (36) and (37) denote the same expression, and there is no necessity for two numbers.

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On the Separability of Dirac Equation

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In a previous paper[1] where we have investigated the formulation[2] of quantum mechanics which describes a quantum-mechanical motion of a system by an ensemble of its corresponding classical motions modified by certain 'quantum force', we have especially clarified the following point: For the applicability of this formulation, first, the Hamiltonian of the system must be such as is expressible as a differential operator (not a general integral operator), and second, if the original Schrödinger equation is simultaneous equations for several component wave functions as is the case for a particle with spin, it is required to transform beforehand the original equation into a set of separate equations, in so far as we should be free from resorting to the picture of two or more *mutually transmuting* trajectory ensembles or 'fluids'. Bearing these considerations in mind, we shall now treat a Dirac particle.[3]

The separation of free Dirac equation,

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+H\right)\psi=0\,,\quad H=\boldsymbol{\alpha}\boldsymbol{p}+m\beta\,,\qquad(1)$$

may be attained by the diagonalization of the Hamiltonian with respect spinor indices by means of a unitary transformation such that

$$\psi=U\psi'\,,\quad U^{-1}HU=W=\text{diag.}\qquad(2)$$

The problem is then clearly equivalent to obtaining the simultaneous energy eigen-states $\varphi(\boldsymbol{p})=\chi(\boldsymbol{p})\,\delta(\boldsymbol{p}-\boldsymbol{p}')$ for a specified value of momentum $\boldsymbol{p}$ to be determined from

$$H\chi=W\chi\,,\qquad(3)$$

whose four independent spinor solutions, $\chi^{(\rho)}$'s ($\rho=1,2,3,4$), construct the unitary matrix U of (2) according to

$$U_{\sigma\rho}=\chi_\sigma{}^{(\rho)}.\qquad(4)$$

Here $\chi^{(\rho)}$'s correspond to the energy eigen-values

$$W_\rho=\pm\mathcal{p}_0\,,\quad(\mathcal{p}_0\equiv(\mathcal{p}^2+m^2)^{1/2})\qquad(5)$$

each of which is doubly degenerate.

Though we considered above in the $\mathcal{p}$-representation, we may now regard the argument $\boldsymbol{p}$ in (4) or (5) as an abstract operator, and by this unitary transformation, (1) turns into separate equations

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t}+W_\rho\right)\psi_\rho'=0\,.\qquad(6)$$

On account of the degeneracy of W_ρ, we may take, for U, any set of normalized solutions $\chi^{(\rho)}$'s under the orthogonality condition between $\chi^{(1)}$ and $\chi^{(2)}$, and between $\chi^{(3)}$ and $\chi^{(4)}$. If we specifically pick up the customary solutions,[4] the corresponding unitary operator just agrees with the one due to Foldy[5] and others:

$$U=U_F\equiv\left(\frac{\mathcal{p}_0+m}{2\mathcal{p}_0}\right)^{1/2}\left(1-\frac{\beta\boldsymbol{\alpha}\boldsymbol{p}}{\mathcal{p}_0+m}\right),\qquad(7)$$

in the usual representation for Dirac matrices making β diagonal. However, corresponding to the general set of solutions we obtain the more general unitary operator:

$$U=U_F\cdot U_\alpha\,,\quad U_\alpha=\begin{pmatrix}u&0\\0&u'\end{pmatrix},\qquad(8)$$

where u and u' are any unitary matrices of two rows and columns.

As a simple example, taking $U_\alpha=\beta$, we get a Hermitian operator

$$U=U^{-1}=\{2\mathcal{p}_0(\mathcal{p}_0+m)\}^{-1/2}(H+\beta\mathcal{p}_0).$$

Next example is the unitary transformation, which diagonalizes $\boldsymbol{\sigma}\boldsymbol{p}$ as well as H. It is to be determined by

$$U_\alpha{}^{-1}\,\boldsymbol{\sigma}\boldsymbol{p}\,U_\alpha=\text{diag.},\qquad(9)$$

because the Foldy transformation U_F is the one which diagonalizes H leaving $\boldsymbol{\sigma}\boldsymbol{p}$ invariant. From (9) we get

$$u=u'\cdot\frac{1}{\sqrt{2}\mathcal{p}}\begin{Bmatrix}\sqrt{\mathcal{p}+\mathcal{p}_3}&-\sqrt{\mathcal{p}-\mathcal{p}_3}\\ \mathcal{p}_1+i\mathcal{p}_2&\mathcal{p}_1+i\mathcal{p}_2\\ \sqrt{\mathcal{p}+\mathcal{p}_3}&\sqrt{\mathcal{p}-\mathcal{p}_3}\end{Bmatrix},\qquad(10)$$

which satisfies $\boldsymbol{\sigma}\boldsymbol{p}u=\mathcal{p}u\sigma_3$.

In any case, as the result of transformations the original wave equation separates into the form (6), where the new Hamiltonian W_ρ becomes, in return, an *integral* operator in the ordinary space representation, and we find again that our previous formulation of quantum mechanics is inapplicable to this case as was remarked in a previous note.[6]

Next we shall investigate another procedure for the separation of the Dirac equation, which is feasible also for the case of a particle in a constant electromagnetic field. Generally the Dirac equation in an external potential A_μ,

$$(\gamma_\mu D_\mu+\varkappa)\psi=0\,,\quad(D_\mu\equiv\partial_\mu-ie\hbar^{-1}A_\mu)\qquad(11)$$

is transformed[7] by introducing a '*potential*' spinor Ψ such as

$$\psi=(\gamma_\mu D_\mu-\varkappa)\,\Psi\,,\qquad(12)$$

into

$$(D_\mu{}^2-\varkappa^2+e\hbar^{-1}\Omega)\,\Psi=0\,,\qquad(13)$$

$$\Omega\equiv-(i/2)\sum_{\mu\nu}\gamma_\mu\gamma_\nu F_{\mu\nu}=\boldsymbol{\sigma}\boldsymbol{H}-i\boldsymbol{\alpha}\boldsymbol{E}\,.\qquad(14)$$

In this form we could thus separate γ_μ's from D_μ's in the operator operating on the wave function. Then for the separation of (13), it is required only to diagonalize the momentum-independent term $e\hbar^{-1}\Omega$, provided the field strength $F_{\mu\nu}$ be constant. That is, by the transformation $\Psi=U\Psi'$, which makes $\Omega'=U^{-1}\Omega U$ diagonal, the transformed equation will become four separate *local* wave equations:

$$(D_\mu{}^2-\varkappa^2)\,\Psi_\rho'+e\hbar^{-1}\Omega_{\rho'}\,\Psi_{\rho'}=0\,.\qquad(15)$$

For our problem it is convenient to exploit the '*spinor representation*' that makes γ_5 diagonal, for we have then the 'Stufenmatrix'

$$\Omega=\begin{pmatrix}\boldsymbol{\sigma}\boldsymbol{B}&0\\0&\boldsymbol{\sigma}\boldsymbol{C}\end{pmatrix},\qquad(16)$$

where $\boldsymbol{B}=\boldsymbol{H}-i\boldsymbol{E}$ and $\boldsymbol{C}=\boldsymbol{H}+i\boldsymbol{E}$ are self-dual tensors. Now the problem to diagonalize Ω becomes formally analogous with (9), but the essential dif-

ference lies in the fact that the operator $\mathcal{Q}$ is not Hermitian due to the imaginary property of the space-time components of $F_{\mu\nu}$, and hence that its 'eigenvalues' are complex and the transformation matrix is no longer unitary. The secular equation for $\mathcal{Q}U=U\mathcal{Q}'$ gives the 'eigenvalues' of $\mathcal{Q}$ to be

$$\mathcal{Q}_\rho'=\pm B,\ \pm C, \tag{17}$$

where $B\equiv(\boldsymbol{B}^2)^{1/2}$ and $C\equiv(\boldsymbol{C}^2)^{1/2}$ can be expressed with a scalar $\mathfrak{F}=\boldsymbol{H}^2-\boldsymbol{E}^2$ and a pseudoscalar $\mathfrak{G}=2\boldsymbol{H}\boldsymbol{E}$, as

$$B=(\mathfrak{F}-i\mathfrak{G})^{1/2},\quad C=(\mathfrak{F}+i\mathfrak{G})^{1/2}.$$

These eigenvalues may also be obtainable from the relation,[8] $\mathcal{Q}^2=\mathfrak{F}+i\gamma_5\mathfrak{G}$. The transformation matrix is obtained to be

$$U=\begin{pmatrix} u_B & 0 \\ 0 & u_{C'} \end{pmatrix},\quad u_B\propto\begin{pmatrix} B+B_3 & -B+B_3 \\ B^+ & B^+ \end{pmatrix},\ \text{etc.} \tag{18}$$

where $B^+=B_1+iB_2=H^+-iE^+$, etc. By such transformation we get (15), each of which is of the wave equation for Klein-Gordon field coupled with an *extra complex scalar field*, $\mathcal{Q}_\rho'$.

At this point, putting further

$$\Psi_\rho'=R_\rho e^{iS_\rho/\hbar},\quad \mathcal{Q}_\rho'=a_\rho+ib_\rho,$$
$$(R_\rho,\ S_\rho,\ a_\rho,\ b_\rho:\ \text{real})$$

each of (15) turns into

$$\begin{cases}(\partial_\mu S-eA_\mu)^2+m^2-\hbar^2\Box R_\rho/R_\rho-e\hbar\, a_\rho=0,\\ \partial_\mu\{R_\rho^2(\partial_\mu S_\rho-eA_\mu)\}=-eb_\rho R_\rho^2,\end{cases}$$

with

$$\begin{cases}a_\rho=\pm 2^{-1/2}\{(\mathfrak{F}^2+\mathfrak{G}^2)^{1/2}+\mathfrak{F}\}^{1/2},\\ b_\rho=\pm 2^{-1/2}\{(\mathfrak{F}^2+\mathfrak{G}^2)^{1/2}-\mathfrak{F}\}^{1/2}.\end{cases}$$

Though this form corresponds to the picture of a set of four mutually independent trajectory ensembles,[9] we must there regard that the electromagnetic field acts upon particle not only in a directly classical manner but also through an extra scalar field,[9] and moreover that each ensemble has *source* unless $\boldsymbol{H}\boldsymbol{E}=0$. Further it is to be noted that this correspondence is more indirect in so far as it is to be intermediated through the relations (12) and (18).

Details and further considerations will soon be published.

1) T. Takabayasi, Prog. Theor. Phys. **8** (1952), 143.
2) Also see, T. Takabayasi, Prog. Theor. Phys. **9** (1953), 187.
3) As for another treatment, cf., D. Bohm, Prog. Theor. Phys. **9** (1953), 273; Reference 2), Appendix F.
4) See e.g., W. Heitler, *The Quantum Theory of Radiation* (1936), p. 86.
5) L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78** (1950), 29; S. Tani, Prog. Theor. Phys. **6** (1951), 267; also see Reference 6).
6) T. Takabayasi, Prog. Theor. Phys. **9** (1953), 557.
7) V. Fock, Phys. ZS. Sowjetunion **12** (1937), 404.
8) See, J. Schwinger, Phys Rev. **82** (1951), 664.
9) See, Reference 2).

On the D State Probability of Deuteron

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May 19, 1953

From the experimental values of the magnetic moments of proton, neutron and deuteron, we can only say, about the D state probability of deuteron (hereafter denoted as P_D), that

$$0\% < P_D < 10\%. \tag{1}$$

This large uncertainty comes from the possible existence of about 6% uncertain corrections to the non-relativistically calculated value (4%), 2% being due to the relativistic corrections[1] and 4% to the dependence of the magnetic moment of one nucleon on the proximity of another.[2]

It is the object of this note to show that we can obtain more knowledge about P_D utilizing the experimental ratio of the hyperfine structure separations of the ground states of deuterium and hydrogen besides the experimental values of the magnetic moments. The discrepancy between the experimental ratio of the hyperfine structure in deuterium and hydrogen and that obtained from Fermi's formula is expressed by

$$\Delta=1-\frac{(\nu_D/\nu_H)_{\exp}}{(\nu_D/\nu_H)_F}, \tag{2}$$

$$(\nu_D/\nu_H)_F=(3/4)(m_D/m_H)^2(\mu_D/\mu_P), \tag{3}$$

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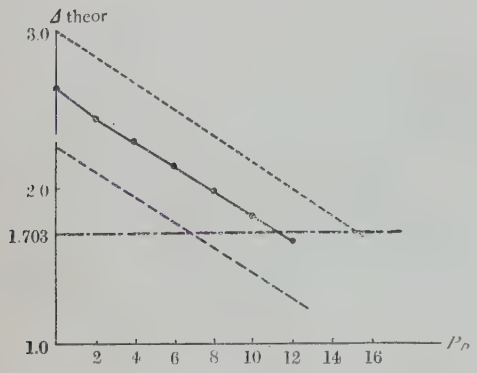

Fig. 1. Δ_{theor} calculated assuming a point nucleon. —•— and – – – – represent theoretical values and its uncertainties, and — · — · — represents experimental value respectively.

m_D and m_H being reduced masses of deuterium and hydrogen atoms, μ_D and μ_P being the magnetic moments of deuteron and proton respectively. From the experimental values for ν_D/ν_H[3], μ_D/μ_P[4], and m_D/m_H, we obtain

$$\Delta = (1.703 \pm 0.008) \times 10^{-4} . \qquad (4)$$

Although the major part of this discrepancy has been explained by A. Bohr[5] and more in detail by Low[6] as the effect of the structure of deuteron, the experimental-value, taking into account the relativistic effects for nucleons and the radiative corrections,[7]

$$\Delta_{\text{theor}} = (2.30 \pm 0.40) \times 10^{-4} \quad [8] \qquad (5)$$

is well outside the range of experimental uncertainties.

The value (5) obtained for Δ was based on the assumption that (a) each nucleon behaves as a point particle with both its electric charge and magnetic moment concentrated in a point and (b) P_D is just 4%. Since it is meaningless to adhere to the value 4% for P_D, it seems to be the most reasonable approach to derive a reliable restriction to P_D from the experimental value (4) for Δ, taking into account the probable effect of the structure of nucleons.

We have calculated Δ_{theor} varying P_D and assuming point nucleons. From the results given in Fig. 1., we obtain a restriction to P_D, i.e. $7\% < P_D \lesssim 15\%$. If the anomalous magnetic moment of a nucleon is assumed to be spread uniformly over a certain finite volume, the radius of which being of the order of the π meson Compton wave length, this restriction becomes to $5\% \sim P_D \lesssim 13\%$. So we obtain, from the experimental value (4), a restriction to P_D,

$$5\% < P_D \lesssim 15\% . \qquad (6)$$

Combining this value with (1), obtained from the magnetic moments, we arrive at the most reliable value of P_D,

$$5\% < P_D \lesssim 10\% . \qquad (7)$$

It seems worth while to note that the adiabatic nuclear forces derived from the pseudoscalar meson theory with pseudovector coupling up to fourth order in coupling constant are able to reproduce the all low energy data for neutron-proton system including the above obtained value for P_D.[9]

The author wishes to express his sincere thanks to Professor M. Taketani for his stimulating discussions, and to the Yukawa Hall and Professor M. Kobayasi for the hospitality.

1) G. Breit and R. M. Thaler, Phys. Rev. **89** (1953), 1177, and earlier works cited there.
2) H. Miyazawa, Prog. Theor. Phys. **7** (1952), 207.
3) A. G. Prodell and P. Kusch, Phys. Rev. **88** (1952), 184.
4) B. Smaller, E. Yasaitis and H. I. Anderson, Phys. Rev. **81** (1951), 896.
5) A. Bohr, Phys. Rev. **73** (1948), 1109.
6) F. Low, Phys. Rev. **77** (1950), 361.
7) R. Karplus and A. Klein, Phys. Rev. **85** (1952), 972; N. M. Kroll and F. Pollock, Phys. Rev. **86** (1952), 876.
8) E. E. Salpeter and W. A. Newcomb, Phys. Rev. **87** (1952), 150.
9) M. Taketani, S. Machida, and S. O-numa, Prog. Theor. Phys. **7** (1952), 45. In the case of pseudoscalar coupling, the fourth order potential obtained in the above mentioned paper, eq. (22), must be multiplied by factor (−4). No other calculations nor conclusions in that paper, however, are altered by this error, since we have confined our calculations to the case of pseudovector coupling only, throughout that paper. Non-adiabatic corrections to the fourth order potentials have been calculated by Nakabayashi and Sato (K. Nakabayashi and I. Sato, Phys. Rev. **88** (1952), 144; and I. Sato, Prog. Theor. Phys. to be published), and it has been shown that these corrections are not large for the case of pseudovector coupling, although they are very large for the case of pseudoscalar coupling. (M. M. Lévy has also calculated the fourth order non-adiabatic nuclear potentials for the case of pseudoscalar coupling. But his result is in serious error. M. M. Lévy, Phys. Rev. **88** (1952), 725.

On the Families of Spinor Fields*

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Usually, anti-symmetrical commutation relations are adopted for the different kinds of spinor fields. In the case of commutation relations of (+)-type between two different Fermi fields they are not regarded as kinematically independent, but as the different states of the identical field obeying the Fermi statistics. In this sense, we may say that the assembly of the spinor fields which obey anti-commutation relations among them constitutes a "family" and the fields of different "family" are independent and commutable with eaeh other.[1] In this paper the word "family" is used in the above sense. We should like to classify the spinor fields into several families from the formal requirement of quantized field theory[2] and the recently accumulated, rather temporary knowledges on elementary particles.

We summarize our requirement of quantized field theory as follows: "The present quantum field theory is written in the canonical formalism which has the interaction representation as its special one".[3] Therefore there exists an unitary transformation $S(\sigma)$ which combines between the interaction- and the Heisenberg-representations. The interaction Hamiltonian $H(x)$ is defined by

$$i\frac{\delta S(\sigma)}{\delta\sigma(x)}=H(x)S(\sigma). \qquad (1)$$

In general $H(x)$ is the sum of various interactions:

$$H(x)=\Sigma_i H_i(x) \quad (i=1, 2, 3, \cdots). \qquad (2)$$

$\psi^\alpha(x)$ $(\alpha=1, 2, \cdots)$ describing α-th spinor field satisfies the well-known anti-commutation relation in the interaction representation[4]

$$[\psi^\alpha(x), \bar{\psi}^\alpha(x')]_+=iR^\alpha(\partial)\Delta(x-x'). \qquad (3)$$

From the above requirement, $\psi^\alpha(x)$ and the interaction $H_i(x)$ must satisfy the following remarkable relations[5]

$$[\psi^\alpha(x), H_i(x')]_-=-iR^\alpha(\partial)\Delta(x-x')j_i{}^\alpha(x'),$$
$$j_i{}^\alpha=-\partial H_i/\partial\bar{\psi}^\alpha, \quad (i, \alpha=1, 2, 3, \cdots). \qquad (4)$$

From eq. (4) the following important theorems can be derived:

Theorem (I)

"If we take any spinor** field ψ^α, every interaction $H_i(i=1, 2, \cdots)$ should involve even number $(0, 2, \cdots)$ of spinor** fields which have anti-commutation relations with ψ^α".

Since all real or virtual processes are realized by the repetition of interactions $H_i(i=1, 2, 3, \cdots)$, we get

Theorem (II)

"In any transmutation process (real or virtual) $A_1+A_2+\cdots \rightleftarrows B_1+B_2+\cdots$ there should appear even number $(0, 2, \cdots)$ of spinor fields which satisfy (+) type commutation relations with any spinor field ψ^α".

Theorem (III)

"If there is the following transmutation reaction (virtual or real) involving only two spinor fields A and B:

$$A \rightleftarrows B+C_1+C_2+\cdots \quad (C_1, C_2, \cdots \text{ are some bosons}) \qquad (5)$$

then A must be anti-commutable with B. That is, A and B belong to the same family".

Theorem (IV)

"For the transmutation process (5) (real or virtual) any spinor field ψ^α which is anti-commutable with A must be also anti-commutable with B".

According to these theorems and realized processes, we will classify the spinor fields into various families. The nucleon-π-meson interaction and V^0-decay

$$P\rightarrow N+\pi^+ \text{ and } V^0\rightarrow P+\pi^- \qquad (6)$$

indicate from theorem (III) and (IV) that, (N, P, V^0) constitutes a "family" which may be called as "nucleon family". From β-decay interaction $(N\rightarrow P+e^-+\nu)$ and (6), we get the process

$$\pi^\pm\rightarrow e^\pm+\nu, \qquad (7)$$

which proves that electron and neutrino belong to a group named "lepton family" tentatively. In the same manner the π-decay or μ^--capture by nuclei

$$\pi^\pm\rightarrow\mu^\pm+\mu^0 \text{ (or } \mu^-+P\rightarrow N+\mu^0) \qquad (8)$$

shows the existence of the family $(\mu^\pm, \mu^0)$. If μ^0

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* Read at the Kyoto Meeting of the Physical Society of Japan, May 3, 1953.

** When we take into account the families including integer spin fields, it is unnecessary to confine to the spinor fields from only the requirement of the condition (4).

is identified with neutrino, $\mu^{\pm}$-meson belong to the "lepton family" by theorem (IV). If μ_1^0 is turned out to be neutrino in the μ-decay process

$$\mu^{\pm} \to e^{\pm} + \mu_1^0 + \mu_2^0, \qquad (9)$$

$(\mu^{\pm}, \mu_2^0)$ belong to the same family and if, moreover, μ_2^0 is also neutrino, $\mu^{\pm}$-meson belongs to the "lepton family". Further, for the recently observed χ-decay,[6] several possibilities can be considered:

$$\chi^{\pm} \to \mu^{\pm} + \chi_1^0 + \chi_2^0. \qquad (10)$$

If both χ_1^0 and χ_2^0 are identified with neutrino, χ-meson is also a member of the "lepton family". These arguments will be improved by the accumulation of the precise data (especially of the decay processes) and we expect to obtain accurate knowledges about the families of spinor fields in future. At present, it might be possible that "lepton family" includes μ or χ-meson. There are, however, no definite experimental suggestions that leptons are included in "nucleon family". It must be noted that from our theorems we can not directly assert the conservation of the number of particles of each family in any reaction, but only point out the change of the members of each family by even number. Summarizing above results, we have the following several possibilities for the families of spinor fields.

Case (*A*)

"Spinor fields are divided into only "nucleon" and "lepton families".

In order to obtain the stability of matter, one of the authors[7] defined a sort of charge (which may be called as nucleonic charge) and postulated the conservation of this charge in any reaction. He also defined nucleon family as those fields (N, P, V^0) which have unit nucleonic charge and electron family as those spinor fields $(e, \nu, \mu, \chi, \cdots)$ with zero nucleonic charge. If bosons with unit nucleonic charge (such as $V^0 \to P + \pi^- + \nu$ or $V^+ \to N + \mu^+$)[8] do not exist, the conservation of nucleonic charge means that the number of the members of "nucleon family" appearing in any process is always "even" and thus case (A) is consistent with our criterion. On the other hand, if we admit the case (A), there will never occur the transition process such as $V^0 \to P + \pi^- + \nu$ or $V^0 \to P + \mu^-$ which implies that (P, ν) or (P, μ) belongs to the same family.[9]

In this standpoint (A), it is interesting to search for the intrinsic property which characterizes each family. For example, each family might have different properties of the time, space-inversion or charge conjugation. But also in such prescriptions it can be only concluded that in any reaction the number of spinor particles of each family changes by even number. It is interesting to inquire what corresponds to nucleonic charge.

Case (*B*)

"All fermions belong to the same family".

This possibility suggests that all fermions may correspond to the different internal states of the same particle. However, it seems still necessary to find the another reason which explain the experimental essential difference between the properties of heavy particles and leptons. In this case (B), the existence of heavy bosons with unit nucleonic charge is allowed contrary to the case (A).

Case (*C*)

"Except nucleon family (N, P, V^0), fermions constitute the set of families by pairs (charged and neutral fermions)".

For instance, if $\pi^{\pm} \to \mu^{\pm} + \mu^0$, $\mu^- + P \to N + \mu^0$, $\mu^{\pm} \to \mu^0 + e^{\pm} + \nu$, and $\chi^{\pm} \to \chi^0 + \mu^{\pm} + \mu^0$ are realized, such set of fameills as $(e^{\pm}, \nu)$, $(\mu^{\pm}, \mu^0)$ and $(\chi^{\pm}, \chi^0)$ may exist. At present, this possibility cannot always be denied, but is essentially dependent on the identification of neutral particles. The exception of the nucleon family (N, P, V^0) is the weak point of the case (C).

While we cannot a priori deny the other complicated structures of the families of spinor fields, the case (A) seems to be favoured by the present phenomenological informations of the elementary particles. It is hoped that these approach will give some insight into the future concept of elementary particles.

1) From only intuitive considerations, there seems to be no physical distinctions whether we regard two spinor fields as the different particles of Bose statistics or as the different states of particles obeying Fermi statistics. However, this intuitive picture is based on the existence of the assembly of free particles with canonical formalism. On the other hand, it is shown that the possibility of canonical formalism restricts the commutation relations between different spinor particles. For example, if spinor particles belonging to the same family correspond to the various isobar states of one particle, it is necessary to require Fermi statistics among them.

2) From the integrability conditions of Tomonaga-Schwinger equations, Nishijima discussed on the possible restrictions upon the commutation relations between different spinor fields. K. Nishijima, Prog. Theor. Phys. **5** (1950), 187. His arguments, however, are based on the promise

of the existence of interaction Hamiltonian and, as we shall show, further restriction involving Nishijima's conditions are derived from the requirement of the existence of interaction Hamiltonian.

3) This corresponds to the demand that the description of the state of field as the assembly of free particles is possible.

4) In this paper we call fields with "half-odd" integer spin as spinor particles. When ψ^α has spin 1/2, $R^\alpha(\partial)\Delta(x-x')=-S(x-x')$.

5) The detailed explanations of eq. (4) are given in H. Umezawa and Y. Takahashi, Prog. Theor. Phys. in press and Y. Takahashi and H. Umezawa, Prog. Theor. Phys. **9** (1953), 14. Eq. (4) expresses the intimate relations between commutation relations and Green functions of the fields and is the complete condition that unitary transformation $S(\sigma)$ connects the eq. of motion in the interaction representation with that in the Heisenberg representation.

6) We assume that the processes (9) and (10) occur in the four fermion fields.

7) S. Ōneda, Prog. Theor. Phys. **9** (1953), 327.

8) If the coplanarity is not certain in V^0-decay, the possible decay schemes $V^0\to P+\pi^-+\nu$ will be proposed. In this case neutral boson field V^0 must be complex field from the requirement of the conservation of heavy particles. See reference (7).

9) Besides $V^0\to P+\pi^-$, $V^0\to P+\mu^-(e^-)+\nu$ will be allowed in this standpoint (A).

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